ON THE CHARACTER RING
OF A FINITE GROUP

by

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Abstract. — Let $G$ be a finite group and let $k$ be a sufficiently large finite field. Let $R(G)$ denote the character ring of $G$ (i.e. the Grothendieck ring of the category of $\mathbb{C}G$-modules). We study the structure and the representations of the commutative algebra $k \otimes \mathbb{Z} R(G)$.

Contents

Introduction ................................................................. 5
1. Preliminaries .......................................................... 6
2. Modules for $k R(G)$ and $k R(kG)$ .............................. 7
3. Principal block ......................................................... 11
4. Some invariants ....................................................... 13
5. The symmetric group .................................................. 15
6. Dihedral groups ....................................................... 17
7. Some tables ............................................................ 19
References ................................................................. 23

Introduction

Let $G$ be a finite group. We denote by $R(G)$ the Grothendieck ring of the category of $\mathbb{C}G$-modules (it is usually called the character ring of $G$). It is a natural question to try to recover properties of $G$ from the knowledge of $R(G)$. It is clear that two finite groups having the same character table have the same Grothendieck rings and it is a Theorem of Saksonov [S] that the converse also holds. So the problem is reduced to an intensively studied question in character theory: recover properties of the group through properties of its character table.

In this paper, we study the $k$-algebra $k R(G) = k \otimes \mathbb{Z} R(G)$, where $k$ is a splitting field for $G$ of positive characteristic $p$. It is clear that the knowledge of $k R(G)$ is a much weaker information than the knowledge of $R(G)$. The aim of this paper is to gather results on the representation theory of the algebra $k R(G)$: although most of the results are certainly well-known, we have not found any general treatment of these questions. The blocks of $k R(G)$ are local algebras which are parametrized by conjugacy classes of $p$-regular elements of $G$. So the simple $k R(G)$-modules are parametrized by conjugacy classes of $p$-regular elements of $G$. Moreover, the dimension of the projective cover of the simple module associated to the conjugacy class of the $p$-regular element $g \in G$ is equal to the number of conjugacy classes of $p$-elements in the centralizer $C_G(g)$. We also prove that the radical of $k R(G)$ is the kernel of the decomposition map $k R(G) \to k \otimes \mathbb{Z} R(kG)$, where $R(kG)$ is the Grothendieck ring of the category of $kG$-modules (i.e. the ring of virtual Brauer characters of $G$).

2000 Mathematics Subject Classification. — primary 19A31; secondary 19A22.
We prove that the block of $kR(G)$ associated to the $p'$-element $g$ is isomorphic to the block of $kR(C_G(g))$ associated to 1 (such a block is called the principal block). This shows that the study of blocks of $kR(G)$ is reduced to the study of principal blocks. We also show that the principal block of $kR(G)$ is isomorphic to the principal block of $kR(H)$ whenever $H$ is a subgroup of $p'$-index which controls the fusion of $p$-elements or whenever $H$ is the quotient of $G$ by a normal $p'$-subgroup.

We also introduce several numerical invariants (Loewy length, dimension of Ext-groups) that are partly related to the structure of $G$. These numerical invariants are computed completely whenever $G$ is the symmetric group $S_n$ (this relies on previous work of the author: the descending Loewy series of $kR(S_n)$ was entirely computed in [B]) or $G$ is a dihedral group and $p = 2$. We also provide tables for these invariants for small groups (alternating groups $A_n$ with $n \leq 12$, some small simple groups, groups $PSL(2,q)$ with $q$ a prime power $\leq 27$, exceptional finite Coxeter groups).

**Notation** - Let $O$ be a Dedekind domain of characteristic zero, let $p$ be a maximal ideal of $O$, let $K$ be the fraction field of $O$ and let $k = O/p$. Let $O_p$ be the localization of $O$ at $p$: then $k = O_p/pO_p$. If $x \in O_p$, we denote by $\bar{x}$ its image in $O_p/pO_p = k$. Throughout this paper, we assume that $k$ has characteristic $p > 0$ and that $K$ and $k$ are splitting fields for all the finite groups involved in this paper. If $n$ is a non-zero natural number, $n_p$ denotes the largest divisor of $n$ prime to $p$ and we set $n_p = n/n_p$.

If $F$ is a field and if $A$ is a finite dimensional $F$-algebra, we denote by $\mathcal{R}(A)$ its Grothendieck group. If $M$ is an $A$-module, the radical of $M$ is denoted by $\text{Rad}M$ and the class of $M$ in $\mathcal{R}(A)$ is denoted by $[M]$. If $S$ is a simple $A$-module, we denote by $[M : S]$ the multiplicity of $S$ as a chief factor of a Jordan-Hölder series of $M$. The set of irreducible characters of $A$ is denoted by $\text{Irr}A$.

We fix all along this paper a finite group $G$. For simplification, we set $\mathcal{R}(G) = \mathcal{R}(KG)$ and $\text{Irr} G = \text{Irr} KG$ (recall that $K$ is a splitting field for $G$). The abelian group $\mathcal{R}(G)$ is endowed with a structure of ring induced by the tensor product. If $\chi \in \mathcal{R}(G)$, we denote by $\chi^*$ its dual (as a class function on $G$, we have $\chi^*(g) = \chi(g^{-1})$ for any $g \in G$). If $R$ is any commutative ring, we denote by $\text{Class}_{R}(G)$ the space of class functions $G \to R$ and we set $\mathcal{R}(G) = R \otimes \mathcal{R}(G)$. If $X$ is a subset of $G$, we denote by $1_X^G : G \to R$ the characteristic function of $X$. If $R$ is a subring of $K$, then we simply write $1_X = 1_X^K$. Note that $1_G$ is the trivial character of $G$. If $f, f' \in \text{Class}_{K}(G)$, we set

$$\langle f, f' \rangle_G = \frac{1}{|G|} \sum_{g \in G} f(g^{-1})f'(g).$$

Then $\text{Irr} G$ is an orthonormal basis of $\text{Class}_{K}(G)$. We shall identify $\mathcal{R}(G)$ with the sub-$\mathbb{Z}$-module (or sub-$\mathbb{Z}$-algebra) of $\text{Class}_{K}(G)$ generated by $\text{Irr} G$, and $K \mathcal{R}(G)$ with $\text{Class}_{K}(G)$. If $f \in O_p \mathcal{R}(G)$, we denote by $f$ its image in $k \mathcal{R}(G)$.

If $g$ and $h$ are two elements of $G$, we write $g \sim h$ (or $g \sim_G h$ if we need to emphasize the group) if they are conjugate in $G$. We denote by $g_p$ (resp. $g'_p$) the $p$-part (resp. the $p'$-part) of $g$. If $X$ is a subset of $G$, we set $X_{p'} = \{g_{p'} \mid g \in X\}$ and $X_p = \{g_p \mid g \in X\}$. If moreover $X$ is closed under conjugacy, the set of conjugacy classes contained in $X$ is denoted by $X/\sim$. In this case, $1_X^G \in \text{Class}_{R}(G)$. The centre of $G$ is denoted by $Z(G)$.

**Remark** - We have recently discovered that some of the questions investigated in this paper were already studied by M. Deiml in his Ph.D. Thesis [D, Chapter 3]. More precisely, most of the results of our Section 2 were already proved by M. Deiml.

1. Preliminaries

1.A. Symmetrizing form. — Let

$$\tau_G : \mathcal{R}(G) \longrightarrow \mathbb{Z}$$

$$\chi \quad \mapsto \quad \langle \chi, 1_G \rangle_G$$
denote the canonical symmetrizing form on $R(G)$. The dual basis of $\text{Irr} G$ is $(\chi^*)_{\chi \in \text{Irr} G}$. It is then readily seen that $(R(G), \text{Irr} G)$ is a based ring (in the sense of Lusztig [L, Page 236]).

If $R$ is any ring, we denote by $\tau^R_G : R R(G) \to R$ the symmetrizing form $\text{Id}_R \otimes \tau G$.

1.B. Translation by the centre. — If $\chi \in \text{Irr} G$, we denote by $\omega : Z(G) \to \mathbb{O}^\times$ the linear character such that $\chi(z g) = \omega(z) \chi(g)$ for all $z \in Z(G)$ and $g \in G$. If $z \in Z(G)$, we denote by $t_z : K R(G) \to K R(G)$ the linear map defined by $(t_z f)(g) = f(z g)$ for all $f \in K R(G)$ and $g \in G$.

It is clear that $t_z t_{z'} = t_z \circ t_{z'}$ for all $z$, $z' \in Z(G)$ and that $t_z$ is an automorphism of algebra.

Moreover,

$$t_z \chi = \omega(z) \chi$$

for every $\chi \in \text{Irr} G$. Therefore, $t_z$ is an isometry which stabilizes $\mathcal{O} R(G)$. If $R$ is a subring of $K$ such that $O \subset R \subset K$, we still denote by $t_z : R R(G) \to R R(G)$ the restriction of $t_z$. Let $\ell_z = \text{Id}_k \otimes t_z : k R(G) \to k R(G)$.

This is again an automorphism of $k$-algebra. If $z$ is a $p$-element, then $t_z = \text{Id}_{k R(G)}$.

1.C. Restriction. — If $\pi : H \to G$ is a morphism of groups, then the restriction through $\pi$ induces a morphism of rings $\text{Res}_\pi : R R(G) \to R R(H)$. If $R$ is a subring of $K$, we still denote by $\text{Res}_\pi : R R(G) \to R R(H)$ the morphism $\text{Id}_R \otimes \text{Res}_\pi$. We denote by $\overline{\text{Res}_\pi} : k R(G) \to k R(H)$ the reduction modulo $p$ of $\text{Res}_\pi : \mathcal{O} R(G) \to \mathcal{O} R(H)$. Recall that, if $H$ is a subgroup of $G$ and $\pi$ is the canonical injection, then $\text{Res}_\pi$ is just $\text{Res}_H^G$. In this case, $\overline{\text{Res}_\pi}$ will be denoted by $\overline{\text{Res}_H}$. Note the following fact:

(1.1) If $\pi$ is surjective, then $\overline{\text{Res}_\pi}$ is injective.

Proof of 1.1. — Indeed, if $\pi$ is surjective, then $\text{Res}_\pi : R R(G) \to R R(H)$ is injective and its image is a direct summand of $R R(H)$. \hfill \square

1.D. Radical. — First, note that, since $k R(G)$ is commutative, we have

(1.2) $\text{Rad} k R(G)$ is the ideal of nilpotent elements of $k R(G)$.

So, if $\pi : H \to G$ is a morphism of finite groups, then

(1.3) $\text{Res}_\pi (\text{Rad} k R(G)) \subset \text{Rad} k R(H)$.

The Loewy length of the algebra $k R(G)$ is defined as the smallest natural number $n$ such that $(\text{Rad} k R(G))^n = 0$. We denote it by $\ell_p(G)$. By 1.1 and 1.3, we have:

(1.4) If $\pi$ is surjective, then $\ell_p(G) \leq \ell_p(H)$.

2. Modules for $K R(G)$ and $k R(G)$

2.A. Semisimplicity. — Recall that $K R(G)$ is identified with the algebra of class functions on $G$. If $C \in G/\sim$ and $f \in K R(G)$, we denote by $f(C)$ the constant value of $f$ on $C$. We now define $e v_C : K R(G) \to K$, $f \mapsto f(C)$. It is a morphism of $K$-algebras. In other words, it is an irreducible representation (or character) of $K R(G)$. We denote by $\mathcal{D}_C$ the corresponding simple $K R(G)$-module $(\dim_K \mathcal{D}_C = 1$ and an element $f \in K R(G)$ acts on $\mathcal{D}_C$ by multiplication by $e v_C(f) = f(C)$). Now, $1_C$ is a primitive idempotent of $K R(G)$ and it is easily checked that

(2.1) $K R(G) 1_C \simeq \mathcal{D}_C$.

Recall that

(2.2) $1_C = \frac{|C|}{|G|} \sum_{\chi \in \text{Irr} G} \chi(C^{-1}) \chi$. 


and
\[(2.3) \quad \sum_{C \in G/\sim} 1_C = 1_G.\]

Therefore:

**Proposition 2.4.** — We have:

(a) \((DC)_{C \in G/\sim}\) is a family of representatives of isomorphy classes of simple \(KR(G)\)-modules.

(b) \(\text{Irr} \, KR(G) = \{ev_C \mid C \in G/\sim\}\).

(c) \(KR(G)\) is split semisimple.

We conclude this section by the computation of the Schur elements (see [GP, 7.2] for the definition) associated to each irreducible character of \(KR(G)\). Since
\[(2.5) \quad t^K_G = \sum_{C \in G/\sim} |C| |G| ev_C,
\]
we have by [GP, Theorem 7.2.6]:

**Corollary 2.6.** — Let \(C \in G/\sim\). Then the Schur element associated with the irreducible character \(ev_C\) is \(|G|/|C|\).

**Remark 2.7** - If \(z \in Z(G)\), then \(t_z\) induces an isomorphism of algebras \(KR(G)_{1C} \cong KR(G)_{1z^{-1}C}\).

**Remark 2.8** - If \(f \in KR(G)\), then \(f = \sum_{C \in G/\sim} f(C)_{1C}\).

**Example 2.9** - The map \(ev_1\) will sometimes be denoted by \(\text{deg}\), since it sends a character to its degree.

2.B. **Decomposition map.** — Let \(d_p : R(G) \to R(kG)\) denote the decomposition map. If \(R\) is any commutative ring, we denote by \(d_p^R : RR(G) \to RR(kG)\) the induced map. Note that \(R(kG)\) is also a ring (for the multiplication given by tensor product) and that \(d_p\) is a morphism of ring. Also, by [CR, Corollary 18.14],
\[(2.10) \quad d_p \text{ is surjective.}\]

Since \(\text{Irr}(kG)\) is a linearly independent family of class functions \(G \to k\) (see [CR, Theorem 17.4]), the map \(\chi : kR(kG) \to \text{Class}_k(G)\) that sends the class of a \(kG\)-module to its character is (well-defined and) injective. This is a morphism of \(k\)-algebras.

Now, if \(C\) is a conjugacy class of \(p\)-regular elements (i.e. \(C \in G_{p'/\sim}\)), we define
\[S'_p(C) = \{g \in G \mid g_{p'} \in C\}\]
(for instance, \(S'_p(1) = C_p\)). Then \(S'_p(C)\) is called the \(p'\)-section of \(C\): this is a union of conjugacy classes of \(G\). Let \(\text{Class}^p_k(G)\) be the space of class functions \(G \to k\) which are constant on \(p'\)-sections. Then, by [CR, Lemma 17.8], \(\text{Irr}(kG) \subset \text{Class}^p_k(G)\), so the image of \(\chi\) is contained in \(\text{Class}^p_k(G)\). But, \(\chi\) is injective, \(|\text{Irr}(kG)| = |G_{p'/\sim}|\) (see [CR, Corollary 17.11]) and \(\dim_k \text{Class}^p_k(G) = |G_{p'/\sim}|\). Therefore, we can identify, through \(\chi\), the \(k\)-algebras \(kR(kG)\) and \(\text{Class}^p_k(G)\). In particular,
\[(2.11) \quad kR(kG) \text{ is split semisimple.}\]
2.C. Simple $k\mathcal{R}(G)$-modules. — If $C \in G/\sim$, we still denote by $\text{ev}_C : \mathcal{O}_R(G) \to \mathcal{O}$ the restriction of $\text{ev}_C$ and we denote by $\overline{\text{ev}}_C : k\mathcal{R}(G) \to k$ the reduction modulo $p$ of $\text{ev}_C$. It is easily checked that $\overline{\text{ev}}_C$ factorizes through the decomposition map $d_p$. Indeed, if $\text{ev}_C^b : k\mathcal{R}(kG) \to k$ denote the evaluation at $C$ (recall that $k\mathcal{R}(kG)$ is identified, via the map $\chi$ of the previous subsection, to $\text{Class}_{k'}(G)$), then

$$\overline{\text{ev}}_C = \text{ev}_C^b \circ d_p^b.$$  

Let $D_C$ be the corresponding simple $k\mathcal{R}(G)$-module. Let $\delta_p : \mathcal{R}(k\mathcal{R}(G)) \to \mathcal{R}(k\mathcal{R}(G))$ denote the decomposition map (see [GP, 7.4] for the definition). Then

$$\delta_p[D_C] = [D_C].$$

The following facts are well-known:

**Proposition 2.14.** — Let $C, C' \in G/\sim$. Then $D_C \simeq D_{C'}$ if and only if $C_p' = C'_p$.

**Proof.** — The “if” part follows from the following classical fact [CR, Proposition 17.5 (ii) and (iv)] and Lemma 17.8: if $\chi \in \mathcal{R}(G)$ and if $g \in G$, then

$$\chi(g) \equiv \chi(gp') \mod p.$$  

The “only if” part follows from 2.12 and from the surjectivity of the decomposition map $d_p$. □

**Corollary 2.15.** — We have:

(a) $(D_C)_{C \in G/\sim}$ is a family of representatives of isomorphy classes of simple $k\mathcal{R}(G)$-modules.

(b) $\text{Irr} k\mathcal{R}(G) = \{\overline{\text{ev}}_C | C \in G_p'/\sim\}.$

(c) $\text{Rad} k\mathcal{R}(G) = \text{Ker} d_p^k.$

(d) $k\mathcal{R}(G)$ is split.

**Proof.** — (a) follows from 2.13 and from the fact that the isomorphy class of any simple $k\mathcal{R}(G)$-modules must occur in some $\delta_p[S]$, where $S$ is a simple $K\mathcal{R}(G)$-module. (b) follows from (a). (c) and (d) follow from (a), (b), 2.12 and 2.11. □

**Corollary 2.16.** — $\dim_k \text{Rad}(k\mathcal{R}(G)) = |G/\sim| - |G_p'/\sim|.$

**Corollary 2.17.** — $k\mathcal{R}(G)$ is semisimple if and only if $p$ does not divide $|G|$.

**Example 2.18** — Since $\text{ev}_1$ is also denoted by deg, we shall sometimes denote by $\overline{\text{deg}}$ the morphism $\overline{\text{ev}}_1$. If $G$ is a $p$-group, then Corollary 2.15 shows that $\text{Rad} k\mathcal{R}(G) = \text{Ker}(\overline{\text{deg}})$. In this case, if $1, \lambda_1, \ldots, \lambda_r$ denote the linear characters of $G$ and $\chi_1, \ldots, \chi_s$ denote the non-linear irreducible characters of $G$, then $(\overline{\chi}_1 - 1, \ldots, \overline{\chi}_r - 1, \overline{\lambda}_1, \overline{\lambda}_s)$ is a $k$-basis of $\text{Rad} k\mathcal{R}(G)$.

2.D. Projective modules. — We now fix a conjugacy class $C$ of $p$-regular elements (i.e. $C \in G_p'/\sim$). Let

$$e_C = 1_{S_{p'}(C)} = \sum_{D \in S_{p'}(C)/\sim} 1_D.$$  

If necessary, $e_C$ will be denoted by $e_C^G$. If $H$ is a subgroup of $G$, then

$$\text{Res}_H^G e_C^G = \sum_{D \in (C \cap H)/\sim_H} e_D^H.$$  

**Proposition 2.20.** — Let $C \in G_p'/\sim$. Then $e_C \in \mathcal{O}_p \mathcal{R}(G)$.  

Proof. — Using Brauer’s Theorem, we only need to prove that \( \text{Res}_N^G e_C^G \in \mathcal{O}_p \mathcal{R}(N) \) for every nilpotent subgroup \( N \) of \( G \). By 2.19, this amounts to prove the lemma whenever \( G \) is nilpotent. So we assume that \( G \) is nilpotent. Then \( G = G' \times G_p \), and \( G_p \) and \( G_p' \) are subgroups of \( G \). Moreover, \( C \subseteq G_p' \) and \( S_p(G) = C \times G_p \). If we identify \( K \mathcal{R}(G) \) and \( K \mathcal{R}(G_p') \otimes K K \mathcal{R}(G_p) \), we have \( e_C^G = 1_{C_p'} \otimes \alpha_s e_1^G \). But, by 2.2, we have that \( e_C^G \in \mathcal{O}_p \mathcal{R}(G_p') \). On the other hand, \( e_1^G = 1_{G_p} \in \mathcal{R}(G_p) \). The proof of the lemma is complete.

Corollary 2.21. — Let \( C \in G_p'/\sim \). Then \( e_C \) is a primitive idempotent of \( \mathcal{O}_p \mathcal{R}(G) \).

Proof. — By Proposition 2.15 (a), the number of primitive idempotents of \( k \mathcal{R}(G) \) is \( |G_p'/\sim| \). So the number of primitive idempotents of \( \mathcal{O}_p \mathcal{R}(G) \) is also \( |G_p'/\sim| \) (here, \( \mathcal{O}_p \) denotes the completion of \( \mathcal{O}_p \) at its maximal ideal). Now, \( (e_C)_{C \in G_p'/\sim} \) is a family of orthogonal idempotents of \( \mathcal{O}_p \mathcal{R}(G) \) (see Proposition 2.20) and \( 1_G = \sum_{C \in G_p'/\sim} e_C \). The proof of the lemma is complete.

Let \( e_C \in k \mathcal{R}(G) \) denote the reduction modulo \( p \mathcal{O}_p \) of \( e_C \). Then it follows from 2.12 that

\[
(2.22) \quad d_C^k e_C = 1^k_{S_p'(C)} \in k \mathcal{R}(kG) \simeq \text{Class}_k^p(G).
\]

Let \( P_C = \mathcal{O}_p \mathcal{R}(G) e_C \) and \( \bar{P}_C = k \mathcal{R}(G) e_C \): they are indecomposable projective modules for \( \mathcal{O}_p \mathcal{R}(G) \) and \( k \mathcal{R}(G) \) respectively. Then

\[
\mathcal{O}_p \mathcal{R}(G) = \bigoplus_{C \in G_p'/\sim} P_C
\]

and

\[
k \mathcal{R}(G) = \bigoplus_{C \in G_p'/\sim} \bar{P}_C.
\]

Note also that

\[
(2.23) \quad \dim k \mathcal{R}(G) e_C = \text{rank}_{\mathcal{O}_p} \mathcal{O}_p \mathcal{R}(G) e_C = |S_p'(G)/\sim|.
\]

Proposition 2.24. — Let \( C \) and \( C' \) be two conjugacy classes of \( p' \)-regular elements of \( G \). Then:

(a) \( |\bar{P}_C : \mathcal{D}_C| = \begin{cases} |S_p'(C)/\sim| & \text{if } C = C', \\ 0 & \text{otherwise}. \end{cases} \)

(b) \( \bar{P}_C / \text{Rad} \bar{P}_C \simeq \mathcal{D}_C \).

Proof. — Let us first prove (a). By definition of \( e_C \), we have

\[
[K \otimes_{\mathcal{O}_p} P_C] = \sum_{D \in S_p'(G)/\sim} [P_D].
\]

Also, by definition of the decomposition map \( \delta_p : \mathcal{R}(K \mathcal{R}(G)) \to \mathcal{R}(k \mathcal{R}(G)) \), we have

\[
\delta_p[K \otimes_{\mathcal{O}_p} P_C] = [\mathcal{P}_C].
\]

So the result follows from these observations and from 2.13. Now, (b) follows easily from (a).

2.E. More on the radical. — Let \( \text{Rad}_p(G) \) denote the set of functions \( f \in \mathcal{O}_p \mathcal{R}(G) \) whose restriction to \( G_p' \) is zero. Note that \( \text{Rad}_p(G) \) is a direct summand of the \( \mathcal{O}_p \)-module \( \mathcal{O}_p \mathcal{R}(G) \). So, \( k \text{Rad}_p(G) = k \otimes_{\mathcal{O}_p} \text{Rad}_p(G) \) is a sub-\( k \)-vector space of \( k \mathcal{R}(G) \).

Proposition 2.25. — We have:

(a) \( \dim_k k \text{Rad}_p(G) = |G/\sim| - |G_p'/\sim| \).

(b) \( k \text{Rad}_p(G) \) is the radical of \( k \mathcal{R}(G) \).

Proof. — (a) is clear. (b) follows from 2.12 and from Corollary 2.15.

Corollary 2.26. — Let \( e \) be the number such that \( p^e \) is the exponent of a Sylow \( p \)-subgroup of \( G \). If \( f \in \text{Rad} k \mathcal{R}(G) \), then \( f^{p^e} = 0 \).

Proof. — Let \( e = e_p(G) \). If \( f \in K\mathcal{R}(G) \) and if \( n \geq 1 \), we denote by \( f^{(n)} : G \to K \), \( g \mapsto f(g^n) \). Then the map \( K\mathcal{R}(G) \to K\mathcal{R}(G) \), \( f \mapsto f^{(n)} \) is a morphism of \( K \)-algebras. Moreover (see for instance [CR, Corollary 12.10]), we have

\[
(2.27) \quad \text{If } f \in \mathcal{R}(G), \text{ then } f^{(n)} \in \mathcal{R}(G).
\]

Therefore, it induces a morphism of \( k \)-algebras \( \theta_n : k\mathcal{R}(G) \to k\mathcal{R}(G) \). Now, let \( F : k\mathcal{R}(G) \to k\mathcal{R}(G) \), \( \lambda \otimes z \mapsto \lambda^p \otimes z \). Then \( F \) is an injective endomorphism of the ring \( k\mathcal{R}(G) \). Moreover (see for instance [I, Problem 4.7]), we have

\[
(2.28) \quad F \circ \theta_p(f) = f^p
\]

for every \( f \in k\mathcal{R}(G) \). Since \( F \) and \( \theta_p \) commute, we have \( F^e \circ \theta_p^e(f) = f^{p^e} \) for every \( f \in k\mathcal{R}(G) \). Therefore, if \( \chi \in \text{Rad}_p(G) \), we have

\[
\tilde{\chi}^{p^e} = F^e(\chi^{p^e}).
\]

But, by hypothesis, \( g^{p^e} \in G^{p^e} \) for every \( g \in G \). So, if \( f \in \text{Rad}_p(G) \), then \( f^{(p^e)} = 0 \). Therefore, \( \tilde{f}^{p^e} = 0 \). The corollary follows from this observation and from Proposition 2.25. \( \square \)

3. Principal block

If \( C \in G_{p^e} / \sim \), we denote by \( \mathcal{R}_p(G,C) \) the \( \mathcal{O}_p \)-algebra \( \mathcal{O}_p \mathcal{R}(G) e_C \). As an \( \mathcal{O}_p \mathcal{R}(G) \)-module, this is just \( \mathcal{P} \mathcal{C} \), but we want to study here its structure as a ring, so that is why we use a different notation. If \( R \) is a commutative \( \mathcal{O}_p \)-algebra, we set \( H\mathcal{R}_p(G,C) = R \otimes_{\mathcal{O}_p} \mathcal{R}_p(G,C) \). For instance, \( k\mathcal{R}_p(G,C) \) and \( K\mathcal{R}_p(G,C) \) can be identified with the algebra of class functions on \( S_p(G) \).

The algebra \( \mathcal{R}_p(G,1) \) (resp. \( k\mathcal{R}_p(G,1) \)) will be called the principal block of \( \mathcal{O}_p \mathcal{R}(G) \) (resp. \( k\mathcal{R}(G) \)). The aim of this section is to construct an isomorphism \( \mathcal{R}_p(G,C) \cong \mathcal{R}_p(G,C,G) \), where \( g \) is any element of \( C \). We also emphasize the functorial properties of the principal block.

Remark 3.1 - If \( C \in G_{p^e} / \sim \) and if \( z \in Z(G) \), then \( t_z \) induces an isomorphism of algebras \( \mathcal{R}_p(G,C) \cong \mathcal{R}_p(G,z^{-1}C) \) (see Remark 2.7). Consequently, \( \tilde{t}_z \) induces an isomorphism of algebras \( k\mathcal{R}_p(G,C) \cong k\mathcal{R}_p(G,z^{-1}C) \).

3. A. Centralizers. — Let \( C \in G_{p^e} / \sim \). Let \( \text{proj}^G_p : K\mathcal{R}(G) \to K\mathcal{R}_p(G,C) \), \( x \mapsto x e_C \) denote the canonical projection. We still denote by \( \text{proj}^G_p : \mathcal{O}_p \mathcal{R}(G) \to \mathcal{R}_p(G,C) \), the restriction of \( \text{proj}^G_p \) and we denote by \( \text{proj}^G_p \mathcal{R}(G) \to \mathcal{R}_p(G,C) \) its reduction modulo \( p\mathcal{C} \).

Let us now fix \( g \in C \). It is well-known (and easy) that the map \( C_G(g)_p / \sim_{C_G(g)} \to S_p(G) / \sim_G \) that sends the \( C_G(g) \)-conjugacy class \( D \in C_G(g)_p / \sim_{C_G(g)} \) to the \( G \)-conjugacy class containing \( gD \) is bijective. In particular,

\[
|S_p(G) / \sim_G| = |C_G(g)_p / \sim_{C_G(g)}|.
\]

Now, let \( d^G_g : K\mathcal{R}(G) \to K\mathcal{R}_p(C_G(g)) \) be the map defined by:

\[
d^G_g(f)(h) = \begin{cases} f(gh) & \text{if } h \in C_G(g)_p, \\ 0 & \text{otherwise,} \end{cases}
\]

for all \( f \in K\mathcal{R}(G) \) and \( h \in C_G(g) \). Then \( d^G_g f \in K\mathcal{R}_p(C_G(g),1) \). It must be noticed that

\[
(3.3) \quad d^G_g = \text{proj}^G_{C_G(g)} \circ d^C_g \circ \text{Res}^G_{C_G(g)} = t^{C_G(g)} \circ \text{proj}^G_{C_G(g)} \circ \text{Res}^G_{C_G(g)}.
\]

In particular, \( d^G_g \) sends \( \mathcal{O}_p \mathcal{R}(G) \) to \( \mathcal{R}_p(C_G(g),1) \). We denote by \( \text{res}_g : \mathcal{R}_p(G,C) \to \mathcal{R}_p(C_G(g),1) \) the restriction of \( d^G_g \) to \( \mathcal{R}_p(G,C) \). Let \( \text{ind}_g : K\mathcal{R}_p(C_G(g),1) \to K\mathcal{R}_p(G,C) \) be the map defined by

\[
\text{ind}_g f = \text{Ind}^G_{C_G(g)}(t^{C_G(g)}_g f)
\]
for every $f \in KR_p(C_G(g), 1)$. It is clear that $\text{ind}_g f \in R_p(G, C)$ if $f \in R_p(C_G(g), 1)$. Thus we have defined two maps

$$\text{res}_g : R_p(G, C) \rightarrow R_p(C_G(g), 1)$$

and

$$\text{ind}_g : R_p(C_G(g), 1) \rightarrow R_p(G, C).$$

We have:

**Theorem 3.4.** — If $g \in G_p'$, then $\text{res}_g$ and $\text{ind}_g$ are isomorphisms of $O_p$-algebras inverse to each other.

**Proof.** — We first want to prove that $\text{res}_g \circ \text{ind}_g$ is the identity morphism. Let $f \in KR_p(C_G(g), 1)$. Let $f' = t_{g^{-1}} f$ and let $x \in C_G(g)_p$. We just need to prove that

$$(\text{Ind}^G_{C_G(g)} f')(gx) = f'(gx).$$

But, by definition,

$$(\text{Ind}^G_{C_G(g)} f')(gx) = \sum_{h \in [G/C_G(g)]} f'(h(gx)h^{-1}).$$

Here, $[G/C_G(g)]$ denotes a set of representatives of $G/C_G(g)$. Since $f'$ has support in $gC_G(g)_p$, we have $f'(h(gx)h^{-1}) \neq 0$ only if the $p'$-part of $h(gx)h^{-1}$ is equal to $g$, which happens if and only if $h \in C_G(g)$. This shows (7).

The fact that $\text{ind}_g \circ \text{res}_g$ is the identity can be proved similarly, or can be proved by using a trivial dimension argument. Since $\text{res}_g$ is a morphism of algebras, we get that $\text{ind}_g$ is also a morphism of algebras. 

\[ \square \]

### 3.B. Subgroups of index prime to $p$.

If $H$ is a subgroup of $G$, then the restriction map $\text{Res}^G_H$ sends $R_p(G, 1)$ to $R_p(H, 1)$ (indeed, by 2.19, we have $\text{Res}^G_H e_f^G = e_f^H$).

**Theorem 3.5.** — If $H$ is a subgroup of $G$ of index prime to $p$, then $\text{Res}^G_H : R_p(G, 1) \rightarrow R_p(H, 1)$ is a split injection of $O_p$-modules.

**Proof.** — Let us first prove that $\text{Res}^G_H$ is injective. For this, we only need to prove that the map $\text{Res}^G_H : KR_p(G, 1) \rightarrow KR_p(H, 1)$. But $KR_p(G, 1)$ is the space of functions whose support is contained in $G_p$. Since the index of $H$ is prime to $p$, every conjugacy class of $p$-elements of $G$ meets $H$. This shows that $\text{Res}^G_H$ is injective.

In order to prove that it is a split injection, we only need to prove that the $O_p$-module $R_p(H, 1)/\text{Res}^G_H(R_p(G, 1))$ is torsion-free. Let $\pi$ be a generator of the ideal $pO_p$. Let $\gamma \in R_p(G, 1)$ and $\eta \in R_p(H, 1)$ be such that $\pi \eta = \text{Res}^G_H \gamma$. We only need to prove that $\gamma/\pi \in R_p(G, 1)$. By Brauer’s Theorem, it is sufficient to show that, for any nilpotent subgroup $N$ of $G$, we have $\text{Res}^G_N \gamma \in \pi O_p \mathcal{R}(N)$.

So let $N$ be a nilpotent subgroup. We have $N = N_p \times N_{p'}$ and, since the index of $H$ in $G$ is prime to $p$, we may assume that $N_p \subset H$. Since $\text{Res}^G_N \psi \in R_p(N, 1) = O_p \mathcal{R}(N_p) \otimes_{O_p} e_1^{N_{p'}}$, we have

$$\text{Res}^G_N \gamma = (\text{Res}^G_{N_p} \gamma) \otimes_{O_p} e_1^{N_{p'}}$$

$$= (\pi \text{Res}^H_{N_p} \eta) \otimes_{O_p} e_1^{N_{p'}} \in \pi O_p \mathcal{R}(N),$$

as expected. 

\[ \square \]

**Corollary 3.6.** — If $H$ is a subgroup of $G$ of index prime to $p$, then the map $\overline{\text{Res}}^G_H : kR_p(G, 1) \rightarrow kR_p(H, 1)$ is an injective morphism of $k$-algebras.

**Corollary 3.7.** — If $H$ is a subgroup of $G$ of index prime to $p$ which controls the fusion of $p$-elements, then $\text{Res}^G_H : R_p(G, 1) \rightarrow R_p(H, 1)$ is an isomorphism of $O_p$-algebras.
Proof. — In this case, \( \dim_K K \mathcal{R}_p(G, 1) = \dim_K K \mathcal{R}_p(H, 1) \), so the result follows from Corollary 3.6.

Example 3.8 - Let \( P \) be a Sylow \( p \)-subgroup of \( G \) and assume in this example that \( P \) is abelian. Then \( N_G(P) \) controls the fusion of \( p \)-elements. It then follows from Corollary 3.7 that the restriction from \( G \) to \( N_G(P) \) induces isomorphisms of algebras \( \mathcal{R}_p(G, 1) \cong \mathcal{R}_p(N_G(P), 1) \) and \( k \mathcal{R}_p(G, 1) \cong k \mathcal{R}_p(N_G(P), 1) \). In particular, \( \ell_p(G, 1) = \ell_p(N_G(P), 1) \).

Example 3.9 - Let \( N \) be a \( p' \)-group, let \( H \) be a group acting on \( N \) and let \( G = H \rtimes N \). Then \( H \) is of index prime to \( p \) and controls the fusion of \( p \)-elements of \( G \). So \( \text{Res}_N^G \) induces isomorphisms of algebras \( \mathcal{R}_p(G, 1) \cong \mathcal{R}_p(H, 1) \) and \( k \mathcal{R}_p(G, 1) \cong k \mathcal{R}_p(H, 1) \). In particular, \( \ell_p(G, 1) = \ell_p(H, 1) \).

3.C. Quotient by a normal \( p' \)-subgroup. — Let \( N \) be a normal subgroup of \( G \). Let \( \pi : G \to G/N \) denote the canonical morphism. Then the morphism of algebras \( \text{Res}_\pi : \mathcal{R}_p(G/N) \to \mathcal{R}_p(G) \) induces a morphism of algebras \( \text{Res}_\pi(1) : \mathcal{R}_p(G/N, 1) \to \mathcal{R}_p(G, 1), f \mapsto (\text{Res}_\pi f)e_1^G \). Note that \( \text{Res}_\pi(1)e_1^{G/N} = e_1^G \). We denote by \( \overline{\text{Res}_\pi(1)} : k \mathcal{R}_p(G/N, 1) \to k \mathcal{R}_p(G, 1) \) the morphism induced by \( \text{Res}_\pi(1) \). Then:

**Theorem 3.10.** — With the above notation, we have:

(a) \( \text{Res}_\pi(1) \) is a split injection of \( \mathcal{O}_p \)-modules.
(b) If \( N \) is prime to \( p \), then \( \text{Res}_\pi(1) \) is an isomorphism.

Proof. — (a) The injectivity of \( \text{Res}_\pi(1) \) follows from the fact that \( (G/N)_p = G_p/N/N \). Now, let \( I \) denote the image of \( \text{Res}_\pi(1) \). Since \( \text{Res}_\pi(\mathcal{O}_p \mathcal{R}(G/N)) \) is a direct summand of \( \mathcal{O}_p \mathcal{R}(G) \), we get that \( \text{Res}_\pi(\mathcal{R}_p(G/N, 1)) \) is a direct summand of \( \mathcal{O}_p \mathcal{R}(G) \). Since \( I = e_1^G \text{Res}_\pi(\mathcal{R}_p(G/N, 1)) \) and \( e_1^G = e_1^G \text{Res}_\pi(e_1^{G/N}) \), we get that \( I = e_1^G \text{Res}_\pi(\mathcal{O}_p \mathcal{R}(G/N)) \) is a direct summand of \( \mathcal{O}_p \mathcal{R}(G) \), as desired.

(b) now follows from (a) and from the fact that the map \( \pi \) induces a bijection between \( G_p/\sim_G \) and \( (G/N)_p/\sim_{G/N} \) whenever \( N \) is a normal \( p' \)-subgroup.

4. Some invariants

We introduce in this section some numerical invariants of the \( k \)-algebra \( k \mathcal{R}(G) \) (more precisely, of the algebras \( k \mathcal{R}_p(G, C) \)): Loewy length, dimension of the \( \text{Ext} \)-groups.

4.A. Loewy length. — If \( C \in G_p/\sim \), we denote by \( \ell_p(G, C) \) the Loewy length of the \( k \)-algebra \( k \mathcal{R}_p(G, C) \). Then, by definition, we have

\[
\ell_p(G) = \max_{C \in G_p/\sim} \ell_p(G, C).
\]

On the other hand, by Theorem 3.4, we have

\[
\text{If } C \in G_p/\sim \text{ and if } g \in C, \text{ then } \ell_p(G, C) = \ell_p(C_G(g), 1).
\]

The following bound on the Loewy length of \( k \mathcal{R}(G) \) is obtained immediately from 2.23 and 3.2:

\[
\ell_p(G) \leq \max_{C \in G_p/\sim} |S_p(C)/\sim| = \max_{g \in G_p} |C_G(g)_p/\sim_{C_G(g)}|.
\]

We set \( S_p(G) = \max_{C \in G_p/\sim} |S_p(C)/\sim| \).

Example 4.4 - The inequality 4.3 might be strict. Indeed, if \( G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), then \( \ell_2(G) = 3 < 4 = S_2(G) \).
Example 4.5 - If $S_p(G) = 2$, then $\ell_p(G) = 2$. Indeed, in this case, we have that $p$ divides $|G|$, so $kR(G)$ is not semisimple by Corollary 2.17, so $\ell_p(G) \geq 2$. The result then follows from 4.3.

4.B. Ext-groups. — If $i \geq 0$ and if $C \in G_{p'}/\sim$, we set

$$\text{ext}^i_p(G, C) = \dim_k \text{Ext}_k^i(G, C)(\bar{\mathcal{D}}, \bar{\mathcal{D}}).$$

Note that $\text{ext}^i_p(G, C) = \dim_k \text{Ext}_k^i(kR(G, C)(\bar{\mathcal{D}}, \bar{\mathcal{D}}))$. So, if $g \in C$, it follows from Theorem 3.4 that

$$(4.6) \quad \text{ext}^i_p(G, C) = \text{ext}^i_p(C_G(g), 1).$$

4.C. Subgroups, quotients. — The next results follows respectively from Corollaries 3.6, 3.7 and from Theorem 3.10:

**Proposition 4.7.** — Let $H$ be a subgroup of $G$ of index prime to $p$ and let $N$ be a normal subgroup of $G$.

(a) $\ell_p(G, 1) \leq \ell_p(H, 1)$.
(b) If $H$ controls the fusion of $p$-elements, then $\ell_p(G, 1) = \ell_p(H, 1)$ and $\text{ext}^i_p(G, 1) = \text{ext}^i_p(H, 1)$ for every $i \geq 0$.
(c) $\ell_p(G/N, 1) \leq \ell_p(G, 1)$.
(d) If $|N|$ is prime to $p$, then $\ell_p(G, 1) = \ell_p(H, 1)$ and $\text{ext}^i_p(G, 1) = \text{ext}^i_p(H, 1)$ for every $i \geq 0$.

4.D. Direct products. — We study here the behaviour of the invariants $\ell_p(G, C)$ and $\text{ext}^i_p(G, C)$ with respect to taking direct products. We first recall the following result on finite dimensional algebras:

**Proposition 4.8.** — Let $A$ and $B$ be two finite dimensional $k$-algebras. Then:

(a) $\text{Rad}(A \otimes_k B) = A \otimes_k (\text{Rad} B) + (\text{Rad} A) \otimes_k B$.
(b) If $A/\text{Rad} A \simeq k$ and $B/\text{Rad} B \simeq k$, then

$$\text{Rad}(A \otimes_k B)/\text{Rad}(A \otimes_k B)^2 \simeq (\text{Rad} A)/(\text{Rad} A)^2 \oplus (\text{Rad} B)/(\text{Rad} B)^2.$$

**Proof.** — (a) is proved for instance in [CR, Proof of 10.39]. Let us now prove (b). Let $\theta : (\text{Rad} A) \oplus (\text{Rad} B) \rightarrow \text{Rad}(A \otimes_k B)/\text{Rad}(A \otimes_k B)^2$, $a \oplus b \mapsto \bar{a} \otimes_k 1 + 1 \otimes_k \bar{b}$. By (a), $\theta$ is surjective and $(\text{Rad} A)^2 \oplus (\text{Rad} B)^2$ is contained in the kernel of $\theta$. Now the result follows from dimension reasons (using (a)).

**Proposition 4.9.** — Let $G$ and $H$ be two finite groups and let $C \in G_{p'}/\sim$ and $D \in H_{p'}/\sim$. Then

$$\ell_p(G \times H, C \times D) = \ell_p(G, C) + \ell_p(H, D) - 1$$

and

$$\text{ext}^i_p(G \times H, C \times D) = \text{ext}^i_p(G, C) + \text{ext}^i_p(H, D).$$

**Proof.** — Write $A = kR_p(G, C)$ and $B = kR_p(H, D)$. It is easily checked that $kR_p(G \times H, C \times D) = A \otimes_k B$. So the first equality follows from Propositon 4.8 (a) and from the commutativity of $A$ and $B$. Moreover $A/(\text{Rad} A) \simeq k$ and $B/(\text{Rad} B) \simeq k$. In particular

$$\dim_k \text{Ext}_k^i(A/\text{Rad} A, A/\text{Rad} A) = \dim_k (\text{Rad} A)/(\text{Rad} A)^2.$$

So the second equality follows from Proposition 4.8 (b).
4.E. Abelian groups. — We compute here the invariants $\ell_p(G, 1)$ and $\text{ext}^1_p(G, 1)$ whenever $G$ is abelian. If $G$ is abelian, then there is a (non-canonical) isomorphism of algebras $kR(G) \simeq kG$.

Let us first start with the cyclic case:

(4.10) if $G$ is cyclic, then $\ell_p(G) = |G|_p + 1$ and $\text{ext}^1_p(G, 1) = \begin{cases} 1 & \text{if } p \text{ divides } |G|, \\ 0 & \text{otherwise}. \end{cases}$

Therefore, by Proposition 4.9, we have: if $G_1, \ldots, G_n$ are cyclic, then

(4.11) $\ell_p(G_1 \times \cdots \times G_n) = |G_1|_p + \cdots + |G_n|_p - n + 1.$

and

(4.12) $\text{ext}^1_p(G_1 \times \cdots \times G_n) = \{|1 \leq i \leq n \mid p \text{ divides } G_i\}$.

5. The symmetric group

In this section, and only in this section, we fix a non-zero natural number $n$ and a prime number $p$ and we assume that $G = \mathfrak{S}_n$, that $O = \mathbb{Z}$ and that $p = p\mathbb{Z}$. Let $\overline{F}_p = k$. It is well-known that $\mathbb{Q}$ and $\overline{F}_p$ are splitting fields for $\mathfrak{S}_n$. For simplification, we set $R_n = R(\mathfrak{S}_n)$ and $\overline{R}_n = \overline{F}_p R(\mathfrak{S}_n)$. We investigate further the structure of $\overline{R}_n$. This is a continuation of the work started in [B] in which the description of the descending Loewy series of $\overline{R}_n$ was obtained.

We first introduce some notation. Let $\text{Part}(n)$ denote the set of partitions of $n$. If $\lambda = (\lambda_1, \ldots, \lambda_r) \in \text{Part}(n)$ and if $1 \leq i \leq n$, we denote by $r_i(\lambda)$ the number of occurences of $i$ as a part of $\lambda$. We set

$$\pi_p(\lambda) = \sum_{i=1}^{n} \left[ \frac{r_i(\lambda)}{p} \right]$$

where, for $x \in \mathbb{R}$, $x \geq 0$, we denote by $[x]$ the unique natural number $m \geq 0$ such that $m \leq x < m + 1$. Note that $\pi_p(\lambda) \in \{0, 1, 2, \ldots, [n/p]\}$ and recall that $\lambda$ is $p$-regular (resp. $p$-singular) if and only if $\pi_p(\lambda) = 0$ (resp. $\pi_p(\lambda) \geq 1$). We denote by $\mathfrak{S}_{\lambda}$ the Young subgroup canonically isomorphic to $\mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_r}$, by $1_{\lambda}$ the trivial character of $\mathfrak{S}_{\lambda}$, and by $c_{\lambda}$ an element of $\mathfrak{S}_{\lambda}$ with only $r$ orbit in $\{1, 2, \ldots, n\}$. Let $C_{\lambda}$ denote the conjugacy class of $c_{\lambda}$ in $\mathfrak{S}_n$. Then the map $\text{Part}(n) \rightarrow \mathfrak{S}_n/\sim$, $\lambda \mapsto C_{\lambda}$ is a bijection. Let $W(\lambda) = N_{\mathfrak{S}_n}(\mathfrak{S}_{\lambda})/\mathfrak{S}_{\lambda}$. Then

(5.1) $W(\lambda) \simeq \prod_{i=1}^{n} \mathfrak{S}_{r_i(\lambda)}.$

In particular, $\pi_p(\lambda)$ is the $p$-rank of $W(\lambda)$, where the $p$-rank of a finite group is the maximal rank of an elementary abelian subgroup. Now, we set $\varphi_\lambda = \text{Ind}_{\mathfrak{S}_{\lambda}}^{\mathfrak{S}_n} 1_{\lambda}$. An old result of Frobenius says that

(5.2) $(\varphi_\lambda)_{\lambda \in \text{Part}(n)}$ is a $\mathbb{Z}$-basis of $R_n$

(see for instance [GP, Theorem 5.4.5 (b)]). Now, if $i \geq 1$, let

$$\text{Part}^i_p(n) = \{ \lambda \in \text{Part}(n) \mid \pi_p(\lambda) \geq i \}$$

and

$$\text{Part}^i_p(n) = \{ \lambda \in \text{Part}(n) \mid \pi_p(\lambda) = i \}.$$

Then, by [B, Theorem A], we have

(5.3) $$(\text{Rad } \overline{R}_n)^i = \bigoplus_{\lambda \in \text{Part}^i_p(n)} F_p \varphi_\lambda.$$
Let \( \text{Part}_{p^e}(n) \) denote the set of partitions of \( n \) whose parts are prime to \( p \). Then the map \( \text{Part}_{p^e}(n) \to G_{p^e}/\sim, \lambda \mapsto C_\lambda \) is bijective. We denote by \( \tau_{p^e}(\lambda) \) the unique partition of \( n \) such that \((c_\lambda)_p \in C_{\tau_{p^e}(\lambda)}\). If \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \), the partition \( \tau_{p^e}(\lambda) \) is obtained as follows. Let
\[
\lambda' = ((\lambda_1)_p', \ldots, (\lambda_1)_p', \ldots, (\lambda_r)_p', \ldots, (\lambda_r)_p').
\]
Then \( \tau_{p^e}(\lambda) \) is obtained from \( \lambda' \) by reordering the parts. The map \( \tau_{p^e} : \text{Part}(n) \to \text{Part}_{p^e}(n) \) is obviously surjective. If \( \lambda \in \text{Part}_{p^e}(n) \), we set for simplification \( \mathcal{R}_{n,p}(\lambda) = \mathcal{R}_{p^e}(\mathfrak{S}_n, C_\lambda) \) and \( \mathcal{R}_n(\lambda) = \mathcal{R}_{p^e}(\mathfrak{S}_n, C_\lambda) \). In other words,
\[
\mathcal{Z}_{p\mathbb{Z}} \mathcal{R}_n = \bigoplus_{\lambda \in \text{Part}_{p^e}(n)} \mathcal{R}_{n,p}(\lambda)
\]
and
\[
\mathcal{R}_n = \bigoplus_{\lambda \in \text{Part}_{p^e}(n)} \mathcal{R}_n(\lambda)
\]
are the decomposition of \( \mathcal{Z}_{p\mathbb{Z}} \mathcal{R}_n \) and \( \mathcal{R}_n \) as a sum of blocks. We now make the result 5.3 more precise:

**Proposition 5.4.** — If \( \lambda \in \text{Part}_{p^e}(n) \) and if \( i \geq 0 \), then
\[
\dim_{\mathbb{F}_p} (\text{Rad} \mathcal{R}_n(\lambda))^i = |\tau_{p^e}^{-1}(\lambda) \cap \text{Part}_{p^e}(n)|^i.
\]

**Proof.** — If \( \lambda \) and \( \mu \) are two partitions of \( n \), we write \( \lambda \subset \mu \) if \( \mathfrak{S}_\lambda \) is conjugate to a subgroup of \( \mathfrak{S}_\mu \). This defines an order on \( \text{Part}(n) \). On the other hand, if \( d \in \mathfrak{S}_n \), we denote by \( \lambda \cap d\mu \) the unique partition \( \nu \) of \( n \) such that \( \mathfrak{S}_\lambda \cap d\mathfrak{S}_\mu \) is conjugate to \( \mathfrak{S}_\nu \). Then, by the Mackey formula for tensor product (see for instance [CR, Theorem 10.18]), we have
\[
\varphi_\lambda \otimes \varphi_\mu = \sum_{d \in [\mathfrak{S}_\lambda \cap d\mathfrak{S}_\mu]} \varphi_{\lambda \cap d\mu}.
\]
Here, \([\mathfrak{S}_\lambda / \mathfrak{S}_\mu]\) denotes a set of representatives of the \((\mathfrak{S}_\lambda, \mathfrak{S}_\mu)\)-double cosets in \( \mathfrak{S}_n \). This shows that, if we fix \( \lambda_0 \in \text{Part}(n) \), then \( \bigoplus_{\lambda \subseteq \lambda_0} \mathcal{Z} \varphi_\lambda \) and \( \bigoplus_{\lambda \subseteq \lambda_0} \mathcal{Z} \varphi_\lambda \) are sub-\( \mathcal{R}(G) \)-module of \( \mathcal{R}(G) \). We denote by \( \mathcal{D}_\lambda^{\mathbb{Z}} \) the quotient of these two modules. Then
\[
K \otimes_{\mathbb{Z}} \mathcal{D}_\lambda^{\mathbb{Z}} \simeq \mathcal{D}_C, \lambda.
\]
This follows for instance from [GP, Proposition 2.4.4]. Consequently,
\[
k \otimes_{\mathbb{Z}} \mathcal{D}_\lambda^{\mathbb{Z}} \simeq \mathcal{D}_C, \lambda.
\]
It then follows from Proposition 2.14 that
\[
k \otimes_{\mathbb{Z}} \mathcal{D}_\mu^{\mathbb{Z}} \simeq k \otimes_{\mathbb{Z}} \mathcal{D}_\mu^{\mathbb{Z}} \quad \text{if and only if} \quad \tau_{p^e}(\lambda) = \tau_{p^e}(\mu).
\]
Now the Theorem follows from easily from (3), (4) and 5.3. \( \square \)

Now, if \( \lambda \in \text{Part}_{p^e}(n) \), then \( C_{\phi_n}(\psi_\lambda) \) contains a normal \( p^e \)-subgroup \( N_\lambda \) such that \( C_{\phi_n}(w_\lambda)/N_\lambda \) is isomorphic to \( W(\lambda) \). We denote by \( 1^n \) the partition \((1, 1, \ldots, 1)\) of \( n \). It follows from Theorem 3.4 and Theorem 3.10 that
\[
\mathcal{R}_{n,p}(\lambda) \simeq \mathcal{R}_{p^e}(W(\lambda), 1) \simeq \bigoplus_{i=1}^n \mathcal{R}_{\tau_{p^e}(\lambda), p}(1^{r_i(\lambda)})
\]
and
\[
\mathcal{R}_n(\lambda) \simeq \mathcal{R}(W(\lambda), 1) \simeq \bigoplus_{i=1}^n \mathcal{R}_{\tau(\lambda), 1}(1^{r_i(\lambda)}).
\]
We denote by \( \text{Log}_{p^e} n \) the real number \( x \) such that \( p^x = n \). Then:

**Corollary 5.7.** — If \( \lambda \in \text{Part}_{p^e}(n) \), then
\[
\text{ext}_{\mathbb{F}_p}(\mathfrak{S}_n, C_\lambda) \simeq \sum_{i=1}^n \text{Log}_{p^e} r_i(\lambda)
\]
and
\[
\ell_{p}(\mathfrak{S}_n, C_\lambda) = \pi_{p}(\lambda) + 1.
\]
Proof. — By 5.6 and by Proposition 4.9, both equalities need only to be proved whenever \( \lambda = (1^n) \).
So we assume that \( \lambda = (1^n) \).

Let us show the first equality. By Proposition 5.4, we are reduced to show that \( |\tau_{p'}^{-1}(1^n) \cap \text{Part}_p^1(n)| = [\text{Log}_p n] \). Let \( r = [\text{Log}_p n] \). In other words, we have \( p^r \leq n < p^{r+1} \).
If \( 1 \leq i \leq r \), write \( n - p^i = \sum_{j=0}^{r} a_{ij} p^j \) with \( 0 \leq a_{ij} < p - 1 \) (the \( a_{ij}'s \) are uniquely determined). Let
\[
\lambda(i) = (p^i, \ldots, p^i, p^{i-1}, \ldots, p^{i-1}, p^{i-2}, \ldots, 1, \ldots, 1).
\]
The result will follow from the following equality
\[
(\ast)
\]
So let us now prove (\ast\). Let \( I = \{\lambda(1), \lambda(2), \ldots, \lambda(r)\} \). It is clear that \( I \subset \tau_{p'}^{-1}(1^n) \cap \text{Part}_p^1(n) \).

Now, let \( \lambda \in \tau_{p'}^{-1}(1^n) \cap \text{Part}_p^1(n) \). Then there exists a unique \( i \in \{1, 2, \ldots, r\} \) such that \( r_{p'}^{-1}(\lambda) \geq p \).
Moreover, \( r_{p'}^{-1}(\lambda) < 2p \). So, if we set \( r_{p'} = r_{p'}(\lambda) \) if \( j \neq i \) and \( r_{p'}^{-1} = r_{p'}^{-1}(\lambda) - p \), we get that \( 0 \leq r_{p'} \leq p - 1 \) and \( n - p^i = \sum_{j=0}^{r} r_{p'} p^j \). This shows that \( \tau_{p'} = a_{ij}, \) so \( \lambda = \lambda(i) \).

Let us now show the second equality by the Corollary. By Proposition 5.4, we only need to show that \( |\tau_{p'}^{-1}(1^n) \cap \text{Part}_p^{[n/p]}(n)| \geq 1 \). But in fact, it is clear that \( \tau_{p'}^{-1}(1^n) \cap \text{Part}_p^{[n/p]}(n) = \{1^n\} \).

**Corollary 5.8.** — We have
\[
\dim_p (\text{Rad} \mathcal{R}_p(n))^{|n/p|} = 1
\]
and
\[
\dim_p \text{Ext}_p^1 \mathcal{R}_p(D_1^n, D_1^n) = [\text{Log}_p n].
\]
In particular, \( \ell_p(\mathfrak{G}_n, 1) = \ell_p(\mathcal{R}_p) = |n/p| \).

**Proof.** — This is just a particular case of the previous corollary. The first equality has been obtained in the course of the proof of the previous corollary.

6. Dihedral groups

Let \( n \geq 1 \) and \( m \geq 0 \) be two natural numbers. We assume in this section, and only in this subsection, that \( G = D_{2^n(2m+1)} \) is the dihedral group of order \( 2^n(2m+1) \) and that \( p = 2 \).

**Proposition 6.1.** — If \( n \geq 1 \) and \( m \geq 0 \) are natural numbers, then
\[
\ell_2(D_{2^n(2m+1)}, 1) = \begin{cases} 
2 & \text{if } n = 1, \\
3 & \text{if } n = 2, \\
2^n - 1 & \text{if } n \geq 3.
\end{cases}
\]
and
\[
\text{ext}_2^1(D_{2^n(2m+1)}, 1) = \begin{cases} 
1 & \text{if } n = 1, \\
2 & \text{if } n = 2, \\
3 & \text{if } n \geq 3.
\end{cases}
\]

**Proof.** — Let \( N \) be the normal subgroup of \( G \) of order \( 2m+1 \). Then \( G \simeq D_{2m+1} \times N \). So, by Proposition 4.7 (d), we may, and we will, assume that \( m = 0 \). If \( n = 1 \) or 2 the result is easily checked. Therefore, we may, and we will, assume that \( n \geq 3 \).

Write \( h = 2^{n-1} \). We have
\[
G = \langle s, t \mid s^2 = t^2 = (st)^h = 1 \rangle.
\]
Let \( H = \langle s, t \rangle \) and \( S = \langle s \rangle \). Then \( |H| = 2^{n-1} = h \) and \( G = S \rtimes H \). We fix a primitive \( h \)-th root of unity \( \xi \in \mathbb{C}^\times \). If \( i \in \mathbb{Z} \), we denote by \( \xi_i \) the unique linear character of \( H \) such that \( \xi_i(st) = \xi^i \).
Then \( \text{Irr} H = \{\xi_0, \xi_1, \ldots, \xi_{h-1}\} \), and \( \xi_0 = 1_H \).
Since \( n \geq 3 \), \( h \) is even and, if we write \( h = 2h' \), then \( h' = 2^{n-2} \) is also even. For \( i \in \mathbb{Z} \), we set
\[
\chi_i = \text{Ind}^G_H \xi_i.
\]
It is readily seen that \( \chi_1 = \chi_{-i} \), that \( \chi_{i+h} = \chi_i \) and that
\[
(6.2) \quad \chi_i \chi_j = \chi_{i+j} + \chi_{i-j}.
\]
Let \( \varepsilon \) (resp. \( \varepsilon_s \), resp. \( \varepsilon_t \)) be the unique linear character of order 2 such that \( \varepsilon(st) = 1 \) (resp. \( \varepsilon_s(s) = 1 \), resp. \( \varepsilon_t(t) = 1 \)). Then
\[
\begin{align*}
\chi_0 &= 1_G + \varepsilon, \\
\chi_h' &= \varepsilon_s + \varepsilon_t,
\end{align*}
\]
and, if \( h' \) does not divide \( i \),
\[
\chi_i \in \text{Irr} G.
\]
Moreover, \( |\text{Irr} G| = h' + 3 \) and
\[
\text{Irr} G = \{1_G, \varepsilon, \varepsilon_s, \varepsilon_t, \chi_1, \chi_2, \ldots, \chi_{h'-1}\}.
\]
Finally, note that
\[
(6.3) \quad \varepsilon_s \chi_i = \varepsilon_t \chi_i = \chi_{i+h'}.
\]
Let us start by finding a lower bound for \( \ell_2(G) \). First, notice that the following equality holds: for all \( i, j \in \mathbb{Z} \) and every \( r \geq 0 \), we have
\[
(6.4) \quad (\bar{\chi}_i + \bar{\chi}_j)^{2^r} = \bar{\chi}_{2^r i} + \bar{\chi}_{2^r j}.
\]

Proof of 6.4. — Recall that \( \bar{\chi}_i \) denotes the image of \( \chi_i \) in \( k\mathcal{R}(G) \). We proceed by induction on \( r \). The case \( r = 0 \) is trivial. The induction step is an immediate consequence of 6.2. \( \square \)

Note also the following fact (which follows from Example 2.18):
\[
(6.5) \quad \text{If } i \in \mathbb{Z}, \text{ then } \bar{\chi}_i \in \text{Rad} k\mathcal{R}(G).
\]
Therefore,
\[
(6.6) \quad \ell_2(G) \geq 2^{n-2} + 1.
\]

Proof of 6.6. — By 6.4, we have immediately that \( (\bar{\chi}_0 + \bar{\chi}_1)^{2^{n-2}} = \bar{\chi}_0 + \bar{\chi}_{h'} \neq 0 \)
and, by 6.5, \( \bar{\chi}_0 + \bar{\chi}_1 \in \text{Rad} k\mathcal{R}(G) \). \( \square \)

By Example 2.18, we have
\[
(6.7) \quad (\bar{1}_G + \bar{\varepsilon}_s, \bar{\chi}_0, \bar{\chi}_1, \ldots, \bar{\chi}_{h'}) \text{ is a } k\text{-basis of } \text{Rad} k\mathcal{R}(G).
\]
By 6.3 and 6.2, we get that
\[
(6.8) \quad (\bar{\chi}_i + \bar{\chi}_{i+2})_{0 \leq i \leq h'-2} \text{ is a } k\text{-basis of } (\text{Rad} k\mathcal{R}(G))^2.
\]
This shows that \( \text{ext}_p^k(G) = 3 \), as expected. It follows that, if \( n \geq 3 \) and \( 2 \leq i \leq 2^{n-2} + 1 \), then
\[
(6.9) \quad \dim_k (\text{Rad} k\mathcal{R}(D_{2^n}))^i = 2^{n-2} + 1 - i
\]

Proof of 6.9. — Let \( d_i = \dim_k (\text{Rad} k\mathcal{R}(D_{2^n}))^i \). By 6.8, we have \( d_2 = 2^{n-2} - 1 \).
By 6.6, we have \( d_{2^n-2} \geq 1 \). Moreover, \( d_1 > d_2 > d_3 > \ldots \) So the proof of 6.9 is complete. \( \square \)

In particular, we get:
\[
(6.10) \quad \text{If } n \geq 3, \text{ then } (\text{Rad} k\mathcal{R}(D_{2^n}))^{2^{n-2}} = k(\bar{1}_{D_{2^n}} + \varepsilon + \bar{\varepsilon}_s + \bar{\varepsilon}_t),
\]
and \( \ell_2(D_{2^n}) = 2^{n-2} + 1 \), as expected. \( \square \)
7. Some tables

For $0 \leq i \leq \ell_p(G) - 1$, we set $d_i = \dim_k(\text{Rad} kR(G))^i$. Note that $d_0 = |G/\sim|$ and $d_0 - d_1 = |G_P/\sim|$. In this section, we give tables containing the values $\ell_p(G)$, $\ell_p(G,1)$, $S_p(G)$, $\text{ext}_p^1(G,1)$ and the sequence $(d_0, d_1, d_2, \ldots)$ for various groups. These computations have been made using GAP3 [GAP3].

These computations show that, if $G$ satisfies at least one of the following conditions:

1. $|G| \leq 200$;
2. $G$ is a subgroup of $S_8$;
3. $G$ is one of the groups contained in the next tables;

then $\ell_p(G,1) = \ell_p(N_G(P),1)$ (here, $P$ denotes a Sylow $p$-subgroup of $G$). Note also that this equality holds if $P$ is abelian (see Example 3.8).

**Question.** Is it true that $\ell_p(G,1) = \ell_p(N_G(P),1)$?

The first table contains the data for the the exceptional Weyl groups, the second table is for the alternating groups $A_n$ for $5 \leq n \leq 12$, the third table is for some small finite simple groups, and the last table is for the groups $PSL(2,q)$ for $q$ a prime power $\leq 27$.

| $G$     | $|G|$  | $p$ | $\ell_p(G)$ | $S_p(G)$ | $d_0, d_1, d_2, \ldots$ | $\ell_p(G,1)$ | $\text{ext}_p^1(G,1)$ |
|---------|-------|-----|-------------|----------|--------------------------|--------------|------------------------|
| $W(E_6)$ | 51840 | 2   | 5           | 10       | 25, 19, 9, 3, 1          | 5            | 3                      |
|         | 2^7.3^4.5 | 3   | 4           | 5        | 25, 13, 4, 1            | 4            | 2                      |
|         |       | 5   | 2           | 2        | 25, 2                   | 2            | 1                      |
| $W(E_7)$ | 2903040 | 2   | 7           | 24       | 60, 52, 35, 18, 7, 3, 1 | 7            | 4                      |
|         | 2^10.3^4.5.7 | 3   | 4           | 5        | 60, 30, 8, 2            | 4            | 2                      |
|         |       | 5   | 2           | 2        | 60, 6                   | 2            | 1                      |
|         |       | 7   | 2           | 2        | 60, 2                   | 2            | 1                      |
| $W(E_8)$ | 696729600 | 2   | 8           | 32       | 112, 100, 68, 36, 17, 7, 3, 1 | 8            | 5                      |
|         | 2^11.3^5.5^2.7 | 3   | 5           | 8        | 112, 65, 24, 7, 2       | 5            | 2                      |
|         |       | 5   | 3           | 3        | 112, 17, 2              | 3            | 1                      |
|         |       | 7   | 2           | 2        | 112, 4                  | 2            | 1                      |
| $W(F_4)$ | 1152 | 2   | 5           | 14       | 25, 21, 12, 4, 1        | 5            | 4                      |
|         | 2^7.3^2 | 3   | 3           | 4        | 25, 11, 2               | 3            | 2                      |
| $W(H_3)$ | 120 | 2   | 3           | 4        | 10, 6, 1                | 3            | 2                      |
|         | 2^3.3.5 | 3   | 2           | 2        | 10, 2                   | 2            | 1                      |
|         |       | 5   | 3           | 3        | 10, 4, 2                | 3            | 1                      |
| $W(H_4)$ | 14400 | 2   | 4           | 7        | 34, 24, 9, 1            | 4            | 3                      |
|         | 2^6.3^2.5^2 | 3   | 3           | 3        | 34, 11, 2               | 3            | 1                      |
|         |       | 5   | 5           | 6        | 34, 20, 11, 4, 2        | 5            | 2                      |
| $G$ | $|G|$ | $p$ | $\ell_p(G)$ | $S_p(G)$ | $d_0, d_1, d_2, \ldots$ | $\ell_p(G, 1)$ | $\text{ext}_p^1(G, 1)$ |
|-----|-----|-----|-------------|--------|-----------------|-------------|----------------|
| $A_5$ | 60  | 2   | 2           | 2      | 5, 1            | 2           | 1             |
|      | $2^2.3.5$ | 3   | 2           | 2      | 5, 1            | 2           | 1             |
|      |      | 5   | 3           | 3      | 5, 2, 1         | 3           | 1             |
| $A_6$ | 360 | 2   | 3           | 3      | 7, 2, 1         | 3           | 1             |
|      | $2^3.3^2.5$ | 3   | 3           | 3      | 7, 2, 1         | 3           | 1             |
|      |      | 5   | 3           | 3      | 7, 2, 1         | 3           | 1             |
| $A_7$ | 2520 | 2   | 3           | 3      | 9, 3, 1         | 3           | 1             |
|      | $2^3.3^2.5.7$ | 5   | 2           | 2      | 9, 1           | 2           | 1             |
|      |      | 7   | 3           | 3      | 9, 2, 1         | 3           | 1             |
| $A_8$ | 20160 | 2   | 4           | 5      | 14, 6, 2, 1     | 4           | 2             |
|      | $2^6.3^2.5.7$ | 3   | 3           | 3      | 14, 6, 2       | 3           | 1             |
|      |      | 5   | 3           | 3      | 14, 3, 1       | 2           | 1             |
|      |      | 7   | 3           | 3      | 14, 2, 1       | 3           | 1             |
| $A_9$ | 181440 | 2   | 4           | 5      | 18, 8, 3, 1    | 4           | 2             |
|      | $2^6.3^4.5.7$ | 3   | 4           | 6      | 18, 10, 3, 1   | 4           | 3             |
|      |      | 5   | 3           | 3      | 18, 4, 1       | 2           | 1             |
|      |      | 7   | 3           | 3      | 14, 2, 1       | 3           | 1             |
| $A_{10}$ | 1814400 | 2   | 5           | 7      | 24, 12, 6, 2, 1 | 5           | 2             |
|      | $2^7.3^4.5^2.7$ | 3   | 4           | 6      | 24, 13, 4, 1   | 4           | 3             |
|      |      | 5   | 3           | 3      | 24, 4, 1       | 3           | 1             |
|      |      | 7   | 3           | 3      | 24, 3, 1       | 2           | 1             |
| $A_{11}$ | 19958400 | 2   | 5           | 7      | 31, 17, 8, 3, 1 | 5           | 2             |
|      | $2^7.3^4.5^2.7.11$ | 3   | 4           | 5      | 31, 16, 6, 1   | 4           | 2             |
|      |      | 5   | 3           | 3      | 31, 6, 1       | 3           | 1             |
|      |      | 7   | 3           | 3      | 31, 4, 1       | 2           | 1             |
|      |      | 11  | 3           | 3      | 31, 2, 1       | 3           | 1             |
| $A_{12}$ | 239500800 | 2   | 6           | 10     | 43, 25, 13, 6, 2, 1 | 6           | 2             |
|      | $2^9.3^6.5^2.7.11$ | 3   | 5           | 8      | 43, 22, 9, 2, 1 | 5           | 3             |
|      |      | 5   | 3           | 3      | 43, 10, 2      | 3           | 1             |
|      |      | 7   | 3           | 3      | 43, 5, 1       | 2           | 1             |
|      |      | 11  | 3           | 3      | 43, 2, 1       | 3           | 1             |
| \( G \) | \(| G |\) | \( p \) | \( \ell_p(G) \) | \( S_p(G) \) | \( d_0, d_1, d_2, \ldots \) | \( \ell_p(G, 1) \) | \( \text{ext}_p^1(G, 1) \) |
| --- | --- | --- | --- | --- | --- | --- | --- |
| \( GL(3, 2)\) | 168 | 2 | 3 | 3 | 6, 2, 1 | 3 | 1 |
| & \( 2^3 \cdot 3.7 \) | & 3 | 2 | 2 | 6, 1 | 2 | 1 |
| & \( 7 \) | & 3 | 3 | 6, 2, 1 | 3 | 1 |
| \( SL(2, 8) \) | 504 | 2 | 2 | 2 | 9, 1 | 2 | 1 |
| & \( 2^3 \cdot 3^2.7 \) | & 3 | 5 | 5 | 9, 4, 3, 2, 1 | 5 | 1 |
| & \( 7 \) | & 4 | 4 | 9, 3, 2, 1 | 4 | 1 |
| \( SL(3, 3) \) | 5616 | 2 | 5 | 5 | 12, 5, 3, 2, 1 | 5 | 1 |
| & \( 2^4 \cdot 3^3.13 \) | & 3 | 3 | 3 | 12, 3, 1 | 3 | 1 |
| & \( 13 \) | & 5 | 5 | 12, 4, 3, 2, 1 | 5 | 1 |
| \( SU(3, 3) \) | 6048 | 2 | 6 | 7 | 14, 9, 6, 4, 2, 1 | 6 | 2 |
| & \( 2^6 \cdot 3^3.7 \) | & 3 | 3 | 3 | 14, 5, 1 | 3 | 1 |
| & \( 7 \) & 3 | 3 | 14, 2, 1 | 3 | 1 |
| \( M_{11} \) | 7920 | 2 | 5 | 5 | 10, 5, 3, 2, 1 | 5 | 1 |
| & \( 2^4 \cdot 3^2.5.11 \) | & 3 | 2 | 2 | 10, 2 | 2 | 1 |
| & \( 5 \) | & 2 | 2 | 10, 1 | 2 | 1 |
| & \( 11 \) | & 3 | 3 | 10, 2, 1 | 3 | 1 |
| \( PSp(4, 3) \) | 25920 | 2 | 4 | 5 | 20, 12, 5, 1 | 4 | 2 |
| & \( 2^6 \cdot 3^4.5 \) | & 3 | 5 | 7 | 20, 14, 8, 3, 1 | 5 | 2 |
| & \( 5 \) | & 2 | 2 | 20 | 2 | 1 |
| \( M_{12} \) | 95040 | 2 | 4 | 7 | 15, 9, 3, 1 | 4 | 3 |
| & \( 2^6 \cdot 3^3.5.11 \) | & 3 | 3 | 3 | 15, 4, 1 | 3 | 1 |
| & \( 5 \) | & 2 | 2 | 15, 2 | 2 | 1 |
| & \( 11 \) | & 3 | 3 | 15, 2, 1 | 3 | 1 |
| \( J_1 \) | 175560 | 2 | 2 | 2 | 15, 4 | 2 | 1 |
| & \( 2^5 \cdot 3^3.5.7.11.19 \) | & 3 | 2 | 2 | 15, 4 | 2 | 1 |
| & \( 5 \) | & 3 | 3 | 15, 6, 3 | 3 | 1 |
| & \( 7 \) | & 2 | 2 | 15, 1 | 2 | 1 |
| & \( 11 \) | & 2 | 2 | 15, 1 | 2 | 1 |
| & \( 19 \) | & 4 | 4 | 15, 3, 2, 1 | 4 | 1 |
| \( M_{22} \) | 443520 | 2 | 4 | 5 | 12, 5, 2, 1 | 4 | 2 |
| & \( 2^7 \cdot 3^3.5.7.11 \) | & 3 | 2 | 2 | 12, 2 | 2 | 1 |
| & \( 5 \) | & 2 | 2 | 12, 1 | 2 | 1 |
| & \( 7 \) | & 3 | 3 | 12, 2, 1 | 3 | 1 |
| & \( 11 \) | & 3 | 3 | 12, 2, 1 | 3 | 1 |
| \( J_2 \) | 604800 | 2 | 4 | 5 | 21, 11, 3, 1 | 4 | 2 |
| & \( 2^7 \cdot 3^3.5^2.7 \) | & 3 | 3 | 3 | 21, 7, 1, | 3 | 1 |
| & \( 5 \) | & 5 | 5 | 21, 10, 6, 2, 1 | 5 | 1 |
| & \( 7 \) | & 2 | 2 | 21 | 2 | 1 |
| \( HS \) | 44352000 | 2 | 5 | 9 | 24, 15, 8, 3, 1 | 5 | 3 |
| & \( 2^9 \cdot 3^3.5^3.7.11 \) | & 3 | 2 | 2 | 24, 5 | 2 | 1 |
| & \( 5 \) | & 3 | 4 | 24, 8, 2 | 3 | 2 |
| & \( 7 \) | & 2 | 2 | 24, 1 | 2 | 1 |
| & \( 11 \) | & 3 | 3 | 24, 2, 1 | 3 | 1 |
| $G$ | $|G|$ | $p$ | $\ell_p(G)$ | $S_p(G)$ | $d_0, d_1, d_2, \ldots$ | $\ell_p(G, 1)$ | $\text{ext}_p^1(G, 1)$ |
|-----|-----|-----|------------|--------|----------------|-------------|------------------|
| $\text{PSL}(2, 2)$ | 6 | 2 | 2 | 2 | 3, 1 | 2 | 1 |
| $\simeq \mathfrak{S}_3$ | 2.3 | 3 | 2 | 2 | 3, 1 | 2 | 1 |
| $\text{PSL}(2, 3)$ | 12 | 2 | 2 | 2 | 4, 1 | 2 | 1 |
| $\simeq \mathfrak{A}_4$ | 2.3.3 | 3 | 3 | 3 | 4, 2, 1 | 3 | 1 |
| $\text{PSL}(2, 4)$ | 60 | 2 | 2 | 2 | 5, 1 | 2 | 1 |
| $\simeq \text{PSL}(2, 5)$ | 2.3.5 | 3 | 2 | 2 | 5, 1 | 2 | 1 |
| $\simeq \mathfrak{A}_5$ | 5 | 3 | 3 | 5, 2, 1 | 3 | 1 |
| $\text{PSL}(2, 7)$ | 168 | 2 | 3 | 3 | 6, 2, 1 | 3 | 1 |
| | $2.3.7$ | 3 | 2 | 2 | 6, 1 | 2 | 1 |
| | 7 | 3 | 3 | 6, 2, 1 | 3 | 1 |
| $\text{PSL}(2, 8)$ | 504 | 2 | 2 | 2 | 9, 1 | 2 | 1 |
| | $2.3.7.7$ | 3 | 5 | 5 | 9, 4, 3, 2, 1 | 5 | 1 |
| | 7 | 4 | 4 | 9, 3, 2, 1 | 4 | 1 |
| $\text{PSL}(2, 9)$ | 360 | 2 | 3 | 3 | 7, 2, 1 | 3 | 1 |
| $\simeq \mathfrak{A}_6$ | 2.3.5.7 | 3 | 3 | 3 | 7, 2, 1 | 3 | 1 |
| | 5 | 3 | 3 | 7, 2, 1 | 3 | 1 |
| $\text{PSL}(2, 11)$ | 660 | 2 | 2 | 2 | 8, 2 | 2 | 1 |
| | $2.3.5.11$ | 3 | 2 | 2 | 8, 2 | 2 | 1 |
| | 5 | 3 | 3 | 8, 2, 1 | 3 | 1 |
| | 11 | 3 | 3 | 8, 2, 1 | 3 | 1 |
| $\text{PSL}(2, 13)$ | 1092 | 2 | 2 | 2 | 9, 2 | 2 | 1 |
| | $2.3.7.13$ | 3 | 2 | 2 | 9, 2 | 2 | 1 |
| | 7 | 4 | 4 | 9, 3, 2, 1 | 4 | 1 |
| | 13 | 3 | 3 | 9, 2, 1 | 3 | 1 |
| $\text{PSL}(2, 16)$ | 4080 | 2 | 2 | 2 | 17, 1 | 2 | 1 |
| | $2.3.5.17$ | 3 | 3 | 3 | 17, 5, 2 | 2 | 1 |
| | 5 | 5 | 5 | 17, 6, 4, 2, 1 | 3 | 1 |
| | 17 | 9 | 9 | 17, 8, 7, 6, 5, 4, 3, 2, 1 | 9 | 1 |
| $\text{PSL}(2, 17)$ | 2448 | 2 | 5 | 5 | 11, 4, 3, 2, 1 | 5 | 1 |
| | $2.3.5.17$ | 3 | 5 | 5 | 11, 4, 3, 2, 1 | 5 | 1 |
| | 17 | 3 | 3 | 11, 2, 1 | 3 | 1 |
| $\text{PSL}(2, 19)$ | 3420 | 2 | 2 | 2 | 12, 3 | 2 | 1 |
| | $2.3.5.19$ | 3 | 5 | 5 | 12, 4, 3, 2, 1 | 5 | 1 |
| | 5 | 3 | 3 | 12, 4, 2 | 3 | 1 |
| | 19 | 3 | 3 | 12, 2, 1 | 3 | 1 |
| $\text{PSL}(2, 23)$ | 6072 | 2 | 4 | 4 | 14, 5, 3, 1 | 3 | 1 |
| | $2.3.11.23$ | 3 | 3 | 3 | 14, 4, 1 | 2 | 1 |
| | 11 | 6 | 6 | 14, 5, 4, 3, 2, 1 | 6 | 1 |
| | 23 | 3 | 3 | 14, 2, 1 | 3 | 1 |
| $\text{PSL}(2, 25)$ | 7800 | 2 | 4 | 4 | 15, 5, 3, 1 | 3 | 1 |
| | $2.3.5.13$ | 3 | 3 | 3 | 15, 4, 1 | 2 | 1 |
| | 5 | 3 | 3 | 15, 2, 1 | 3 | 1 |
| | 13 | 7 | 7 | 15, 6, 5, 4, 3, 2, 1 | 7 | 1 |
| $\text{PSL}(2, 27)$ | 9828 | 2 | 2 | 2 | 16, 4 | 2 | 1 |
| | $2.3.7.13$ | 3 | 3 | 3 | 16, 2, 1 | 3 | 1 |
| | 7 | 4 | 4 | 16, 6, 4, 2 | 4 | 1 |
| | 13 | 7 | 7 | 16, 6, 5, 4, 3, 2, 1 | 7 | 1 |
References


November 8, 2006

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