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*Dedicated to Professor Jerzy Browkin on the occasion of his 60th birthday*

**Abstract**

The paper gives a further generalization of congruences of the K. Hardy and K.S. Williams [3] type among the values of 2-adic $L$-functions $L_2(k, \chi\omega^{1-k})$ for quadratic Dirichlet characters $\chi$ and for $-1 \leq k \leq 2$ which produce some new congruences between the conjectured orders of $K_2$-groups of the integers and class numbers of appropriate quadratic fields. These congruences extend results of [2], [5], [3], [6] and are of the same type as congruences of [8] and [7]. We apply ideas of R.F. Coleman [1] and methods of T. Uehara [5].

## 1 Introduction

Let $k$ be an integer. If $p$ is a prime number, let $\mathbf{C}_p$ stand for completion of an algebraic closure of $\mathbf{Q}$ at some place above $p$. Consider the formal series:

$$l_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}.$$

This series determines an analytic function on the open unit ball in $\mathbf{C}_p$. Using "the action of Frobenius" on some differential equations, R.F. Coleman [1] extended $l_k$ to a locally analytic function $l_{k,p}$ on $\mathbf{C}_p - \{1\}$. He gave a $p$-adic analogue of some well-known analytic formulas for the values of $p$-adic $L$-functions at integers

$$L_p(k, \chi\omega^{1-k}) = (1 - \chi(p)p^{-k})g(\chi)M^{-1} \sum_{a=1}^{M-1} \overline{\chi}(a) l_{k,p}(\zeta^{-a}), \tag{1}$$

extending Leopoldt's formulas for $L_p(1, \chi)$, where $\chi$ denotes a primitive Dirichlet character modulo $M$ ($M > 1$) with values in $\mathbf{C}_p$ (for positive $k$ see formula (2) of [1] and for non-positive $k$ see Theorem 5.11 and Lemma 5.20 [9]). Here $\zeta := \exp(2\pi i/M)$, $g(\chi)$ stands for the Gauss sum attached to $\chi$ and $\omega := \omega_p$ denotes the Teichmüller character at $p$. It is well-known that for Dirichlet characters $\chi_1$ and $\chi_2$ with relatively prime conductors we have $g(\chi_1\chi_2) = g(\chi_1)g(\chi_2)$. The formula (1) is true for primitive characters. However note also that if the character $\chi$ is induced from a character $\chi_1$ modulo some divisor of $M$, then

$$B_{n,\chi} = B_{n,\chi_1} \prod_{p|M} (1 - \chi_1(p)p^{n-1}),$$

where $B_{n,\chi}$ is the $n$th generalized Bernoulli number belonging to the character $\chi$ (cf. the proof of Theorem [8]). Therefore by

$$L_p(k, \chi) = \lim L_p(1 - n, \chi),$$

where $1 - n \to k$ $p$-adically and $n \to \infty$, and by Theorem 5.11 [9] we get

$$\begin{aligned} L_p(k, \chi) &= -\lim (1 - \chi(p)p^{n-1}) \frac{B_{n,\chi}}{n} = -\lim \frac{B_{n,\chi}}{n} \\ &= -\lim \left( \frac{B_{n,\chi_1}}{n} \prod_{q|M,\, q-\text{prime}} (1 - \chi_1(q)q^{n-1}) \right) \\ &= L_p(k, \chi_1) \prod_{q|M,\, q \neq p} (1 - \chi_1(q)\omega^{k-1}(q)q^{-k}), \end{aligned}$$

because

$$\lim q^n = \lim (\omega(q) < q >)^n = \lim < q >^n = < q >^{1-k} = q^{1-k}\omega^{k-1}(q),$$

if $q \neq p$. Consequently, we obtain

$$L_p(k, \chi\omega^{1-k}) = L_p(k, \chi_1\omega^{1-k}) \prod_{q|M,\, q \neq p} (1 - \chi_1(q)q^{-k}),$$

but we shall not use this in the paper.

Following R.F. Coleman [1], the functions $l_k := l_{k,p}$ are called the multilogarithms. For $-1 \le k \le 1$ by definition we get explicit formulas for $l_k$:

$$l_{-1}(z) = \frac{z}{(1-z)^2}\,,$$

$$l_0(z) = \frac{z}{1-z}\,,$$

$$l_1(z) = -\log_p(1-z)\,,$$

where $\log_p$ denotes the $p$-adic logarithm. The function $l_2$ is related to the so-called $p$-adic dilogarithm function defined by the formula

$$D(z) = l_2(z) + \frac{1}{2}\log_p(z)\log_p(1-z)\,.$$

It is well-known (see Proposition 6.4 [1]) that

$$l_k(z) + (-1)^k l_k(z^{-1}) = \frac{-1}{k!}\log_p^k(z), \tag{2}$$

($1/k! := 0$, if $k < 0$) and for any positive integer $m$ the functions $l_k$ satisfy the identity

$$\frac{1}{m}\sum_{\zeta^m=1} l_k(\zeta z) = \frac{l_k(z^m)}{m^k} \tag{3}$$

(see Proposition 6.1 [1] with $z$ replaced by $z^m$ on the right hand side of the equation).

Let $\chi$ be a primitive non-trivial Dirichlet character. Then, it follows from (2) that for $k \neq 0$

$$\sum_{a=1}^{M-1} \overline{\chi}(a) l_k(\zeta^{-a}) = \left(1 + (-1)^{k+1}\overline{\chi}(-1)\right) \sum_{a=1}^{[M/2]} \overline{\chi}(a) l_k(\zeta^{-a}),$$

where $[x]$ denotes the integral part of $x$. Thus for $k \neq 0$ and for primitive non-trivial characters $\chi$, by (1) we get

$$L_p(k, \chi\omega^{1-k}) = 0\,, \tag{4}$$

if $\overline{\chi}(-1) = (-1)^k$ (i.e., if $\chi$ and $k$ are of the same parity). If $k = 0$ then by (2) we have

$$l_k(z) + l_k(z^{-1}) = -1$$

and

$$\sum_{a=1}^{M-1} \overline{\chi}(a) l_k(\zeta^{-a}) = \left(1-\overline{\chi}(-1)\right) \sum_{a=1}^{[M/2]} \overline{\chi}(a) l_k(\zeta^{-a}) - \overline{\chi}(-1) \sum_{a=1}^{[M/2]} \overline{\chi}(a),$$

which gives (4) for an even non-trivial Dirichlet character $\chi$ at once. For $k=0$, by (1) the equation (4) holds for $\chi(p)=1$ too but we shall not use this.

Let $\chi$ be trivial and let $k \neq 1$. If $k \leq 0$ then, by Theorem 5.11 [9] we have

$$L_p(k, \omega^{1-k}) = -(1-p^{-k}) \frac{B_{1-k,\chi}}{1-k}.$$

Thus if $k=0$ then (4) holds for the trivial character $\chi$ too because of the Euler factor equals 0. If $k \leq -1$ then (4) holds if $B_{1-k}=0$, i.e., if $k$ is even.

Let $p$ be finite and let $E_p$ be a finitely ramified extension of $\mathbf{Q}_p$ in $\mathbf{C}_p$. If $k \geq 2$ then

$$L_p(k, \omega^{1-k}) = (1-p^{-k}) \lim l_{k,p}(z),$$

where $z \to 1$ and elements $z$ lie in $E_p - \{1\}$ (see the formula (4) in [1]). Thus if $k \geq 2$ is even then (2) implies

$$2L_p(k, \omega^{1-k}) = -\frac{1}{k!}(1-p^{-k}) \lim \log_p^k(z) = 0.$$

Summarizing, if $\chi$ is trivial then (4) holds if $k \neq 1$ is even, i.e., if $k$ has the same parity as $\chi$ again.

Following R.F. Coleman [1], write

$$l_k^{(p)}(z) = l_k(z) - p^{-k} l_k(z^p),$$

where $l_k = l_{k,p}$. The functions $l_k^{(p)}(z)$ are called $p$-adic multilogarithms. In particular, in view of (3), we have

$$l_k^{(2)}(z) = \frac{1}{2}\left(l_k(z) - l_k(-z)\right). \tag{5}$$

Let $A$ be an integer. For any Dirichlet character $\psi$ modulo $A$, any integer $k$ and $z \in \mathbf{C}_2$, set

$$\mathcal{L}_{k,\psi}(z) = (-1)^{k+1} l_k^{(2)}(z) \qquad (z \neq \pm 1),$$

if $\psi$ is the trivial character modulo $A$ and

$$\mathcal{L}_{k,\psi}(z) = (-1)^{k+1} g(\overline{\psi}) A^{-1} \sum_{a=1}^{A}{}' \psi(a) l_k(\zeta_A^a z) \quad (z \neq \zeta_A^a,\ (a,A)=1),$$

otherwise.

In particular, if $A$ is even and $\psi$ is a quadratic character modulo $A$ then by (5) we have

$$\mathcal{L}_{k,\psi}(z) = (-1)^{k+1} 2 g(\psi) A^{-1} \sum_{a=1}^{A/2}{}' \psi(a) l_k^{(2)}(\zeta_A^a z)$$

because

$$\psi\Big(\frac{A}{2} + a\Big) = -\psi(a) .$$

For any odd natural number $b$, let $r(b)$ denote the number of prime factors of $b$. Set $b^* = b$ (resp. $-b$), if $b \equiv 1 \pmod 4$ (resp. $b \equiv 3 \pmod 4$). These numbers are examples of the so-called fundamental discriminants (which can be described as the set of square-free numbers of the form $4n+1$ and 4 times square-free numbers not of this form). For any fundamental discriminant $d$, denote by $\chi_d$ the primitive quadratic character modulo $|d|$ (in this notation $\chi_1$ is the primitive trivial character). Write $\mathcal{L}_{k,d} = \mathcal{L}_{k,\chi_d}$. For a natural number $m$, denote by $\mathcal{T}_m$ the set of all fundamental discriminants dividing $m$. Let us adopt the notations $\prod_{p|c}$ (resp. $\prod_{a=1}^{c}{}'$, $\sum_{a=1}^{c}{}'$) to stand for a product taken over all primes dividing $c$ (resp. a product or a sum taken over integers $a$ prime to $c$).

## 2 The Main Theorem

Let $K = \{-1, 0, 1, 2\}$. Let us consider a sequence of 2-adic integers $\{x_{k,e}\}$, $k \in K$, $e \in \mathcal{T}_8$. For any $L \subset K$ this sequence is said to be defined on $L$, if $x_{k,e} = 0$ for $k \notin L$. Given $\{x_{k,e}\}$, let us define a sequence $\{z_n\}_{n=0,1,\ldots}$ by the following:

$$z_0 = \sum_{k \in K,\, e \in \mathcal{T}_8} x_{k,e} , \qquad z_1 = 2 \sum_{\substack{k \in K,\, e \in \mathcal{T}_8, \\ \operatorname{sgn} e = (-1)^k}} x_{k,e} ,$$

$$\begin{aligned} z_{2l+\rho} &= 2^{l+\rho}\Big(2^l(2l+1)^2\big((1-\rho)x_{-1,1} + x_{-1,-4}\big) \\ &- (2l-1)(2l+1)^2\big((1-\rho)x_{-1,8} + x_{-1,-8}\big) + (2l+1)^2\big(x_{0,1} + (1-\rho)x_{0,-4}\big) \\ &+ 2^l(2l+1)\big((1-\rho)x_{1,1} + x_{1,-4}\big) + (2l+1)\big((1-\rho)x_{1,8} + x_{1,-8}\big) \\ &+ 2^{3l}\binom{2l}{l}^{-1}\Big(\big(x_{2,1} + (1-\rho)x_{2,-4}\big) + \sum_{k=0}^{l}\binom{2k}{k}2^{-3k}\big(x_{2,8} + (1-\rho)x_{2,-8}\big)\Big)\Big) , \end{aligned} \tag{6}$$

where $l \geq 1$ and $\rho \in \{0, 1\}$.

It is evident that the numbers $z_n$, $n \geq 0$ are 2-adic integers. Indeed, it is well known that

$$\mathrm{ord}_2 \binom{2t}{t} = s_2(t),$$

where $s_2(t)$ denotes the sum of digits in the 2-adic expansion of $t$. Thus we have

$$\mathrm{ord}_2 \left( 2^{3l} \binom{2l}{l}^{-1} \right) = 3l - s_2(l) \geq 2l \,. \tag{7}$$

Moreover, we observe that

$$\mathrm{ord}_2 \left( 2^{3l} \binom{2l}{l}^{-1} \sum_{k=0}^{l} \binom{2k}{k} 2^{-3k} \right) = \mathrm{ord}_2 \left( \sum_{k=0}^{l} \frac{l!((2k-1)!)_2}{k!((2l-1)!)_2} 2^{2(l-k)} \right) = 0 \,, \tag{8}$$

where $(t!)_2 := 1 \cdot 3 \cdots t$ ($t$ odd) denotes the 2-adic factorial.

**DEFINITION.** *For any non-empty subset $L \subset K$, let $c := c(L) \geq 0$ be an integer such that:*

(i) *there exists a sequence of 2-adic integers $\{x_{k,e}\}$ defined on $L$, not all being even, satisfying*

$$z_n \equiv 0 \pmod{2^c},$$

*if $n = 0, 1, 2, \ldots$*

(ii) *if for some sequence of 2-adic integers $\{x_{k,e}\}$ defined on $L$ we have*

$$z_n \equiv 0 \pmod{2^{c+1}}$$

*then all the numbers $x_{k,e}$ are even.*

If $\chi$ is a primitive Dirichlet character and $M > 1$ is any natural number then for $k \in \mathbf{Z}$ we set

$$L_2^{[M]}(k, \chi\omega^{1-k}) = 0 \,,$$

if $k = 1$ and $\chi$ is trivial, and

$$L_2^{[M]}(k, \chi\omega^{1-k}) = \prod_{p|M,\, p-\mathrm{prime}} (1 - \chi(p)p^{1-k}) L_2(k, \chi\omega^{1-k}) \,,$$

otherwise.

Our purpose is to prove the following theorem:

**THEOREM.** *Let $m > 1$ be a square-free odd natural number having $r := r(m)$ prime factors and let $\Psi\colon \mathbf{N} \to \mathbf{C}_2$ be a multiplicative function satisfying $\Psi(s) \equiv 1 \pmod 2$, if $s|m$. Set $K = \{-1, 0, 1, 2\}$. Let $L$ be a non-empty subset of $K$ having $\delta$ elements and let $x := \{x_{k,e}\}_{k \in K, e \in \mathcal{T}_8}$ be a sequence of 2-adic integers not all being even defined on $L$. Write*

$$\mathcal{J}_m = \begin{cases} (\log_2 m)/2, & \text{if } m \text{ is a prime number}, \\ 0, & \text{otherwise.} \end{cases}$$

*Then the number*

$$\Lambda_2(x, m) := \sum_{\substack{e \in \mathcal{T}_8, \\ k \in K}} (-1)^{k+1} x_{k,e} \sum_{d \in \mathcal{T}_m} \Psi(|d|) L_2^{[m]}(k, \chi_{ed}\omega^{1-k}) \; + \; x_{1,1}\mathcal{J}_m$$

*is a 2-adic integer divisible by $2^{r+\lambda}$, where $2^\lambda$ is the greatest common divisor of $2^{c(L)}$ and $z_n$, $0 \le n \le 2c(L) - 2$, and*

$$c(L) = [(7\delta - 3)/2] + \sigma,$$

*$\sigma = 1$, if $L = \{-1, 1\}$ or $\{0, 2\}$, and $\sigma = 0$, otherwise.*

**REMARK.** These congruences are of the same general type as those of [2], [5] and also of [3], [6], [7], [8]. In particular, for $L = \{0, 1\}$ we get Gras-Uehara's congruences for class numbers of quadratic fields which are modulo $2^{r(m)+5}$ and for $L = \{-1, 0\}$ (resp. $L = \{0\}$) we obtain congruences modulo $2^{r(m)+5}$ (resp. modulo $2^{r(m)+2}$) for the same objects as those in [6] (resp. in [3]). These objects are equal to the orders of $K_2$-groups of the rings of integers of real quadratic fields or class numbers of appropriate imaginary quadratic fields. If $2 \in L$ then the obtained congruences are quite new and especially interesting. They produce, via a 2-adic version of the Lichtenbaum conjecture, some new congruences for the orders of $K_2$-groups of the integers of imaginary quadratic fields. For a deeper discussion of this case we refer the reader to the last section of the paper.

# 3 Lemmas

The proof of the Main Theorem is divided into a sequence of lemmas. First, we shall extend Lemma 3 [5].

**LEMMA 1.** *Let $\chi$ be a Dirichlet character modulo $M > 1$ and let $N$ be a multiple of $M$ such that $N/M > 0$ is a rational square-free integer prime to $M$. Set $\zeta_N = \zeta_M \zeta_{N/M}$. Then for any integer $k$ we have*

$$\mathcal{S}_{k,\chi}(N) := \sum_{a=1}^{N}{}' \chi(a) l_k(\zeta_N^a)$$

$$= (-1)^{r(N/M)} \prod_{p|(N/M)} \left(1 - \overline{\chi}(p) p^{1-k}\right) \sum_{b=1}^{M}{}' \chi(b) l_k(\zeta_M^b).$$

*Proof.* Let $q$ be a prime number. Then for any natural number $n$ not divisible by $q$ we have

$$\mathcal{S}_{k,\chi}(nq) = (q^{1-k} - \chi(q)) \mathcal{S}_{k,\chi}(n). \tag{9}$$

Indeed, it is easy to see that

$$\begin{aligned} \mathcal{S}_{k,\chi}(nq) &= \sum_{a=1}^{n}{}' \sum_{c=0}^{q-1} \chi(cn+a) l_k(\zeta_{nq}^{cn+a}) - \sum_{b=1}^{n}{}' \chi(bq) l_k(\zeta_{nq}^{bq}) \\ &= \sum_{a=1}^{n}{}' \chi(a) \sum_{c=0}^{q-1} l_k(\zeta_{nq}^a \zeta_q^c) - \chi(q) \mathcal{S}_{k,\chi}(n) \end{aligned}$$

and (9) is implied by the identity (3) at once.
Now the lemma follows from (9) by induction on the number of prime factors of $N$.□

From now on we regard $l_k$ as the multilogarithms defined on $\mathbf{C}_2 - \{1\}$.

**LEMMA 2.** *Given any odd integer $M$, let $\chi$ be a primitive Dirichlet character modulo $M$. Suppose that $N$ is an odd multiple of $M$ such that $N/M$ is square-free and relatively prime to $M$. Let $\psi$ be either the trivial primitive Dirichlet character or a primitive Dirichlet character of even conductor prime to $N$. Let $\omega$ denote the Teichmüler character at $p = 2$ and set $\zeta_N = \zeta_M \zeta_{N/M}$. Then for $k \in K$ we have*

$$\Lambda_{k,\psi} := \Lambda_{k,\psi}(N, \chi) = g(\overline{\chi}) M^{-1} \sum_{a=1}^{N}{}' \chi(a) \mathcal{L}_{k,\psi}(\zeta_N^a)$$

$$= (-1)^{r(N/M)+k+1} \prod_{p|(N/M)} \left(1 - \overline{\chi}\overline{\psi}(p) p^{1-k}\right) L_2(k, \overline{\chi}\overline{\psi}\omega^{1-k}),$$

*unless $k=1$ and the characters $\chi$ and $\psi$ are trivial, in which case we have*

$$\Lambda_{k,\psi}=\sum_{a=1}^{N}{}'\mathcal{L}_{k,\psi}(\zeta_N^a)=\begin{cases}-(\log_2 N)/2, & \textit{if } N \textit{ is a prime power},\\ 0, & \textit{otherwise.}\end{cases}$$

*Proof.* If $M>1$ and $\psi$ is the trivial character then since $N$ is odd we get

$$\begin{aligned}&(-1)^{k+1}\Lambda_{k,\psi}(N,\chi)\\ &=g(\overline{\chi})M^{-1}(-1)^{k+1}\sum_{a=1}^{N}{}'\chi(a)\mathcal{L}_{k,\psi}(\zeta_N^a)=g(\overline{\chi})M^{-1}\sum_{a=1}^{N}{}'\chi(a)l_k^{(2)}(\zeta_N^a)\\ &=g(\overline{\chi})M^{-1}\Big(\sum_{a=1}^{N}{}'\chi(a)l_k(\zeta_N^a)-2^{-k}\overline{\chi}(2)\sum_{a=1}^{N}{}'\chi(2a)l_k(\zeta_N^{2a})\Big)\\ &=g(\overline{\chi})M^{-1}(1-2^{-k}\overline{\chi}(2))\sum_{a=1}^{N}{}'\chi(a)l_k(\zeta_N^a).\end{aligned}$$

Thus, by Lemma 1 and (1) we obtain

$$\begin{aligned}&(-1)^{r(N/M)+k+1}\Lambda_{k,\psi}(N,\chi)\\ &=g(\overline{\chi})M^{-1}(1-2^{-k}\overline{\chi}(2))\prod_{p|(N/M)}(1-\overline{\chi}(p)p^{1-k})\sum_{b=1}^{M}{}'\chi(b)l_k(\zeta_M^b)\\ &=\prod_{p|(N/M)}(1-\overline{\chi}(p)p^{1-k})L_2(k,\overline{\chi}\omega^{1-k})\end{aligned}$$

and the lemma follows.

If $\psi$ is a nontrivial character modulo $A$ then we have

$$\begin{aligned}&(-1)^{k+1}\Lambda_{k,\psi}(N,\chi)\\ &=g(\overline{\psi})A^{-1}g(\overline{\chi})M^{-1}\sum_{a=1}^{N}{}'\chi(a)\sum_{b=1}^{A}{}'\psi(b)l_k(\zeta_N^a\zeta_A^b)\\ &=g(\overline{\psi}\overline{\chi})(AM)^{-1}\sum_{a=1}^{N}{}'\sum_{b=1}^{A}{}'\chi(a)\psi(b)l_k(\zeta_N^a\zeta_A^b)\end{aligned}$$

$$= g(\overline{\psi}\overline{\chi})(AM)^{-1}\sum_{a=1}^{NA}{}'(\chi\psi)(a)l_k(\zeta_{NA}^a).$$

Thus, by Lemma 1 and (1) we find that

$$\begin{aligned}&(-1)^{r(N/M)+k+1}\Lambda_{k,\psi}(N,\chi)\\ &= g(\overline{\psi}\overline{\chi})(AM)^{-1}\prod_{p|(N/M)}(1-\overline{\chi}\overline{\psi}(p)p^{1-k})\sum_{b=1}^{MA}{}'(\chi\psi)(b)l_k(\zeta_{MA}^b)\\ &= \prod_{p|(N/M)}(1-\overline{\chi}\overline{\psi}(p)p^{1-k})L_2(k,\overline{\chi}\overline{\psi}\omega^{1-k}).\end{aligned}$$

This completes the proof of the lemma in case when either $\chi$ or $\psi$ is not trivial. In order to finish the proof of the lemma it remains to consider the case when both the characters $\chi$ and $\psi$ are trivial. Then, by definition of the functions $\mathcal{L}_{k,\psi}$ we have

$$(-1)^{k+1}\Lambda_{k,\psi} = \sum_{a=1}^{N}{}'l_k^{(2)}(\zeta_N^a) = (1-2^{-k})\sum_{a=1}^{N}{}'l_k(\zeta_N^a).$$

Thus, we get

$$\Lambda_{k,\psi} = 0$$

if $k=0$ or $k=2$ because of (2) and $\log_2(\zeta_N^a)=0$.

On the other hand, $L_2(0,\omega) = L_2(2,\omega^{-1}) = 0$ (see the Introduction) and the right hand side of the equation of the lemma equals 0, too.

If $k=1$ then we have

$$\Lambda_{k,\psi} = -\frac{1}{2}\sum_{a=1}^{N}{}'\log_2(1-\zeta_N^a) = -\frac{1}{2}\log_2\Big(\prod_{a=1}^{N}{}'(1-\zeta_N^a)\Big),$$

and consequently

$$\Lambda_{k,\psi} = -(\log_2 N)/2,$$

if $N$ is a prime power and $\Lambda_{k,\psi}=0$, otherwise.

Finally, if $k=-1$ then we have

$$\Lambda_{k,\psi} = \sum_{a=1}^{N}{}'l_{-1}(\zeta_N^a) = \sum_{a=1}^{N}{}'\frac{\zeta_N^a}{(1-\zeta_N^a)^2} = r_2'(N) - r_1'(N), \tag{10}$$

where

$$r'_k(N) = \sum_{a=1}^{N}{}' \frac{1}{(1-\zeta_N^a)^k}\,.$$

It is easy to see that $r'_1(N) = \frac{1}{2}\phi(N)$ because

$$\frac{1}{1-\zeta_N^a} + \frac{1}{1-\zeta_N^{N-a}} = 1\,.$$

In order to calculate $r'_2(N)$, let us observe that for any arithmetical function $f$ we have

$$\begin{aligned}\sum_{\substack{1\le a\le x\\ (a,N)=1}} f(a) &= \sum_{1\le a\le x} \Big(\sum_{d|(a,N)} \mu(d)\Big) f(a) = \sum_{d|N} \mu(d) \sum_{\substack{1\le a\le x,\\ d|a}} f(a)\\ &= \sum_{d|N} \mu(d) \sum_{1\le a\le x/d} f(ad)\end{aligned}$$

Therefore, putting

$$f(a) = \frac{1}{(1-\zeta_N^a)^k}$$

and $x = N-1$ we get the formula

$$r'_k(N) = \sum_{d|N} \mu(d) r_k(N/d), \tag{11}$$

where

$$r_k(n) = \sum_{a=1}^{n-1} \frac{1}{(1-\zeta_n^a)^k}$$

and $\zeta := \zeta_n = \exp(2\pi i/n)$.

Let us compute $r_2(n)$. Then the numbers $\zeta^a - 1$, $1 \le a \le n-1$ are all the zeros of the polynomial

$$(x+1)^n + \ldots + (x+1) + 1 = \frac{(x+1)^n - 1}{x}\,.$$

Therefore, all the zeros of the polynomial

$$(1+x)^n - x^n = nx^{n-1} + \binom{n}{2}x^{n-2} + \binom{n}{3}x^{n-3} + \ldots$$

are of the form $x_a := 1/(\zeta^a - 1)$, where $1 \le a \le n-1$. Thus

$$\begin{aligned} r_2(n) &= \sum_{a=1}^{n-1} x_a^2 = \Big(\sum_{a=1}^{n-1} x_a\Big)^2 - 2 \sum_{1\le a<b\le n-1} x_a x_b \\ &= \Big(\frac{n-1}{2}\Big)^2 - \frac{(n-1)(n-2)}{3} \\ &= -\frac{(n-1)(n-5)}{12}. \end{aligned}$$

Substituting the above to (11) gives

$$\begin{aligned} r_2'(N) &= -\frac{N^2}{12}\sum_{d|N}\frac{\mu(d)}{d^2} + \frac{N}{2}\sum_{d|N}\frac{\mu(d)}{d} - \frac{5}{12}\sum_{d|N}\mu(d) \\ &= -\frac{N^2}{12}\prod_{p|N}(1-p^{-2}) + \frac{N}{2}\prod_{p|N}(1-p^{-1}) \\ &= -\frac{N^2}{12}\prod_{p|N}(1-p^{-2}) + \frac{\phi(N)}{2}. \end{aligned}$$

Thus the lemma for $k = -1$ follows from (10) and Theorem 5.11 [9]. This completes the proof of Lemma 2. □

**LEMMA 3.** *Let $n \ge 0$ be an integer. Set $\gamma_n = -1$, if $n \equiv 1, 2 \pmod 4$, and $\gamma_n = 1$, otherwise. Then we have*

(i)

$$\sum_{k=0}^{n} \binom{n+k}{n-k} \frac{4^{2k}(-1)^k}{(2k+1)^2} \binom{2k}{k}^{-1} = \frac{1}{(2n+1)^2},$$

(ii)

$$\sum_{k=0}^{n} \binom{n+k}{n-k} \frac{4^{2k}(-1)^k}{(2k+1)^2} \binom{2k}{k}^{-1} \sum_{l=0}^{k} \binom{2l}{l} 2^{-3l} = \frac{\gamma_n}{(2n+1)^2}.$$

*Proof.* (A. Granville) Set

$$\lambda_k = (-1)^k 2^{4k} \binom{n+k}{n-k} \binom{2k}{k}^{-1}.$$

The main idea of the proof is to note the following identity:

$$\lambda_k - \lambda_{k+1} = \left(\frac{2n+1}{2k+1}\right)^2 \lambda_k$$

for all $k \geq 0$. Thus the left hand side of the equation (i) of the lemma equals

$$\sum_{k=0}^{n} \frac{\lambda_k}{(2k+1)^2} = \frac{1}{(2n+1)^2}\sum_{k=0}^{n}(\lambda_k - \lambda_{k+1}) = \frac{\lambda_0}{(2n+1)^2} = \frac{1}{(2n+1)^2}\,.$$

The identity (ii) is a little more subtle. Multiplying through by $(2n+1)^2$ we get

$$(2n+1)^2 \sum_{k=0}^{n} \frac{\lambda_k}{(2k+1)^2} \sum_{l=0}^{k} \binom{2l}{l} 2^{-3l} = \sum_{k=0}^{n}(\lambda_k - \lambda_{k+1}) \sum_{l=0}^{k} \binom{2l}{l} 2^{-3l}$$
$$= \sum_{k=0}^{n} \lambda_k \binom{2k}{k} 2^{-3k} = \sum_{k=0}^{n} \binom{n+k}{n-k} (-2)^k.$$

To evaluate the last sum, note that $\binom{n+k}{n-k}$ is the coefficient of $t^n$ in

$$\frac{t^k}{(1-t)^{2k+1}}\,.$$

Thus our sum is the coefficient of $t^n$ in

$$\sum_{k \geq 0} \frac{(-2t)^k}{(1-t)^{2k+1}} = \frac{1}{(1-t)} \frac{1}{1-((-2t)/(1-t)^2)} = \frac{1-t}{1+t^2} = \frac{1-t-t^2+t^3}{1-t^4}$$

and so equals $-1$, if $n \equiv 1, 2 \pmod 4$ and 1, otherwise. □

Now we extend Lemmas 6 and 7 of [5]. In the two next lemmas let $\xi \neq 1$ be a primitive $N$th root of unity, where $N$ is an odd natural number.

**LEMMA 4.** *For any $e \in \mathcal{T}_8$ write $\alpha = \operatorname{sgn} e$ and set*

$$w_\alpha = \frac{\alpha\xi}{1+\alpha\xi^2}\,.$$

*Then we have*

$$\mathcal{L}_{-1,e}(\xi) = \sum_{k=0}^{\infty}(4\alpha)^k w_\alpha^{2k+1}, \qquad \mathcal{L}_{0,e}(\xi) = w_{-\alpha},$$

$$\mathcal{L}_{1,e}(\xi) = \sum_{k=0}^{\infty}\frac{(4\alpha)^k w_\alpha^{2k+1}}{2k+1}, \qquad \mathcal{L}_{2,e}(\xi) = \sum_{k=0}^{\infty}\frac{(-16\alpha)^k w_{-\alpha}^{2k+1}}{(2k+1)^2}\binom{2k}{k}^{-1},$$

*if* $e \in \mathcal{T}_4$ *and*

$$\mathcal{L}_{-1,e}(\xi) = -\sum_{k=0}^{\infty}(2\alpha)^k(2k-1)w_\alpha^{2k+1}, \qquad \mathcal{L}_{0,e}(\xi) = \sum_{k=0}^{\infty}(-2\alpha)^k w_{-\alpha}^{2k+1},$$

$$\mathcal{L}_{1,e}(\xi) = \sum_{k=0}^{\infty}\frac{(2\alpha)^k w_\alpha^{2k+1}}{2k+1}, \qquad \mathcal{L}_{2,e}(\xi) = \sum_{k=0}^{\infty}\frac{(-16\alpha)^k w_{-\alpha}^{2k+1}}{(2k+1)^2}\binom{2k}{k}^{-1}\sum_{l=0}^{k}\binom{2l}{l}2^{-3l},$$

*if* $e \in \mathcal{T}_8 - \mathcal{T}_4$.

*Proof.* As for the expansions of $\mathcal{L}_{\nu,e}(\xi)$ for $\nu = 0, 1$, we refer the reader to Lemma 6 [5]. Let us consider the case of $\nu = -1$. Then we have

$$\mathcal{L}_{-1,e}(\xi) = -\frac{\alpha\xi(1+\alpha\xi^2)}{(1-\alpha\xi^2)^2},$$

if $e \in \mathcal{T}_4$. In this case it suffices to use the 2-adic expansion

$$\sum_{n\geq 1,\, n-\text{odd}} x^n = \frac{x}{1-x^2} \tag{12}$$

with $x = 2\omega_\alpha\sqrt{\alpha}$. Furthermore, if $e \notin \mathrm{T}_4$ then we have

$$\mathcal{L}_{-1,e}(\xi) = -\frac{\alpha\xi(\xi^2+\alpha)(\xi^4-4\alpha\xi^2+1)}{(1+\xi^4)^2}.$$

In this case it sufficient to apply (12) together with the 2-adic series

$$\sum_{n\geq 1,\, n-\text{odd}} nx^n = \frac{x(1+x^2)}{(1-x^2)^2}$$

with $x = \omega_\alpha\sqrt{2\alpha}$ in both formulas. Indeed, we have

$$\begin{aligned}\sqrt{2\alpha}\sum_{k=0}^{\infty}(2\alpha)^k(2k-1)\omega_\alpha^{2k+1} &= \sum_{k=0}^{\infty}(2k-1)(\omega_\alpha\sqrt{2\alpha})^{2k+1} \\ &= \sum_{n\geq 1,\, n-\mathrm{odd}} n(\omega_\alpha\sqrt{2\alpha})^n - 2\sum_{n\geq 1,\, n-\mathrm{odd}}(\omega_\alpha\sqrt{2\alpha})^n .\end{aligned}$$

It remains to prove the lemma in the case of $\nu = 2$. Then, let us consider the following 2-adic series

$$G_e\colon = \sum_{k=0}^{\infty}\frac{(-16\alpha)^k\eta_{k,e}\omega_{-\alpha}^{2k+1}}{(2k+1)^2},$$

where

$$\eta_{k,e} = \begin{cases} \binom{2k}{k}^{-1}\sum_{l=0}^{k}\binom{2l}{l}2^{-3l}, & \text{if } \ e\notin\mathcal{T}_4 , \\ \binom{2k}{k}^{-1}, & \text{if } \ e\in\mathcal{T}_4 . \end{cases}$$

By (7) and (8) the series $G_e$ converges. Furthermore, setting $\gamma^2 = \alpha$ we have formally

$$\begin{aligned} G_e &= \sum_{k=0}^{\infty}\frac{(-16\alpha)^k\eta_{k,e}}{(2k+1)^2}\Big(\frac{-\alpha\xi}{1-\alpha\xi^2}\Big)^{2k+1} \\ &= \sum_{k=0}^{\infty}\frac{(-16\alpha)^k\eta_{k,e}}{(2k+1)^2}\Big(-\alpha\xi\sum_{l=0}^{\infty}(\gamma\xi)^{2l}\Big)^{2k+1} \\ &= -\gamma\sum_{k=0}^{\infty}\frac{(-16)^k\eta_{k,e}}{(2k+1)^2}\Big(\sum_{l=0}^{\infty}(\gamma\xi)^{2l+1}\Big)^{2k+1} \\ &= -\gamma\sum_{k=0}^{\infty}\frac{(-16)^k\eta_{k,e}}{(2k+1)^2}\sum_{l=0}^{\infty}\binom{2k+l}{l}(\gamma\xi)^{2(k+l)+1} \\ &= -\gamma\sum_{l=0}^{\infty}\sum_{k=0}^{l}\frac{(-16)^k\eta_{k,e}}{(2k+1)^2}\binom{l+k}{l-k}(\gamma\xi)^{2l+1} \\ &= -\gamma\sum_{l=0}^{\infty}(\gamma\xi)^{2l+1}\sum_{k=0}^{l}\binom{l+k}{l-k}\frac{(-16)^k\eta_{k,e}}{(2k+1)^2} \end{aligned}$$

because for any function $f$ we have

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} f(k,l) = \sum_{l=0}^{\infty}\sum_{k=0}^{l} f(k,l-k)\,.$$

Therefore by Lemma 3, we have

$$G_e \;=\; \mathcal{L}_{2,e}(\xi)\,.$$

Since both the series converge Lemma 4 for $\nu = 2$ follows. □

We need some congruences between the numbers $\mathcal{L}_{k,e}(\xi)$, where $k \in \{-1,0,1,2\}$.

**LEMMA 5.** *Set $K = \{-1,0,1,2\}$. Let $\{x_{k,e}\}_{k\in K, e\in\mathcal{T}_8}$ be a sequence of integers in $\mathbf{C}_2$ not all being even defined on a non-empty subset $L$ of $K$ having $\delta$ elements. Then we have*

(i)

$$\sum_{\substack{k\in L,\\ e\in\mathcal{T}_8}} x_{k,e}\mathcal{L}_{k,e}(\xi) \equiv 0 \pmod{2^\lambda}\,,$$

*where $2^\lambda$ is the greatest common divisor of $2^{c(L)}$ and $z_n$, $0 \le n \le 2c(L)-2$ ,*

(ii)

$$c(L) \;=\; [(7\delta-3)/2] \;+\; \sigma\,,$$

*where $\sigma = 1$, if $L = \{-1,1\}$ or $\{0,2\}$, and $\sigma = 0$, otherwise.*

*Proof.* From the previous lemma we get

$$\Lambda := \sum_{\substack{k\in L,\\ e\in\mathcal{T}_8}} x_{k,e}\mathcal{L}_{k,e}(\xi)$$

$$= \sum_{e\in\mathcal{T}_8}\big(x_{-1,e}\mathcal{L}_{-1,e}(\xi) + x_{0,e}\mathcal{L}_{0,e}(\xi) + x_{1,e}\mathcal{L}_{1,e}(\xi) + x_{2,e}\mathcal{L}_{2,e}(\xi)\big)$$

$$= (x_{-1,1} + x_{-1,8} + x_{0,-4} + x_{0,-8} + x_{1,1} + x_{1,8} + x_{2,-4} + x_{2,-8})\omega_1$$

$$+(x_{-1,-4} + x_{-1,-8} + x_{0,1} + x_{0,8} + x_{1,-4} + x_{1,-8} + x_{2,1} + x_{2,8})\omega_{-1}$$

$$+\sum_{k=1}^{\infty}\Big(4^k x_{-1,1} - 2^k(2k-1)x_{-1,8} + 2^k x_{0,-8} + \frac{2^k}{2k+1}x_{1,1} + \frac{2^k}{2k+1}x_{1,8}$$

$$+\frac{16^k}{(2k+1)^2}\binom{2k}{k}^{-1}x_{2,-4}+\frac{16^k}{(2k+1)^2}\binom{2k}{k}^{-1}\sum_{l=0}^{k}\binom{2l}{l}2^{-3l}x_{2,-8}\Big)\omega_1^{2k+1}$$

$$+\sum_{k=1}^{\infty}(-1)^k\Big(4^kx_{-1,-4}-2^k(2k-1)x_{-1,-8}+2^kx_{0,8}+\frac{4^k}{2k+1}x_{1,-4}+\frac{2^k}{2k+1}x_{1,-8}$$

$$+\frac{16^k}{(2k+1)^2}\binom{2k}{k}^{-1}x_{2,1}+\frac{16^k}{(2k+1)^2}\binom{2k}{k}^{-1}\sum_{l=0}^{k}\binom{2l}{l}2^{-3l}x_{2,8}\Big)\omega_{-1}^{2k+1}.$$

Consequently, we obtain

$$\Lambda = z_0\omega_1 + z_1 v_1 + \sum_{k=1}^{\infty}\frac{1}{(2k+1)^2}\left(z_{2k}\omega_1^{2k+1}+z_{2k+1}v_{2k+1}\right),$$

where

$$v_{2k+1} := v_{2k+1}(\xi) = \frac{1}{2}\left((-1)^k\omega_{-1}^{2k+1}-\omega_1^{2k+1}\right)$$

is an integer in $\mathbf{C}_2$ and the coefficients $z_n$, $n \geq 0$ are defined by (6). Without loss of generality we may assume that not all $x_{k,e}$ $(k \in K, e \in \mathcal{T}_8)$ are even. Denote by $2^\lambda$ the highest power of 2 dividing all $z_n$, $n \geq 0$. By definition, part (i) of the lemma follows immediately and we have $\lambda \leq c(L)$.

In order to prove part (ii) we need consider 4 cases:

1. Let $L = K$.

Then we shall prove that $c(L) = 12$. Putting (for example)

$$\begin{aligned} x_{-1,-4} &= x_{-1,-8} = 1, \\ x_{0,1} &= x_{0,8} = -3, \\ x_{1,-4} &= x_{1,-8} = -61, \\ x_{2,1} &= x_{2,8} = 63, \end{aligned}$$

and

$$x_{k,1} = -x_{k,-4}, \quad x_{k,8} = -x_{k,-8},$$

we shall show that $z_n \equiv 0 \pmod{2^{12}}$, $n \geq 0$, i.e., that $c(L) \geq 12$. Indeed, it is easily seen that

$$z_1 = 0, \ z_{2l} = 0 \ (l \geq 0).$$

Moreover, by (6) we have

$$z_3 = 4(18x_{-1,-4} - 9x_{-1,-8} + 9x_{0,8} + 6x_{1,-4} + 3x_{1,-8} + 5x_{2,8} + 4x_{2,1}), \tag{13}$$

and so $z_3 = 0$.

On the other hand, putting

$$\eta_{l,1} = 2^{3l}\binom{2l}{l}^{-1},$$

$$\eta_{l,2} = \eta_{l,1}\sum_{k=0}^{l}\binom{2k}{k}2^{-3k},$$

if $l \geq 1$ we get

$$\begin{aligned}
t_l \ &:= z_{2l+3} - \frac{8(l+1)}{2l+1}z_{2l+1} \\
&= 2^{l+2}\big(2^{l+1}(2l+3)^2x_{-1,-4} - (2l+1)(2l+3)^2x_{-1,-8} + (2l+3)^2x_{0,8} \\
&\quad + 2^{l+1}(2l+3)x_{1,-4} + (2l+3)x_{1,-8} + \eta_{l+1,1}x_{2,1} + \eta_{l+1,2}x_{2,8}\big) \\
&\quad - \frac{2^{l+4}(l+1)}{2l+1}\big(2^l(2l+1)^2x_{-1,-4} - (2l-1)(2l+1)^2x_{-1,-8} \\
&\quad + (2l+1)^2x_{0,8} + 2^l(2l+1)x_{1,-4} + (2l+1)x_{1,-8} + \eta_{l,1}x_{2,1} + \eta_{l,2}x_{2,8}\big).
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
t_l \ &= 2^{l+2}\big(x_{2,8} + (8l^3 - 12l^2 - 34l - 13)x_{-1,-8} + (5 - 4l^2)x_{0,8} - (2l+1)x_{1,-8}\big) \\
&\quad + 2^{2l+3}\big((6l+7)x_{-1,-4} + x_{1,-4}\big)
\end{aligned} \tag{14}$$

because

$$\eta_{l+1,1} - \frac{4(l+1)}{2l+1}\eta_{l,1} = 0$$

and

$$\eta_{l+1,2} - \frac{4(l+1)}{2l+1}\eta_{l,2} = 1.$$

We shall prove that $2^{12}|z_{2l+1}$, $l \geq 1$. If $l \geq 11$ then this is obvious by (6). In order to prove this for $l \leq 10$ it suffices to prove that $2^{12}|t_l$, if $l \leq 9$. For our sequence $\{x_{k,e}\}$, by (14) we find that

$$\begin{aligned}
t_l \ &= 2^{l+2}\big(63 + (8l^3 - 12l^2 - 34l - 13) - 3(5 - 4l^2) + 61(2l+1)\big) \\
&\quad + 2^{2l+3}\big((6l+7) - 61\big) = 2^{l+5}(l^3 + 11l + 12) + 2^{2l+4}(3l - 27),
\end{aligned}$$

and consequently $t_l = 0$, if $l \leq 5$ and $t_l \equiv 0 \pmod{2^{12}}$, if $6 \leq l \leq 9$ as is easy to check. Thus we have $c(L) \geq 12$. In order to prove that $c(L) = 12$ let us assume that $c(L) \geq 13$, i.e., $2^{13} | z_n$, if $n \geq 0$. Therefore, by (14) we get

$$\begin{aligned}
0 &\equiv -2^7 t_1 + 2^7 t_2 - 2^3 t_3 - 2^2 \cdot 7 t_4 + 2 \cdot 5 t_5 - t_6 \\
&\equiv -2^{10}(x_{2,8} + 5x_{-1,-8} + x_{0,8} - 3x_{1,-8}) + 2^{12}(x_{-1,-4} + x_{1,-4}) \\
&+ 2^{11}(x_{2,8} - x_{-1,-8} + x_{0,8} - x_{1,-8}) \\
&- 2^8(x_{2,8} - 7x_{-1,-8} + x_{0,8} - 7x_{1,-8}) + 2^{12}(x_{-1,-4} + x_{1,-4}) \\
&- 2^8 \cdot 7(x_{2,8} + 11x_{-1,-8} + 5x_{0,8} - 9x_{1,-8}) \\
&+ 2^8 \cdot 5(x_{2,8} + 5x_{-1,-8} + x_{0,8} - 11x_{1,-8}) \\
&- 2^8(x_{2,8} - 9x_{-1,-8} - 11x_{0,8} - 13x_{1,-8}) \\
&\equiv 2^{12} x_{0,8} \pmod{2^{13}},
\end{aligned}$$

and consequently $x_{0,8}$ must be even. Since $2^{13} | z_n$ we have $2^{13} | t_l$, and so by $2l+3 \geq l+2$ we get

$$\begin{aligned}
\gamma_l &:= x_{2,8} + (8l^3 - 12l^2 - 34l - 13)x_{-1,-8} + (5 - 4l^2)x_{0,8} - (2l+1)x_{1,-8} \\
&+ 2^{l+1}\big((6l+7)x_{-1,-4} + x_{1,-4}\big) \equiv 0 \pmod{2^{11-l}}.
\end{aligned} \tag{15}$$

Hence, we obtain

$$\gamma_7 \equiv x_{2,8} + x_{-1,-8} + x_{0,8} + x_{1,-8} \equiv 0 \pmod{16}, \tag{16}$$

$$\gamma_8 \equiv x_{2,8} + 3x_{-1,-8} - 3x_{0,8} - x_{1,-8} \equiv 0 \pmod{8}. \tag{17}$$

Therefore we get

$$\gamma_7 + \gamma_8 \equiv 2x_{2,8} + 4x_{-1,-8} - 2x_{0,8} \equiv 0 \pmod{8},$$

and consequently we deduce $2x_{2,8} \equiv 0 \pmod{4}$ because $x_{0,8}$ is even. Thus $x_{2,8}$ must be even too.

Substituting $l = 6, 4$ to (15) gives the congruences

$$\gamma_6 \equiv x_{2,8} - 9x_{-1,-8} - 11x_{0,8} - 13x_{1,-8} \equiv 0 \pmod{32}, \tag{18}$$

$$\gamma_4 \equiv x_{2,8} + 11x_{-1,-8} + 5x_{0,8} - 9x_{1,-8} \equiv 0 \pmod{32}. \tag{19}$$

Consequently, it may be concluded that

$$\gamma_6 - \gamma_4 \equiv -20x_{-1,-8} - 16x_{0,8} - 4x_{1,-8} \equiv 0 \pmod{32},$$

and we get

$$-5x_{-1,-8} - x_{1,-8} \equiv 0 \pmod{8} \tag{20}$$

because $x_{0,8}$ is even.

On the other hand, by (16) and (18) we obtain

$$\gamma_6 - \gamma_7 \equiv 3x_{-1,-8} + 2x_{0,8} + x_{1,-8} \equiv 0 \pmod{8},$$

and consequently we find that

$$-x_{-1,-8} + x_{1,-8} \equiv 0 \pmod{4}. \tag{21}$$

Adding the above and (20) implies

$$-6x_{-1,-8} \equiv 0 \pmod{4},$$

and so

$$x_{-1,-8} \equiv 0 \pmod{2}.$$

The above together with (20) yields

$$x_{1,-8} \equiv 0 \pmod{2}.$$

Substituting $l = 1, 2$ to (15) gives

$$\begin{aligned} \gamma_1 &\equiv x_{2,8} + 13x_{-1,-8} + x_{0,8} - 3x_{1,-8} \\ &\quad +4(5x_{-1,-4} + x_{1,-4}) \equiv 0 \pmod{32}, \end{aligned} \tag{22}$$

$$\begin{aligned} \gamma_2 &\equiv x_{2,8} - x_{-1,-8} - 11x_{0,8} - 5x_{1,-8} \\ &\quad +8(3x_{-1,-4} + x_{1,-4}) \equiv 0 \pmod{32}. \end{aligned}$$

Thus by the above, and by (16) and (18) we deduce that

$$\begin{aligned} 2\gamma_1 - \gamma_2 + \gamma_6 &\equiv 2x_{2,8} - 14x_{-1,-8} + 2x_{0,8} - 14x_{1,-8} - 16x_{-1,-4} \\ &\equiv 2x_{2,8} + 2x_{-1,-8} + 2x_{0,8} + 2x_{1,-8} - 16(x_{-1,-8} + x_{1,-8}) - 16x_{-1,-4} \\ &\equiv 2\gamma_7 - 16x_{-1,-4} \equiv 0 \pmod{32} \end{aligned}$$

because $x_{-1,-8}+x_{1,-8}\equiv 0\,(\mathrm{mod}\ 2)$. Hence $x_{-1,-4}$ must be even. Furthermore, by the above and by (22), (18), (16), (17) we get

$$\gamma_1+\gamma_6-\gamma_7-\gamma_8\equiv 4x_{-1,-4}+4x_{1,-4}\equiv 4x_{1,-4}\ (\mathrm{mod}\,8)\,,$$

and consequently $x_{1,-4}$ must be even.
In order to prove that $x_{2,1}$ is even we shall use the congruence $z_3\equiv 0\ (\mathrm{mod}\,32)$. By (13), we obtain

$$(z_3/4)\equiv 2x_{-1,-4}-x_{-1,-8}+x_{0,8}-2x_{1,-4}+3x_{1,-8}+5x_{2,8}+4x_{2,1}\equiv 0\ (\mathrm{mod}\,8)\,.$$

Therefore, by (17) and (18) we conclude

$$\begin{aligned}&(z_3/4)+\gamma_8-\gamma_6\equiv(2x_{-1,-4}-x_{-1,-8}+x_{0,8}-2x_{1,-4}+3x_{1,-8}+5x_{2,8}+4x_{2,1})\\&+(x_{2,8}+3x_{-1,-8}-3x_{0,8}-x_{1,-8})-2(x_{2,8}-9x_{-1,-8}-11x_{0,8}-13x_{1,-8})\\&\equiv\ 2(x_{-1,-4}-x_{1,-4})+4(x_{2,8}+x_{-1,-8}+x_{0,8}+x_{1,-8})+4x_{2,1}\ (\mathrm{mod}\,8)\,,\end{aligned}$$

and consequently we get

$$2(x_{-1,-4}-x_{1,-4})+4x_{2,1}\equiv 0\ (\mathrm{mod}\,8)\,. \tag{23}$$

On the other hand, by (22) and (16) we obtain

$$\begin{aligned}&\gamma_1-\gamma_7\\&\equiv\ (x_{2,8}-3x_{-1,-8}+x_{0,8}-3x_{1,-8}+4(x_{-1,-4}+x_{1,-4}))-(x_{2,8}+x_{-1,-8}+x_{0,8}+x_{1,-8})\\&\equiv\ -4(x_{-1,-8}+x_{1,-8})+4(x_{-1,-4}+x_{1,-4})\equiv 0\ (\mathrm{mod}\,16)\,,\end{aligned}$$

and so by (21) we get

$$x_{-1,-4}+x_{1,-4}\equiv 0\ (\mathrm{mod}\,4)$$

because $x_{1,-8}$ is even.

The above together with (23) imply $4x_{2,1}\equiv 0\ (\mathrm{mod}\,8)$ because $x_{1,-4}$ is even and $x_{2,1}$ must be even. At last since $(z_1/2)$ is even it may be concluded that $x_{0,1}$ is even too.

Summarizing, we have proved that $x_{k,e}$ is even if $\mathrm{sgn}e=(-1)^k$. In order to prove that $x_{k,e}$ are also even in case $\mathrm{sgn}e\neq(-1)^k$, let us note that

$$2z_{2l}=z_{2l+1}+\tilde{z}_{2l+1}\,, \tag{24}$$

where $\tilde{z}_{2l+1}$ comes into $z_{2l+1}$ by substituting $x_{k,1}$ (resp. $x_{k,-4}$, $x_{k,8}$ or $x_{k,-8}$) instead of $x_{k,-4}$ (resp. $x_{k,1}$, $x_{k,-8}$ or $x_{k,8}$). We have

$$2^{13} \mid z_{2l},\, z_{2l+1}\,,$$

and so

$$2^{13} \mid \tilde{z}_{2l+1}\,.$$

Thus, by the same reasoning as in the case of $\operatorname{sgn} e = (-1)^k$ we get $2|x_{k,e}$ in the other case.

2. Let $\delta = 3$.

In this case we shall prove that $c(L) = 9$. First, we shall show that $c(L) \geq 9$. Indeed, putting (for example)

$$x_{-1,-4} = -x_{-1,-8} = a_{-1}\,,$$

$$x_{0,1} = -x_{0,8} = a_0\,,$$

$$x_{1,-4} = -x_{1,-8} = a_1\,,$$

$$x_{2,1} = -x_{2,8} = a_2\,,$$

and

$$x_{k,1} = -x_{k,-4}\,, \quad x_{k,8} = -x_{k,-8}\ (k \in L)\,,$$

where $a_k := a_k(L)$, $k \in K$, $a_k = 0$, if $k \notin L$ and the remaining $a_k$ are defined by the following:

$$a_{-1} = 1, \quad a_0 = -2, \quad a_1 = -15, \quad \text{if} \quad L = \{-1, 0, 1\}\,,$$

$$a_{-1} = 2, \quad a_0 = -19, \quad a_2 = 225, \quad \text{if} \quad L = \{-1, 0, 2\}\,,$$

$$a_{-1} = 1, \quad a_1 = -19, \quad a_2 = -30, \quad \text{if} \quad L = \{-1, 1, 2\}\,,$$

$$a_0 = -1, \quad a_1 = 2, \quad a_2 = 15, \quad \text{if} \quad L = \{0, 1, 2\}\,,$$

we shall prove that $z_n \equiv 0 \pmod{2^9}$, if $n \geq 0$. In fact, in all these cases we have $z_1 = z_{2n} = 0$, $n \geq 0$. Moreover, by (13) we get

$$z_3 = 4(27a_{-1} - 9a_0 + 3a_1 - a_2)\,,$$

and so $z_3 = 0$, as easy to check. Thus in order to prove that $z_{2l+1} \equiv 0 \pmod{2^9}$ for $2 \leq l \leq 7$, it suffices to show that $t_l \equiv 0 \pmod{2^9}$ for $1 \leq l \leq 6$.

Indeed, by (14) we get

$$t_l = 2^{l+4}(-2l^3 + l^2 + l + 2) + 2^{2l+4}(3l - 4)\,,$$

if $2 \notin L$,

$$t_l = 2^{l+4}(-4l^3 - 13l^2 + 17l + 26) + 2^{2l+4}(6l + 7)\,,$$

if $1 \notin L$,

$$t_l = 2^{l+4}(-2l^3 + 3l^2 - l + 6) + 2^{2l+4}3(l - 2)\,,$$

if $0 \notin L$,

$$t_l = 2^{l+4}(-l^2 + l - 2) + 2^{2l+4}\,,$$

if $-1 \notin L$.

Therefore in all these cases we have $t_l \equiv 0 \pmod{2^9}$, if $l \geq 4$. Furthermore, an easy computation shows that $t_l = 0$, if $l \leq 3$.

In order to prove that $c(L) = 9$, suppose, contrary to our claim, that $c(L) \geq 10$, i.e., $z_n \equiv 0 \pmod{2^{10}}$, if $n \geq 0$. Then we shall prove that all the $x_{k,e}$ must be even. In view of (24), it suffices, similarly as in case $L = K$, to check it for $k \in L$, $e \in \mathcal{T}_8$ satisfying $\operatorname{sgn} e = (-1)^k$.

Since $t_l \equiv 0 \pmod{2^9}$, we can apply the congruence (15) modulo $2^{8-l}$ (instead of modulo $2^{11-l}$). Then, by (14) we get the congruences

$$\begin{gathered}\gamma_l \equiv (8l^3 - 12l^2 - 34l - 13)x_{-1,-8} + (5 - 4l^2)x_{0,8} - (2l + 1)x_{1,-8} \\ +2^{l+1}\big((6l + 7)x_{-1,-4} + x_{1,-4}\big) \equiv 0 \pmod{2^{8-l}}\,,\end{gathered}$$

if $2 \notin L$,

$$\begin{gathered}\gamma_l \equiv x_{2,8} + (8l^3 - 12l^2 - 34l - 13)x_{-1,-8} + (5 - 4l^2)x_{0,8} \\ +2^{l+1}(6l + 7)x_{-1,-4} \equiv 0 \pmod{2^{8-l}}\,,\end{gathered}$$

if $1 \notin L$,

$$\begin{gathered}\gamma_l = x_{2,8} + (8l^3 - 12l^2 - 34l - 13)x_{-1,-8} - (2l + 1)x_{1,-8} \\ +2^{l+1}\big((6l + 7)x_{-1,-4} + x_{1,-4}\big) \equiv 0 \pmod{2^{8-l}}\,,\end{gathered}$$

if $0 \notin L$,

$$\begin{gathered}\gamma_l = x_{2,8} + (5 - 4l^2)x_{0,8} - (2l + 1)x_{1,-8} \\ +2^{l+1}x_{1,-4} \equiv 0 \pmod{2^{8-l}}\,,\end{gathered}$$

if $-1 \notin L$.

Regarding the case $2 \notin L$. Then we have:

$$\gamma_1 \equiv 52x_{-1,-4} - 51x_{-1,-8} + x_{0,8} + 4x_{1,-4} - 3x_{1,-8} \equiv 0 \pmod{128},$$

$$\gamma_2 \equiv 24x_{-1,-4} - x_{-1,-8} - 11x_{0,8} + 8x_{1,-4} - 5x_{1,-8} \equiv 0 \pmod{64},$$

$$\gamma_3 \equiv 16x_{-1,-4} - 7x_{-1,-8} + x_{0,8} + 16x_{1,-4} - 7x_{1,-8} \equiv 0 \pmod{32},$$

$$\gamma_4 \equiv 11x_{-1,-8} + 5x_{0,8} - 9x_{1,-8} \equiv 0 \pmod{16},$$

$$\gamma_5 \equiv 5x_{-1,-8} + x_{0,8} - 3x_{1,-8} \equiv 0 \pmod{8},$$

$$\gamma_6 \equiv -x_{-1,-8} + x_{0,8} - x_{1,-8} \equiv 0 \pmod{4}.$$

Therefore, we get

$$0 \equiv \gamma_1 - \gamma_5 \equiv 4x_{-1,-4} + 4x_{1,-4} \pmod{8},$$

and so $x_{-1,-4} + x_{1,-4}$ must be even.

Consequently, we obtain

$$0 \equiv \gamma_2 - \gamma_4 \equiv 4(x_{-1,-8} + x_{1,-8}) \pmod{16},$$

i.e.,

$$x_{-1,-8} + x_{1,-8} \equiv 0 \pmod{4}. \tag{25}$$

Hence we get

$$0 \equiv \gamma_6 + x_{-1,-8} + x_{1,-8} \equiv x_{0,8} \pmod{4},$$

and consequently

$$0 \equiv \gamma_4 + \gamma_5 \equiv 4x_{1,-8} \pmod{8}.$$

Hence and from (25), $x_{1,-8}$ and $x_{-1,-8}$ must be even. On other hand, by (13) we find that

$$0 \equiv (z_3/4) - \gamma_2 \equiv 2(x_{-1,-4} - x_{1,-4}) \pmod{8}$$

because $x_{0,8}$ is divisible by 4, and so

$$x_{-1,-4} - x_{1,-4} \equiv 0 \pmod{4}. \tag{26}$$

Thus, by $4|x_{0,8}$ and $2|x_{-1,-8}$ we get

$$0 \equiv \gamma_1 + \gamma_3 - 2\gamma_2 \equiv 4(x_{-1,-4} - x_{1,-4}) \equiv 0 \pmod{16},$$

which together with (26) imply $2x_{-1,-4} \equiv 0 \pmod 4$ and consequently $x_{-1,-4}$, $x_{1,-4}$ must be even.

Finally, since $(z_1/2)$ is even $x_{0,1}$ must be even and so all the $x_{k,e}$ are even. Contradiction.

Regarding the case $1 \notin L$. Then we have:

$$\gamma_1 \equiv 52x_{-1,-4} - 51x_{-1,-8} + x_{0,8} + x_{2,8} \equiv 0 \pmod{128},$$

$$\gamma_2 \equiv 24x_{-1,-4} - x_{-1,-8} - 11x_{0,8} + x_{2,8} \equiv 0 \pmod{64},$$

$$\gamma_3 \equiv 16x_{-1,-4} - 7x_{-1,-8} + x_{0,8} + x_{2,8} \equiv 0 \pmod{32},$$

$$\gamma_4 \equiv -5x_{-1,-8} + 5x_{0,8} + x_{2,8} \equiv 0 \pmod{16},$$

$$\gamma_5 \equiv -3x_{-1,-8} + x_{0,8} + x_{2,8} \equiv 0 \pmod 8,$$

$$\gamma_6 \equiv -x_{-1,-8} + x_{0,8} + x_{2,8} \equiv 0 \pmod 4.$$

Therefore, we get

$$0 \equiv \gamma_1 - \gamma_5 \equiv 4x_{-1,-4} \pmod 8,$$

and so $x_{-1,-4}$ must be even.

Furthermore, we get

$$0 \equiv \gamma_2 - \gamma_4 \equiv 4x_{-1,-8} \pmod{16},$$

and consequently $x_{-1,-8} \equiv 0 \pmod 4$, which implies

$$0 \equiv \gamma_4 - \gamma_5 \equiv 4x_{0,8} \pmod 8.$$

Thus $x_{0,8}$ must be even and

$$0 \equiv \gamma_4 + \gamma_3 \equiv 6x_{0,8} + 2x_{2,8} \pmod{16},$$

which gives

$$0 \equiv \gamma_1 + \gamma_4 \equiv 4x_{-1,-4} \pmod{16}$$

because $x_{-1,-8}$ is even.

Consequently, we have $x_{-1,-4} \equiv 0 \pmod 4$, which together with (13) yield

$$0 \equiv \gamma_1 - (z_3/4) \equiv 4x_{2,1} \pmod 8.$$

Thus $x_{2,1}$ and, by $2|(z_1/2)$, $x_{0,1}$ must be even. Summarizing, by the same reasoning as in the previous cases, all the $x_{k,e}$ are even. Contradiction.

Regarding the case $0 \notin L$. Then we have:

$$\gamma_1 \equiv 52x_{-1,-4} - 51x_{-1,-8} + 4x_{1,-4} - 3x_{1,-8} + x_{2,8} \equiv 0 \pmod{128},$$

$$\gamma_2 \equiv 24x_{-1,-4} - x_{-1,-8} + 8x_{1,-4} - 5x_{1,-8} + x_{2,8} \equiv 0 \pmod{64},$$

$$\gamma_3 \equiv 16x_{-1,-4} + 9x_{-1,-8} + 16x_{1,-4} - 7x_{1,-8} + x_{2,8} \equiv 0 \pmod{32},$$

$$\gamma_4 \equiv 11x_{-1,-8} - 9x_{1,-8} + x_{2,8} \equiv 0 \pmod{16},$$

$$\gamma_5 \equiv -3x_{-1,-8} - 3x_{1,-8} + x_{2,8} \equiv 0 \pmod{8},$$

$$\gamma_6 \equiv -x_{-1,-8} - x_{1,-8} + x_{2,8} \equiv 0 \pmod{4}.$$

Consequently, we get

$$0 \equiv \gamma_4 + \gamma_5 \equiv 4x_{1,-8} + 2x_{2,8} \pmod{8},$$

and so $x_{2,8}$ must be even. Next, we have

$$0 \equiv \gamma_1 - \gamma_5 \equiv 4(x_{-1,-4} + x_{1,-4}) \pmod{8},$$

and so $x_{-1,-4} + x_{1,-4}$ must be even, too.
Hence, since $\gamma_6$, $x_{2,8}$ and $(z_1/2)$ are even, $x_{2,1}$ must be even and

$$0 \equiv \gamma_4 - \gamma_2 \equiv -4(x_{-1,-8} + x_{1,-8}) \pmod{16}.$$

Consequently, the congruence (25) in this case holds and $x_{2,8}$ must be even because $\gamma_6 \equiv 0 \pmod{4}$. Moreover, we get

$$0 \equiv \gamma_2 + \gamma_3 \equiv 4x_{1,-8} \pmod{8},$$

and so $x_{1,-8}$ must be even. Furthermore, since $\gamma_6$ and $x_{2,8}$ are even, $x_{-1,-8}$ must be even and, by (13), we deduce

$$0 \equiv (z_3/4) - \gamma_2 \equiv 2(x_{-1,-4} - x_{1,-4}) \pmod{8},$$

because $x_{2,1}$ and $x_{2,8}$ are even. This implies the congruence (26) in this case.

On the other hand, we have

$$0 \equiv \gamma_1 - \gamma_3 \equiv 4(x_{-1,-4} + x_{1,-4}) \pmod{16},$$

and so $x_{-1,-4}$ and $x_{1,-4}$ must be even because of (26).

Summarizing, by the same arguments as in the previous cases all the $x_{k,e}$ must be even. Contradiction.
Regarding the case $-1 \notin L$. Then we have:

$$\gamma_1 \equiv x_{0,8} + 4x_{1,-4} - 3x_{1,-8} + x_{2,8} \equiv 0 \pmod{128},$$

$$\gamma_2 \equiv -11x_{0,8} + 8x_{1,-4} - 5x_{1,-8} + x_{2,8} \equiv 0 \pmod{64},$$

$$\gamma_3 \equiv x_{0,8} + 16x_{1,-4} - 7x_{1,-8} + x_{2,8} \equiv 0 \pmod{32},$$

$$\gamma_4 \equiv 5x_{0,8} - 9x_{1,-8} + x_{2,8} \equiv 0 \pmod{16},$$

$$\gamma_5 \equiv x_{0,8} - 3x_{1,-8} + x_{2,8} \equiv 0 \pmod{8},$$

$$\gamma_6 \equiv x_{0,8} - x_{1,-8} + x_{2,8} \equiv 0 \pmod{4}.$$

Hence, we get

$$0 \equiv \gamma_1 - \gamma_5 \equiv 4x_{1,-4} \pmod{8},$$

and consequently $x_{1,-4}$ must be even. Furthermore, we obtain

$$0 \equiv \gamma_2 - \gamma_4 \equiv 4x_{1,-8} \pmod{16},$$

i.e., $x_{1,-8}$ is divisible by 4. Moreover, we find that

$$0 \equiv \gamma_3 - \gamma_4 \equiv 4x_{0,8} \pmod{8},$$

i.e., $x_{0,8}$, and next $x_{2,8}$ must be even (because $\gamma_6$ is even).

On other hand, we have

$$0 \equiv \gamma_1 - \gamma_3 \equiv 4x_{1,-4} \pmod{16},$$

i.e., $x_{1,-4}$ is divisible by 4.
Hence we get

$$0 \equiv (z_3/4) - \gamma_1 \equiv 4x_{2,1} \equiv 0 \pmod{8}$$

because $x_{1,-8}$ is divisible by 4 and $x_{2,8}$ is even. Consequently, $x_{2,1}$ must be even. This completes the proof of the lemma in case $\delta = 3$.

3. Let $\delta = 2$.

In this case we shall prove that $c(L) = c_0$, where $c_0 = 5$ unless $L = \{-1, 1\}$ or $L = \{0, 2\}$, in which cases $c_0 = 6$.

First, we shall show that $c(L) \geq c_0$. Putting (for example)

$$x_{-1,-4} = x_{-1,-8} = b_{-1}\,,$$
$$x_{0,1} = x_{0,8} = b_0\,,$$
$$x_{1,-4} = x_{1,-8} = b_{-1}\,,$$
$$x_{2,1} = x_{2,8} = b_2\,,$$

and

$$x_{k,1} = -x_{k,-4}\,, \quad x_{k,8} = -x_{k,-8}\ (k \in L),$$

where $b_k := b_k(L)$, $k \in K$, $b_k = 0$, if $k \notin L$ and the remaining $b_k$ are defined by the following:

$$b_k = -b_l = 1\,,$$

if $L = \{k, l\}$, $k < l$, we shall prove that $z_n \equiv 0\,(\mathrm{mod}\ 2^{c_0})$, if $n \geq 0$.
Indeed, in all these cases we have $z_1 = z_{2n} = 0$, $n \geq 0$. Moreover, by (13) we have

$$z_3 = 36(b_{-1} + b_0 + b_1 + b_2)\,,$$

and so $z_3 = 0$, as easy to check.

In order to prove that $z_{2l+1} \equiv 0\ (\mathrm{mod}\,2^{c_0})$ for $l \geq 2$, it suffices to show that $t_l \equiv 0\ (\mathrm{mod}\,2^{c_0})$ for $l \geq 1$. In fact, by (14) we get

$$t_l = 2^{l+4}(2l^3 - 3l^2 - 8l - 3) + 2^{2l+4}3(l+1)\,,$$

if $L = \{-1, 1\}$,

$$t_l = 2^{l+4}(1 - l^2)\,,$$

if $L = \{0, 2\}$,

$$t_l = 2^{l+3}(4l^3 - 4l^2 - 17l - 9) + 2^{2l+3}(6l+7)\,,$$

if $L = \{-1, 0\}$,

$$t_l = 2^{l+3}(4l^3 - 6l^2 - 17l - 7) + 2^{2l+3}(6l+7)\,,$$

if $L = \{-1, 2\}$,

$$t_l = -2^{l+3}(l+1) + 2^{2l+3}\,,$$

if $L = \{1, 2\}$,

$$t_l = 2^{l+3}(-2l^2 + l + 3) - 2^{2l+3}\,,$$

if $L = \{0, 1\}$.

In two the first cases we have $t_1 = 0$ and $2^6 | t_l$, if $l \geq 2$. In the remaining cases it is easily seen that $2^5 | t_l$, too. This gives $c(L) \geq c_0$. In order to prove that $c(L) \leq c_0$ let us suppose, contrary to our claim, that $c(L) \geq c_0 + 1$, i.e., $z_n \equiv 0 \pmod{2^{c_0+1}}$, if $n \geq 0$. Then we shall prove that all the $x_{k,e}$ must be even. Again, it suffices to prove that $x_{k,e}$ are even in case $\operatorname{sgn} e = (-1)^k$.

Since $t_l \equiv 0 \pmod{2^{c_0+1}}$ we can use the congruence (15) modulo $2^{c_0-l-1}$ (instead of $2^{11-l}$). Then, by (14) we get the congruences

$$\begin{aligned}\gamma_l &\equiv (8l^3 - 12l^2 - 34l - 13)x_{-1,-8} - (2l+1)x_{1,-8} \\ &+ 2^{l+1}\big((6l+7)x_{-1,-4} + x_{1,-4}\big) \pmod{2^{5-l}},\end{aligned}$$

if $L = \{-1, 1\}$,

$$\gamma_l \equiv x_{2,8} + (5 - 4l^2)x_{0,8} \pmod{2^{5-l}},$$

if $L = \{0, 2\}$,

$$\gamma_l \equiv (8l^3 - 12l^2 - 34l - 13)x_{-1,-8} + (5 - 4l^2)x_{0,8} + 2^{l+1}(6l+7)x_{-1,-4} \pmod{2^{4-l}},$$

if $L = \{-1, 0\}$,

$$\gamma_l \equiv x_{2,8} + (8l^3 - 12l^2 - 34l - 13)x_{-1,-8} + 2^{l+1}(6l+7)x_{-1,-4} \pmod{2^{4-l}},$$

if $L = \{-1, 2\}$,

$$\gamma_l \equiv x_{2,8} - (2l+1)x_{1,-8} + 2^{l+1}x_{1,-4} \pmod{2^{4-l}},$$

if $L = \{1, 2\}$,

$$\gamma_l \equiv (5 - 4l^2)x_{0,8} - (2l+1)x_{1,-8} + 2^{l+1}x_{1,-4} \pmod{2^{4-l}},$$

if $L = \{0, 1\}$.
Regarding the case $L = \{-1, 1\}$. Then we have:

$$\gamma_1 \equiv 4x_{-1,-4} - 3x_{-1,-8} + 4x_{1,-4} - 3x_{1,-8} \equiv 0 \pmod{16},$$

$$\gamma_2 \equiv -x_{-1,-8} + 3x_{1,-8} \equiv 0 \pmod{8},$$

$$\gamma_3 \equiv -3x_{-1,-8} - 3x_{1,-8} \equiv 0 \pmod{4},$$

and

$$(z_1/2) \equiv x_{-1,-4} + x_{-1,-8} + x_{1,-4} + x_{1,-8} \equiv 0 \pmod{64},$$

$$(z_3/4) \equiv 2x_{-1,-4} - 9x_{-1,-8} + 6x_{1,-4} + 3x_{1,-8} \equiv 0 \pmod{32}.$$

Hence we get

$$0 \equiv (z_3/4) - \gamma_2 \equiv 2(x_{-1,-4} - x_{1,-4}) \pmod 8,$$

i.e., $x_{-1,-4} - x_{1,-4} \equiv 0 \pmod 4$, which implies

$$0 \equiv \gamma_1 + \gamma_2 \equiv 4x_{-1,-8} \pmod 8.$$

Consequently, $x_{-1,-8}$ must be even, and so $x_{1,-8}$ must be even too because $(z_1/2)$ is even. Moreover, we have

$$0 \equiv (z_1/2) - \gamma_3 - (x_{-1,-4} - x_{1,-4}) \equiv 2x_{1,-4} \pmod 4,$$

and so $x_{1,-4} \equiv x_{-1,-4} \equiv 0 \pmod 2$.

Regarding the case $L = \{0, 2\}$. Then we have:

$$\gamma_1 \equiv x_{0,8} + x_{2,8} \pmod{16},$$

$$\gamma_2 \equiv 5x_{0,8} + x_{2,8} \pmod 8,$$

$$(z_3/4) \equiv 9x_{0,8} + 5x_{2,8} + 4x_{2,1} \pmod{32}.$$

Therefore we obtain

$$0 \equiv \gamma_2 - \gamma_1 \equiv 4x_{0,8} \pmod 8,$$

i.e., $x_{0,8}$ and $x_{2,8}$ must be even because $\gamma_1$ is even. Consequently, since $\gamma_1$ is divisible by 8 we find that $0 \equiv (z_3/4) \equiv 4x_{2,1} \pmod 8$, i.e., $x_{2,1}$ must be even.
Regarding the case $L = \{-1, 0\}$. Then we have:

$$\gamma_1 \equiv 4x_{-1,-4} - 3x_{-1,-8} + x_{0,8} \equiv 0 \pmod 8,$$

$$\gamma_2 \equiv -x_{-1,-8} + x_{0,8} \equiv 0 \pmod 4,$$

$$(z_3/4) \equiv 2x_{-1,-4} - 9x_{-1,-8} + 9x_{0,8} \equiv 0 \pmod{16}.$$

Thus we get

$$0 \equiv (z_3/4) - \gamma_2 \equiv 2x_{-1,-4} \pmod 4,$$

i.e., $x_{-1,-4}$ must be even.

On the other hand, we have

$$0 \equiv \gamma_1 + \gamma_2 \equiv 2x_{0,8} \pmod 4,$$

i.e., $x_{0,8}$ must be even, and so $x_{-1,-8}$ must be even too because $\gamma_2$ is even. Regarding the case $L = \{-1, 2\}$. Then we have:

$$\gamma_1 \equiv 4x_{-1,-4} + 5x_{-1,-8} + x_{2,8} \equiv 0 \pmod 8,$$

$$\gamma_2 \equiv -x_{-1,-8} + x_{2,8} \equiv 0 \pmod 4,$$

$$(z_3/4) \equiv 2x_{-1,-4} - 9x_{-1,-8} + 4x_{2,1} + 5x_{2,8} \equiv 0 \pmod{16}.$$

Hence we get

$$0 \equiv \gamma_1 - \gamma_2 \equiv 2x_{-1,-8} \pmod 4,$$

i.e., $x_{-1,-8}$ and $x_{2,8}$ must be even because $\gamma_2$ is even. Consequently, we have

$$0 \equiv (z_3/4) - \gamma_2 \equiv 2x_{-1,-4} \pmod 4,$$

i.e., $x_{-1,-4}$ must be even.

Regarding the case $L = \{1, 2\}$. Then we have:

$$\gamma_1 \equiv 4x_{1,-4} - 3x_{1,-8} + x_{2,8} \equiv 0 \pmod 8,$$

$$\gamma_2 \equiv -x_{1,-8} + x_{2,8} \equiv 0 \pmod 4,$$

$$(z_3/4) \equiv 6x_{1,-4} + 3x_{1,-8} + 5x_{2,8} + 4x_{2,1} \equiv 0 \pmod{16}.$$

Thus we have

$$0 \equiv \gamma_1 + \gamma_2 \equiv 2x_{2,8} \pmod 4,$$

i.e., $x_{2,8}$ and $x_{1,-8}$ must be even because $\gamma_2$ is even. Next, we have

$$0 \equiv (z_3/4) - \gamma_2 \equiv 2x_{1,-4} \pmod 4,$$

and so $x_{1,-4}$ must be even. To finish the proof of the lemma in this case it remains to prove that $x_{2,1}$ is even. But it follows easily because $(z_1/2)$ is even.

Regarding the case $L = \{0, 1\}$. This case was considered in [2] and [5]. Then we have:

$$\gamma_1 \equiv x_{0,8} + 4x_{1,-4} - 3x_{1,-8} \equiv 0 \pmod 8,$$

$$\gamma_2 \equiv x_{0,8} - x_{1,-8} \equiv 0 \pmod 4,$$
$$(z_3/4) \equiv 9x_{0,8} + 6x_{1,-4} + 3x_{1,-8} \equiv 0 \pmod{16}.$$

Hence we get

$$0 \equiv \gamma_1 + \gamma_2 \equiv 2x_{0,8} \pmod 4,$$

i.e., $x_{0,8}$ and $x_{1,-8}$ must be even because $\gamma_2$ is even. Moreover, we have

$$0 \equiv (z_3/4) - \gamma_2 \equiv 2x_{1,-4} \pmod 4,$$

i.e., $x_{1,-4}$ must be even, which completes the proof of the lemma in case $\delta = 2$.
4. Let $\delta = 1$.

In this case we shall prove that $c(L) = 2$. First, we shall show that $c(L) \geq 2$. Let $L = \{k\}$. Set

$$x_{k,1} = -x_{k,8} = 1,$$

if $k$ is even and

$$x_{k,-4} = -x_{k,-8} = 1,$$

if $k$ is odd, and

$$x_{k,1} = -x_{k,-4}, \quad x_{k,8} = -x_{k,-8},$$

and $x_{l,e} = 0$, if $l \in K$, $l \neq k$. For any $k \in K$ we have $z_1 = z_{2n} = 0$, $n \geq 0$. Moreover, by definition $4 | z_{2l+1}$, if $l \geq 0$. Thus we have proved that $c(L) \geq 2$. Let us suppose, contrary to our claim, that $c(L) \geq 3$, i.e., that $8 | z_n$, if $n \geq 0$. We must prove that all the $x_{k,e}$ are even. Again, it suffices to prove it in case $\operatorname{sgn} e = (-1)^k$. By (13) and $4 | (z_1/2)$, we get

$$x_{k,1} + x_{k,8} \equiv x_{k,-4} + x_{k,-8} \equiv 0 \pmod 4,$$

and since $(z_3/4)$ is even, $x_{k,e}$ satisfying $\operatorname{sgn} e = (-1)^k$ must be even. Consequently by the same reasoning as previously all the $x_{k,e}$ ($k \in L$, $e \in \mathcal{T}_8$) must be even and the lemma is proved completely. □

# 4 Proof of Theorem

By Lemma 2 we have

$$\Lambda_2(x,m)$$
$$= (-1)^r \sum_{\substack{k\in K,\\ e\in\mathcal{T}_8}} (-1)^{k+1}(-1)^{k+1} x_{k,e} \sum_{d\in\mathcal{T}_m} \Psi(|d|)\mu(|d|)g(\chi_d)|d|^{-1} \sum_{a=1}^{m}{}' \chi_d(a)\mathcal{L}_{k,e}(\zeta_m^a)$$

$$= (-1)^r \sum_{a=1}^{m}{}' \sum_{\substack{k\in K,\\ e\in\mathcal{T}_8}} x_{k,e}\mathcal{L}_{k,e}(\zeta_m^a) \sum_{d\in\mathcal{T}_m} \Psi(|d|)\mu(|d|)g(\chi_d)|d|^{-1}\chi_d(a)$$

$$= (-1)^r \sum_{a=1}^{m}{}' \Big( \sum_{\substack{k\in K,\\ e\in\mathcal{T}_8}} x_{k,e}\mathcal{L}_{k,e}(\zeta_m^a) \Big) \cdot \Big( \prod_{p|m} (1 - \Psi(p)g(\chi_{p^*})p^{-1}\chi_{p^*}(a)) \Big),$$

where $p^* = (-1)^{(p-1)/2}p$. Therefore it follows from Lemma 4 that the numbers $\Lambda_2(x,m)$ are 2-adic integers and since

$$\Psi(p)g(\chi_{p^*})|p|^{-1}\chi_{p^*}(a) - 1 \equiv 1 + \zeta_p + \cdots \zeta_p^{p-1} \equiv 0 \,(\mathrm{mod}\ 2)$$

by Lemma 5 they are divisible by $2^{r+\lambda}$. The latter lemma implies the rest of the theorem immediately. □

**REMARK.** A similar proof works for the numbers

$$L_2^{[m,\theta]}(k,\chi\omega^{1-k}) = \prod_{p|m,\ p-\text{prime}} (1 - \chi(p)\theta(p)p^{1-k}) L_2(k,\chi\omega^{1-k}),$$

where $\theta\colon \mathbb{N} \to \mathbb{C}_2$ is a multiplicative function (satisfying $\theta(s) \equiv 1 \,(\mathrm{mod}\ 2)$ for $s|m$) instead of the numbers $L_2^{[m]}(k,\chi\omega^{1-k})$.

# 5 Applications

For $k \le 0$, by Theorem 5.11 [9] we get

$$L_p(k,\chi\omega^{1-k}) = -(1 - \chi(p)p^{-k})\frac{B_{1-k,\chi}}{1-k}.$$

Therefore, for $k = -1$ and $0$ we have

$$L_p(-1,\chi\omega^2) = -(1 - \chi(p)p)\frac{B_{2,\chi}}{2},$$

and

$$L_p(0,\chi\omega) = -(1 - \chi(p))B_{1,\chi}.$$

On the other hand, the Mazur-Wiles-Kolster-Greither theorem (earlier the Birch-Tate conjecture) for real quadratic fields $F$ states

$$\eta k_2(D) = B_{2,\chi_D},$$

where $D$ is the discriminant of $F$, $k_2(D) := |K_2(O_F)|$ and $O_F$ (resp. $K_2$) denotes the integers in $F$ (resp. the Milnor functor). Here $\eta(5) := 1/5$, $\eta(8) := 1/2$ and $\eta(D) := 1$, if $D > 8$. If $D = 1$, write $k_2(D) = 2$ and $\eta(D) = 1/12$. Moreover the Dirichlet class number formulas for imaginary quadratic fields $F$ state

$$\xi h(D) = -B_{1,\chi_D},$$

where $h(D)$ stands for the class number of $F$, and $\xi(-3) := 1/3$, $\xi(-4) := 1/2$ and $\xi(D) := 1$, if $D < -4$.

The above formulas give (for $\chi = \chi_D$)

$$L_p(-1, \chi_D\omega^2) = -\frac{1}{2}(1 - \chi_D(p)p)\eta k_2(D), \tag{27}$$

if $D \geq 5$ and

$$L_p(0, \chi\omega) = (1 - \chi(p))\xi h(D). \tag{28}$$

if $D \leq -3$.

If $k = 1$ and $\chi_D$ is an even quadratic character then, by the Leopoldt formulas we obtain

$$L_p(1, \chi_D) = 2(1 - \chi(p)p^{-1})D^{-1/2}h(D)\log_p \varepsilon(D), \tag{29}$$

where $\varepsilon(D)$ denotes the fundamental unit of a quadratic field with the discriminant $D$ (see Theorems 5.18 and 5.24 [9]).

As usual the complex and $p$-adic formulas "differ by an Euler factor". Indeed, the corresponding complex formulas are of the form

$$L(-1, \chi_D) = -\frac{1}{2}\eta k_2(D),$$

if $D \geq 5$,

$$L(0, \chi_D) = \xi h(D),$$

if $D \leq -3$ (for both the formulas see Theorem 4.2 [9]), and

$$L(1, \chi_D) = 2D^{-1/2} h(D) \log \varepsilon(D),$$

if $D \geq 5$ (see Chapter 4 [9]).

If $D \leq -3$ then the modified complex Lichtenbaum conjecture states

$$L(2, \chi_D) = 2R_2 |D|^{-3/2} k_2(D),$$

where $R_2 := R_2(D)$ denotes the second Borel regulator of the corresponding quadratic field (see Notes and comments to §2, p. 199, [4]). Consequently, by analogy the $p$-adic Lichtenbaum conjecture should read

$$L_p(2, \chi_D \omega^{-1}) = 2(1 - \chi_D(p) p^{-2}) R_{2,p} |D|^{-3/2} k_2(D), \tag{30}$$

where $R_{2,p} := R_{2,p}(D)$ denotes the second $p$-adic Borel regulator of the corresponding quadratic field.

For any fundamental discriminant, let

$$H(D) = \begin{cases} \xi(D) h(D), & \text{if } D \leq -3, \\ D^{-1/2} h(D) \log_2 \varepsilon(D), & \text{if } D \geq 5. \end{cases}$$

$$K_2(D) = \begin{cases} \eta(D) k_2(D), & \text{if } D \geq 5, \\ |D|^{-3/2} h(D) R_{2,2}(D) k_2(D), & \text{if } D \leq -3. \end{cases}$$

Then via the $p$-adic Lichtenbaum conjecture for imaginary quadratic fields and by (27), (28), (29) and (30) (for $p = 2$), we can rewrite the main theorem in the form:

**THEOREM.** *Let $m > 1$ be a square-free odd natural number having $r$ prime factors and let $\Psi$: $\mathbf{N} \to \mathbf{C}_2$ be a multiplicative function such that $\Psi(s) \equiv 1 \pmod 2$, if $s|m$. Set $K = \{-1, 0, 1, 2\}$. Let $L$ be a non-empty subset of $K$ having $\delta$ elements and let $\{x_{k,e}\}_{k \in K, e \in \mathcal{T}_8}$ be a sequence of 2-adic integers defined on $L$. Set*

$$\Lambda := \Lambda_{-1} + \Lambda_0 + \Lambda_1 + \Lambda_2 + \Lambda'_{-1} + \Lambda'_1,$$

*where*

$$\Lambda_{-1} = -\frac{1}{2} \sum_{e \in \mathcal{T}_8} x_{-1,e} \sum_{\substack{d \in \mathcal{T}_m, \\ 1 \neq ed > 0}} \Psi(|d|) \prod_{\substack{p|m, \\ p-\text{prime}}} (1 - \chi_{ed}(p) p^2)(1 - \chi_{ed}(2) 2) K_2(ed),$$

$$\Lambda_0 = \sum_{e \in \mathcal{T}_8} x_{0,e} \sum_{\substack{d \in \mathcal{T}_m, \\ ed<0}} \Psi(|d|) \prod_{\substack{p|m, \\ p-\text{prime}}} \big(1 - \chi_{ed}(p)p\big)\big(1 - \chi_{ed}(2)\big)H(ed),$$

$$\Lambda_1 = \sum_{e \in \mathcal{T}_8} x_{1,e} \sum_{\substack{d \in \mathcal{T}_m, \\ 1 \neq ed>0}} \Psi(|d|) \prod_{\substack{p|m, \\ p-\text{prime}}} \big(1 - \chi_{ed}(p)\big)\big(2 - \chi_{ed}(2)\big)H(ed),$$

$$\Lambda_2 = \frac{1}{2} \sum_{e \in \mathcal{T}_8} x_{2,e} \sum_{\substack{d \in \mathcal{T}_m, \\ ed<0}} \Psi(|d|) \prod_{\substack{p|m, \\ p-\text{prime}}} \big(1 - \chi_{ed}(p)p^{-1}\big)\big(4 - \chi_{ed}(2)\big)K_2(ed),$$

*and*

$$\Lambda'_{-1} = \frac{1}{12} x_{-1,1} \prod_{\substack{p|m, \\ p-\text{prime}}} (1 - p^2),$$

$$\Lambda'_1 = \begin{cases} (x_{1,1} \log_2 m)/2, & \text{if } m \text{ is a prime number}, \\ 0, & \text{otherwise}. \end{cases}$$

*Assume in case* $2 \in L$ *that the 2-adic Lichtenbaum conjecture for imaginary quadratic fields holds. Then the number* $\Lambda$ *is a 2-adic integer divisible by* $2^{r+\lambda}$, *where* $\lambda$ *has the same meaning as in the main theorem.* □

**REMARK.** The above theorem produces many new congruences between the orders of $K_2$-groups of the integers and class numbers of appropriate quadratic fields modulo higher powers of 2. We shall deal with such congruences in another paper.

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