EQUIVARIANT L-FUNCTIONS AT $s = 0$ AND $s = 1$

by

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Abstract. — For an abelian extensions of number fields, we review some basic theory and formulate the Stark Conjecture in terms of the 'equivariant' L-function at $s = 0$. After surveying the known cases, we describe some refinements and extensions due to Rubin, Brumer *et al.* and results concerning Fitting ideals of class groups. Finally, we summarise some recent work on minus parts at $s = 1$.

Résumé **(Les Fonctions L Équivariantes en $s = 0$ et en $s = 1$).** — Après quelques rappels, nous énonçons la Conjecture de Stark pour une extension abélienne de corps de nombres, formulée en termes de la fonction L 'équivariante' en $s = 0$. Nous survolons les cas connus et expliquons certaines conjectures plus fines dues à Rubin, Brumer *et al.* ainsi que quelques résultats concernant l'idéal de Fitting du groupe des classes. Enfin, nous résumons certains travaux récents concernant les parties moins en $s = 1$.

1. Introduction

This article is is an expanded version of the notes from four lectures given by the author at the conference 'Fonctions L et Arithmétique' in Besançon, in June 2009. It surveys work on several different conjectures concerning the special values of L-functions attached to characters of Galois extensions K/k of number fields.

In our presentation – and largely in historical fact – the development of such conjectures begins with the seminal work of Stark in [**St**]. We shall, however, consider only the case in which $G = \mathrm{Gal}(K/k)$ is *abelian*, for which the theory is currently richest. In this context, we shall work with the *equivariant* L-function $\Theta_S = \Theta_{S,K/k}(s)$ attached to K/k and a set S of places of k subject to certain conditions. This simply assembles the usual S-truncated L-functions for all irreducible characters of G into a single function taking values in the complex group-ring $\mathbb{C}[G]$. Consequently, the conjectures can be formulated in terms of certain elements and ideals of group-rings $\mathcal{R}[G]$ and 'arithmetic' $\mathcal{R}[G]$-modules attached to K *etc.* Here, $\mathcal{R}$

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is a commutative ring, variously $\mathbb{Q}$ (the rationals), $\mathbb{Z}$ (the integers), $\mathbb{Q}_p$ or $\mathbb{Z}_p$ (the p-adic rationals or integers, for a prime p). The commutativity of $\mathcal{R}[G]$ allows us to use the algebra of determinants, annihilators, Fitting ideals *etc.*

In Sections 2 and 3 we briefly review the definitions and theory of the L-functions concerned, starting with those attached to ray-class characters of k, moving on to characters of G via class-field theory and hence to the equivariant L-function Θ_S mentioned above. More details of the basic theory can be found in standard books on Algebraic Number Theory such as [**La**]. See also [**Ta**, Ch. 0] or [**Ma**] for some of the more advanced facts. In Section 4 we motivate and then state the Stark's basic conjecture in the abelian case, in a formulation due to Rubin. This concerns the leading Taylor coefficient of $\Theta_S(s)$ *at the point* $s = 0$. (Despite the title, the latter half of [**St**] also focusses on $s = 0$, as does the majority of subsequent work.) Section 5 briefly reviews the current state of research on the basic abelian conjecture and its 'integral' refinements. Following on from the latter, Section 6 explains the link – via the Brumer-Stark Conjecture – with Brumer's conjecture on the annihilation of (the minus-part of) the class group of K in the case where K is CM and k totally real. The latter is a conjectural generalisation of Stickelberger's Theorem. Section 7 tells the story of recent work attempting to refine the Brumer Conjecture using Fitting ideals of class groups, much of it due to Greither and Kurihara. The last two sections deal with recent work of the author concerning the minus-part of $\Theta_S(s)$ *at the point* $s = 1$. The lack of a suitable 'equivariant functional equation' means that there is no simple logical connection with the above-mentioned work at $s = 0$. Instead, a fundamental role is played by a certain p-adic logarithmic map $\mathfrak{s}_p$ which is introduced in Section 8. Finally, in Section 9 we explain two conjectures made in [**So1**] and [**So2**]: the Integrality Conjecture concerning the image $\mathfrak{S}_p$ of $\mathfrak{s}_p$ in $\mathbb{Q}_p[G]$ and the Congruence Conjecture. The latter is a sort of conjectural explicit reciprocity law that makes a link with the Stark Conjecture in the plus-part at $s = 0$. We also pose a rather more tentative 'Question' which aims to relate $\mathfrak{S}_p$ to the issues discussed in Section 7.

This survey will suit readers with little previous knowledge of the subject but leaves much out. In particular, we shall not touch on the analogous conjectures at integer values of s different from 0 and 1, nor on Serre's or Gross' p-adic conjectures or their extensions. The function-field case and that of non-abelian G are also hardly mentioned. For more detailed and/or extensive accounts the reader may consult the sources cited in the text or the four earlier survey articles in [**BPSS**] by Dummitt, Flach, Greither and Popescu.

1.1. Basic Notations and Conventions. — In addition to the notations already introduced, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ will denote the natural, real and complex numbers respectively. We shall denote by $\bar{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ and also fix an algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ for each prime number p. A 'number field' L is always a finite extension of $\mathbb{Q}$ within $\bar{\mathbb{Q}}$. Its abelian closure in $\bar{\mathbb{Q}}$ is denoted L^{ab}. Let F be any field, $l \in \mathbb{N}$ and p a prime number. We shall write $\mu(F)$ (*resp.* $\mu_l(F)$, *resp.* $\mu_{p^\infty}(F)$) for the group of all roots of unity (*resp.* all lth roots of unity, *resp.* all p-power roots of unity) in F. When F is omitted it is understood to be $\bar{\mathbb{Q}}$. If

L is a number field, we shall write W_L for the cardinality $|\mu(L)|$ of $\mu(L)$. Finally, we shall use the notation χ_0 for the trivial character of any finite abelian group.

2. Ray-Class L-Functions

2.1. Basic Definitions. — Let k be a number field with ring of integers $\mathcal{O}_k$, r_1 real places and r_2 complex places so that $r_1 + 2r_2 = n := [k : \mathbb{Q}]$. For our purposes, a *cycle* for k will be a formal product $\mathfrak{f} = \mathfrak{f}_0\mathfrak{f}_\infty$ where $\mathfrak{f}_0$ is any non-zero ideal of $\mathcal{O}_k$ and $\mathfrak{f}_\infty$ is the formal product of any subset of the real places of k. Thus $\mathfrak{f}$ can be written uniquely as an infinite product with only finitely many non-trivial terms:

$$\mathfrak{f} = \mathfrak{f}_0\mathfrak{f}_\infty = \prod_{\mathfrak{p}} \mathfrak{p}^{n_\mathfrak{p}(\mathfrak{f})} \prod_{v} v^{n_v(\mathfrak{f})}$$

Here, $\mathfrak{p}$ runs through the set of all non-zero, prime ideals of $\mathcal{O}_k$ and $n_\mathfrak{p}(\mathfrak{f}) = \mathrm{ord}_\mathfrak{p}(\mathfrak{f}_0) \in \mathbb{Z}_{\geq 0}$. Thus $\mathfrak{p}|\mathfrak{f}_0$ if and only if $n_\mathfrak{p}(\mathfrak{f}) \geq 1$. Similarly, v runs through real places of k and $n_v(\mathfrak{f}) \in \{0, 1\}$. By analogy, '$v|\mathfrak{f}_\infty$' will indicate $n_v(\mathfrak{f}) = 1$. Let $I(k)$ denote the group of (non-zero) fractional ideals of k under multiplication and $P(k)$ its subgroup of principal fractional ideals. To each cycle $\mathfrak{f}$ for k there corresponds the subgroup $I_\mathfrak{f}(k)$ of $I(k)$ consisting of those fractional ideals prime to $\mathfrak{f}_0$, and a subgroup $P_\mathfrak{f}(k)$ of $P(k)$ consisting of the principal ideals possessing a generator *'congruent to* 1 *mod* $\mathfrak{f}$*'*:

$$P_\mathfrak{f}(k) := \{(\alpha) \,:\, \alpha \in k^\times,\ \mathrm{ord}_\mathfrak{p}(\alpha - 1) \geq n_\mathfrak{p}\ \forall\, \mathfrak{p}|\mathfrak{f}_0,\ \iota_v(\alpha) > 0\ \forall\, v|\mathfrak{f}_\infty\}$$

where $\iota_v : k \to \mathbb{R}$ is the embedding corresponding to the real place v. Clearly, $P_\mathfrak{f}(k)$ is contained in $I_\mathfrak{f}(k)$ and the quotient $I_\mathfrak{f}(k)/P_\mathfrak{f}(k)$ is, by definition, the *ray-class group* $\mathrm{Cl}_\mathfrak{f}(k)$ *of k modulo* $\mathfrak{f}$. (We shall write $[\mathfrak{a}]_\mathfrak{f}$ for the image in $\mathrm{Cl}_\mathfrak{f}(k)$ of any $\mathfrak{a} \in I_\mathfrak{f}(k)$.) It is finite and, of course, abelian, so its characters may be identified with homomorphisms $\chi : I_f(k) \longrightarrow \mu(\mathbb{C})$ such that $\chi((\alpha)) = 1$ for all $(\alpha) \in P_\mathfrak{f}(k)$. To any such $\mathfrak{f}$ and χ we associate a *ray-class L-function* $L_\mathfrak{f}(s, \chi)$, initially defined on the set $\{s \in \mathbb{C} : \Re(s) > 1\}$ by

(1) $$L_\mathfrak{f}(s, \chi) := \sum_{\substack{\mathfrak{a} \lhd \mathcal{O}_k \\ \mathfrak{a} \in I_\mathfrak{f}(k)}} \chi(\mathfrak{a}) N\mathfrak{a}^{-s} = \prod_{\substack{\mathfrak{p}\ \text{prime} \\ \mathfrak{p} \nmid \mathfrak{f}_0}} \left(1 - \chi(\mathfrak{p}) N\mathfrak{p}^{-s}\right)^{-1}$$

(By standard comparisons, the above sum and Euler product converge absolutely to the same analytic function on this set.)

Example 2.1. — ***The Dedekind Zeta-Function.*** Suppose $\mathfrak{f}$ is trivial *i.e.* $\mathfrak{f}_0 = \mathcal{O}_k$ and $\mathfrak{f}_\infty$ is the empty product, so that $\mathrm{Cl}_\mathfrak{f}(k) = \mathrm{Cl}(k)$, the class group. If also χ is the trivial character χ_0 of $\mathrm{Cl}(k)$, then (1) gives $L_\mathfrak{f}(s, \chi) = \sum_{\mathfrak{a} \lhd \mathcal{O}_k,\ \mathfrak{a} \neq (0)} N\mathfrak{a}^{-s}$ which coincides with the Dedekind zeta function $\zeta_k(s)$ of k.

Example 2.2. — ***Ray-Class L-functions for $k = \mathbb{Q}$.*** Take $\mathfrak{f}$ to be $(f) = f\mathbb{Z}$ for some $f \in \mathbb{Z}_{\geq}1$ and $\mathfrak{f}_\infty$ to be ∞, the unique real place of $\mathbb{Q}$. There is then an isomorphism from $(\mathbb{Z}/f\mathbb{Z})^\times$ to $\mathrm{Cl}_\mathfrak{f}(\mathbb{Q})$ sending $\bar{a}$ to $[(a)]_\mathfrak{f}$, where a is any *positive* integer prime to f. Thus a character χ of $\mathrm{Cl}_{\mathfrak{f}}(\mathbb{Q})$ coincides with a Dirichlet character modulo f and one checks easily from (1) that $L_{\mathfrak{f}}(s,\chi)$ is just the corresponding Dirichlet L-function.

2.2. Primitivity and the Functional Equation. — We shall say that one cycle $\mathfrak{g}$ divides another $\mathfrak{f}$ (written $\mathfrak{g}|\mathfrak{f}$) iff $n_{\mathfrak{p}}(\mathfrak{g}) \leq n_{\mathfrak{p}}(\mathfrak{f})$ for all $\mathfrak{p}$ (*i.e.* $\mathfrak{g}_0|\mathfrak{f}_0$) and $n_v(\mathfrak{g}) \leq n_v(\mathfrak{f})$ for all v. In this case $I_{\mathfrak{f}}(k) \subset I_{\mathfrak{g}}(k)$ and $P_{\mathfrak{f}}(k) \subset P_{\mathfrak{g}}(k)$ so there is a homomorphism

$$\begin{array}{rcccc} \pi_{\mathfrak{f},\mathfrak{g}} & : & \mathrm{Cl}_{\mathfrak{f}}(k) & \longrightarrow & \mathrm{Cl}_{\mathfrak{g}}(k) \\ & & [\mathfrak{a}]_{\mathfrak{f}} & \longmapsto & [\mathfrak{a}]_{\mathfrak{g}} \end{array}$$

Using weak approximation one can show firstly that $\pi_{\mathfrak{f},\mathfrak{g}}$ is surjective and secondly that if $\mathfrak{h}$ also divides $\mathfrak{f}$ then $\ker(\pi_{\mathfrak{f},\mathfrak{g}})\ker(\pi_{\mathfrak{f},\mathfrak{h}}) = \ker(\pi_{\mathfrak{f},(\mathfrak{g},\mathfrak{h})})$. (Here $(\mathfrak{g},\mathfrak{h})$ denotes the cycle that is the h.c.f. of $\mathfrak{g}$ and $\mathfrak{h}$ in the obvious sense). It follows that there exists a unique minimal cycle w.r.t. divisibility, say $\mathfrak{f}_\chi$, such that χ factors through $\pi_{\mathfrak{f},\mathfrak{f}_\chi}$ *i.e.* such that there exists a character $\hat{\chi}$ of $\mathrm{Cl}_{\mathfrak{f}_\chi}(k)$ with $\chi = \hat{\chi} \circ \pi_{\mathfrak{f},\mathfrak{f}_\chi}$. We shall call $\mathfrak{f}_\chi$ the *conductor* of χ and $\hat{\chi}$ the *primitive character* associated to χ.

Remark 2.3. — ***The Idelic Viewpoint.*** Let $\mathrm{Id}(k)$ be the idèle group of k and $C(k) := \mathrm{Id}(k)/k^\times$ the *idèle-class group*. For each cycle $\mathfrak{f}$ one can use weak approximation to define a (surjective) homomorphism $C(k) \to \mathrm{Cl}_{\mathfrak{f}}(k)$. Thus each ray-class character χ modulo $\mathfrak{f}$ gives rise to an idèle-class character that is *continuous* and *of finite order*. All such characters arise in this way. Moreover χ_1 and χ_2 give rise to the same idèle-class character if and only if $\widehat{\chi_1} = \widehat{\chi_2}$.

If χ is *primitive* (*i.e.* $\mathfrak{f}_\chi = \mathfrak{f}$, so $\hat{\chi} = \chi$) we shall write simply $L(s,\chi)$ for the 'primitive L-function' $L_{\mathfrak{f}}(s,\chi)$. If χ is *imprimitive* then $L_{\mathfrak{f}}(s,\chi)$ and $L(s,\hat{\chi})$ differ at most by finitely may Euler factors. More precisely, one clearly has

$$L_{\mathfrak{f}}(s,\chi) = \left(\prod_{\substack{\mathfrak{p}|\mathfrak{f} \\ \mathfrak{p}\nmid\mathfrak{f}_\chi}} \left(1 - \hat{\chi}(\mathfrak{p}) N\mathfrak{p}^{-s}\right) \right) L(s,\hat{\chi}). \tag{2}$$

So suppose χ is a *primitive* ray-class character modulo $\mathfrak{f}$. We summarise the well-known 'analytic continuation' and 'functional equation' for $L(s,\chi)$. Firstly, $L(s,\chi)$ extends to a meromorphic function on $\mathbb{C}$. This is analytic at s except possibly when $s = 1$ where it has a simple pole iff $\chi = \chi_0$ *i.e.*

$$\mathrm{ord}_{s=1}(L_{\mathfrak{f}}(s,\chi)) = -\delta_{\chi,\chi_0} \tag{3}$$

in Kronecker's notation. (It follows easily from (2) that exactly the same is true of $L_{\mathfrak{f}}(s,\chi)$, even when χ is imprimitive.) Secondly, let $a_1(\chi)$ (*resp.* $a_2(\chi)$) denote the number of real places v such that $v \nmid \mathfrak{f}_\infty$ (*resp.* $v|\mathfrak{f}_\infty$) so that $a_1(\chi) + a_2(\chi) = r_1$ and define a *completed L-function*

$$\Lambda(s,\chi) := (|d_k| N\mathfrak{f}_0)^{s/2} 2^{r_2(1-s)} \pi^{-(ns+a_2(\chi))/2} \Gamma(s/2)^{a_1(\chi)} \Gamma((1+s)/2)^{a_2(\chi)} \Gamma(s)^{r_2} L(s,\chi)$$

where d_k denotes the discriminant of k. (The Γ-factors can be considered as Euler factors at infinite places.) Then we have an identity of meromorphic functions:

$$\Lambda(1-s,\chi) = \frac{i^{-a_2(\chi)}\tau(\chi^{-1})}{(N\mathfrak{f}_0)^{1/2}}\Lambda(s,\chi^{-1}) \tag{4}$$

where χ^{-1} denotes the inverse character of χ (which has the same conductor) and $\tau(\chi)$ denotes the Gauss sum (see *e.g.* [**Ma**]. Note that $|\tau(\chi)|^2 = N\mathfrak{f}_0 = (-1)^{a_2(\chi)}\tau(\chi)\tau(\chi^{-1})$.)

3. The Galois Viewpoint

3.1. Set-Up. — Suppose now that K is a finite, Galois extension of the number field k such that $G := \mathrm{Gal}(K/k)$ is *abelian*. Class-field theory associates to K/k a cycle $\mathfrak{f} = \mathfrak{f}_{K/k}$ with the following properties:

(i) A real place v of k divides $\mathfrak{f}_\infty$ iff one (hence any) place above v in K is complex.

(ii) A non-zero prime ideal $\mathfrak{p}$ of $\mathcal{O}_k$ divides $\mathfrak{f}_0$ iff $\mathfrak{p}$ ramifies in K.

(iii) There is a well-defined homomorphism (the *Artin homomorphism*)

$$\mathrm{Cl}_{\mathfrak{f}}(k) \longrightarrow G$$

sending $[\mathfrak{p}]_{\mathfrak{f}}$ to the Frobenius element of G at $\mathfrak{p}$ for each non-zero prime ideal $\mathfrak{p} \nmid \mathfrak{f}_0$.

Note that $\mathfrak{f}_{K/k}$ is the unique minimal cycle for k for which the description in (iii) gives a well-defined homomorphism $\mathrm{Cl}_{\mathfrak{f}}(k) \to G$. It is then unique and surjective and the image of $[\mathfrak{a}]_{\mathfrak{f}}$ (for any ideal $\mathfrak{a}$ prime to $\mathfrak{f}_0$) will be denoted $\sigma_{\mathfrak{a}}$.

Let $\hat{G}$ denote the set of all complex irreducible characters of G, *i.e.* all homomorphisms $\chi : G \to \mathbb{C}^\times$. Composing any such χ with the Artin homomorphism gives rise to a ray-class character $\mathrm{Cl}_{\mathfrak{f}_{K/k}}(k) \to \mathbb{C}^\times$, also denoted χ. Thus $\mathfrak{f}_\chi$ divides $\mathfrak{f}_{K/k}$ and the two cycles are equal iff χ is primitive mod $\mathfrak{f}_{K/k}$. (In fact $\mathfrak{f}_{K/k}$ is always the l.c.m. of the set $\{\mathfrak{f}_\chi : \chi \in \hat{G}\}$.) We have

$$L_{\mathfrak{f}_{K/k}}(s,\chi) := \prod_{\substack{\mathfrak{p}\ \text{prime}\\ \mathfrak{p}\nmid\mathfrak{f}_{K/k,0}}} \left(1-\chi(\sigma_{\mathfrak{p}})N\mathfrak{p}^{-s}\right)^{-1} = \prod_{\mathfrak{p}\notin S_{\mathrm{ram}}} \left(1-\chi(\sigma_{\mathfrak{p}})N\mathfrak{p}^{-s}\right)^{-1}$$

where $S_{\mathrm{ram}} = S_{\mathrm{ram}}(K/k)$ denotes the set of (finite) ramified primes in K/k. It is sometimes convenient to remove further Euler factors from the R.H.S. above. We denote by $S_\infty = S_\infty(k)$ the set of all infinite places of k and by $S_{\min} = S_{\min}(K/k)$ the set $S_{\mathrm{ram}} \cup S_\infty$. For any finite set S of places containing $S_{\min}$ and any $\chi \in \hat{G}$, we define the S-*truncated* L-function to be

$$L_S(s,\chi) := \prod_{\mathfrak{p}\notin S} \left(1-\chi(\sigma_{\mathfrak{p}})N\mathfrak{p}^{-s}\right)^{-1} = \left(\prod_{\substack{\mathfrak{p}\in S\\ \mathfrak{p}\nmid\mathfrak{f}_\chi}} \left(1-\hat{\chi}(\mathfrak{p})N\mathfrak{p}^{-s}\right)\right) L(s,\hat{\chi}) \tag{5}$$

which clearly has a meromorphic continuation to $\mathbb{C}$. Expanding the first product gives

$$L_S(s,\chi) = \sum_{\substack{\mathfrak{a}\lhd\mathcal{O}_k\\ (\mathfrak{a},S)=1}} \chi(\sigma_{\mathfrak{a}})N\mathfrak{a}^{-s} = \sum_{g\in G} \chi(g)\zeta_S(s,g) \quad \text{for } \Re(s) > 1 \tag{6}$$

where $(\mathfrak{a}, S)=1$ indicates that the ideal $\mathfrak{a}$ is prime to every $\mathfrak{p}\in S$ and the *partial zeta-function* $\zeta_S(s,g)$ is the Dirichlet series $\sum N\mathfrak{a}^{-s}$ where $\mathfrak{a}$ ranges through all such integral ideals with $\sigma_{\mathfrak{a}}=g$. As before, it is convergent for $\Re(s)>1$ for any $g\in G$.

Remark 3.1. — **Artin L-Functions.** We consider briefly the more general situation where K/k is Galois but $G=\mathrm{Gal}(K/k)$ is not necessarily abelian, where χ is the character of a d-dimensional complex representation $\rho: G\to \mathrm{GL}(V)$ (but possibly $d>1$) and where S is any set of places of k containing S_∞ (but not necessarily S_{ram}). In this set-up the *Artin* L-function may be defined for $\Re(s)>1$ by generalising the second member of (5):

$$L_{S,\mathrm{Artin}}(s,\chi) := \prod_{\mathfrak{p}\notin S} \det(1-N\mathfrak{p}^{-s}A_{\mathfrak{P}})^{-1}$$

Here, $\mathfrak{P}$ is any prime of K above $\mathfrak{p}$ with inertia group $T_{\mathfrak{P}}\subset G$ say, and $A_{\mathfrak{P}}$ denotes the endomorphism of $V^{\rho(T_{\mathfrak{P}})}$ induced by $\rho(\mathrm{Frob}_{\mathfrak{P}}(K/k))$ (the latter being defined only up to an element of $\rho(T_{\mathfrak{P}})$). If G is abelian and $d=1$ then $\rho=\chi\in\hat{G}$ and it is not hard to show that $L_{S,\mathrm{Artin}}(s,\chi)$ agrees with the third member in (5) (which, of course, makes sense even if $S_{\mathrm{ram}}\not\subset S$). If G is non-abelian, ρ does not in general give rise to-ray class characters over k. However, Brauer induction and the formal properties of Artin L-functions allow us to re-express them in terms of ray-class L-functions for extensions over various intermediate fields k' with $K\supset k'\supset k$. For more details of Artin L-functions, the properties they enjoy and for Stark's conjectures in the nonabelian case, we refer to [**Ta**, Chs. 0,1].

3.2. The Equivariant L-Function. — The rest of this article will be concerned exclusively with the case G abelian and $S\supset S_{\min}$. We can now give three equivalent definitions our basic object of study, the *S-truncated equivariant L-function* $\Theta_S(s)=\Theta_{S,K/k}(s)$. Firstly, for K/k, G and S as above we set

$$\Theta_S(s)=\sum_{g\in G}\zeta_S(s,g)g^{-1} \tag{7}$$

This is *a priori* a function on $\{s\in\mathbb{C}:\Re(s)>1\}$ with values in the group-ring $\mathbb{C}[G]$. Each $\chi\in\hat{G}$ extends $\mathbb{C}$-linearly to a ring homomorphism $\chi:\mathbb{C}[G]\to\mathbb{C}$ and (6) gives

$$\chi(\Theta_S(s))=L_S(s,\chi^{-1})\quad \text{for every } \chi\in\hat{G}. \tag{8}$$

Character theory implies that equations (8) determine $\Theta_S(s)$ uniquely so it follows from (5) that $\Theta_S(s)$ could also have been defined by the Euler product (in $\mathbb{C}[G]$)

$$\Theta_S(s)=\prod_{\mathfrak{p}\notin S}\left(1-\sigma_{\mathfrak{p}}^{-1}N\mathfrak{p}^{-s}\right)^{-1} \tag{9}$$

which makes sense and converges for $\Re(s)>1$. Finally, we can invert equations (8) explicitly to write $\Theta_S(s)$ in terms of L-functions. Let e_χ be the idempotent of $\mathbb{C}[G]$ associated to $\chi\in\hat{G}$, *i.e.* $e_\chi := \frac{1}{|G|}\sum_{g\in G}\chi(g)g^{-1}$. Character theory and (5) give

$$\Theta_S(s) = \sum_{\chi\in\hat{G}} L_S(s,\chi^{-1})e_\chi = \sum_{\chi\in\hat{G}} \Bigg(\prod_{\substack{\mathfrak{p}\in S\\ \mathfrak{p}\nmid\mathfrak{f}_\chi}} \left(1-\hat{\chi}^{-1}(\mathfrak{p})N\mathfrak{p}^{-s}\right)\Bigg) L(s,\hat{\chi}^{-1})e_\chi \tag{10}$$

The properties of $L(s,\hat{\chi})$ now show that Θ_S extends to a meromorphic function on $\mathbb{C}$ that is analytic except at $s=1$ and satisfies (10). The functions $\zeta_S(s,g)$ for $g\in G$ therefore possess similar extensions and equation (3) shows that they all have simple poles at $s=1$ with the same residue, namely $1/|G|$ times that of $L_S(s,\chi_0)$. Note also that $\Theta_S(s)$ is $\mathbb{R}[G]$-valued for $s\in\mathbb{R}_{>1}$, by (9). It follows from the meromorphic continuation that it restricts to an $\mathbb{R}[G]$-valued, analytic function on $\mathbb{R}\setminus\{1\}$.

We note two important 'functorial' properties of Θ_S which follow easily from (9), properties of the Frobenius and analytic continuation. First, if S' is a finite set of places containing S then, clearly

$$\Theta_{S',K/k}(s) = \prod_{\substack{\mathfrak{p}\in S'\\ \mathfrak{p}\notin S}} \left(1-\sigma_{\mathfrak{p}}^{-1}N\mathfrak{p}^{-s}\right)\Theta_{S,K/k} \tag{11}$$

Secondly, let K' be any intermediate field with $K\supset K'\supset k$ and let $\pi_{K/K'}:\mathbb{C}[G]\to\mathbb{C}[\mathrm{Gal}(K'/k)]$ be the natural ring homomorphism induced by the restriction homomorphism $G\to\mathrm{Gal}(K'/k)$. Then

$$\Theta_{S,K'/k} = \pi_{K/K'}\circ\Theta_{S,K/k} \tag{12}$$

as meromorphic functions on $\mathbb{C}$.

Finally, we point out that there are at least two significant obstacles to obtaining a natural 'functional equation' for $\Theta_S(s)$ by combining (10) with (4). Firstly, the Gauss sums in (4) depend on χ. Secondly, the parenthesised *'imprimitivity factor'* on the R.H.S. of (10) not only depends on χ but may vanish at $s=0$ for certain χ. A rather complicated 'functional equation' may nevertheless be constructed along lines suggested in [**So1**, Rem. 2.3(iii)] by involving also $\Theta_{T,K'/k}(s)$ for certain intermediate fields K' as above and subsets T of S.

4. Θ_S at $s=0$ and Stark's Conjecture

4.1. Motivation. — Given K/k and S as above and a certain integer $r\geq 0$ depending on K/k, S, we shall give a formulation of the Basic Abelian Stark Conjecture concerning the rth Taylor coefficient of $\Theta_S(s)$ at $s=0$. This is very similar to that of Conjecture A' in [**Ru**] and may be motivated by consideration of three elementary examples.

Example 4.1. — **Cyclotomic Fields.** We consider the case $k=\mathbb{Q}$ and $K=\mathbb{Q}(\zeta_f)$ where $\zeta_f := \exp(2\pi i/f)$ for some integer $f>2$. We may assume w.l.o.g. that $f\not\equiv 2 \pmod 4$ so that $\mathfrak{f}_{\mathbb{Q}(\zeta_f)/\mathbb{Q}} = \mathfrak{f} := f\mathbb{Z}\infty$ and and $S_{\min}(\mathbb{Q}(\zeta_f)/\mathbb{Q})$ equals $S_f := \{p \text{ prime} : p|f\}\cup\{\infty\}$ which we take for S. Composing the isomorphism $(\mathbb{Z}/f\mathbb{Z})^\times \to \mathrm{Cl}_{\mathfrak{f}}(\mathbb{Q})$ of Example 2.2 with the Artin isomorphism gives the usual isomorphism from $(\mathbb{Z}/f\mathbb{Z})^\times$ to $G=G_f := \mathrm{Gal}(\mathbb{Q}(\zeta_f)/\mathbb{Q})$ sending

$\bar{a}$ to g_a where $g_a(\zeta_f) = \zeta_f^a$ for any integer a with $(a, f) = 1$, and if also $a > 0$ then $\sigma_{a\mathbb{Z}} = g_a$. It follows that

$$\zeta_{S_f}(s, g_a) = \sum_{\substack{n \geq 1 \\ n \equiv a \bmod f}} n^{-s} = f^s \zeta(s, a/f) \quad \text{for } 0 < a \leq f,\ (a, f) = 1 \text{ and } \Re(s) > 1$$

where $\zeta(s, a/f)$ donotes the Hurwitz zeta-function (see [**Wa**, Ch. 4]). The computation of $\zeta(0, a/f)$ in Theorem 4.2 of *loc. cit.* therefore gives

$$\Theta_{S_f, \mathbb{Q}(\zeta_f)/\mathbb{Q}}(0) = -\sum_{\substack{1 \leq a < f \\ (a,f)=1}} \left(\frac{a}{f} - \frac{1}{2}\right) g_a^{-1} \in \mathbb{Q}[G_f] \tag{13}$$

Example 4.2. — **Real Cyclotomic Fields.** Take $k = \mathbb{Q}$, f and other notation as above but now let $K = \mathbb{Q}(\zeta_f)^+ = \mathbb{Q}(\zeta_f + \zeta_f^{-1})$, the maximal real subfield of $\mathbb{Q}(\zeta_f)$. We take S to be S_f as before. (Note that $S_f \supset S_{\min}(\mathbb{Q}(\zeta_f)^+/\mathbb{Q})$ with equality unless $f = 3$ or 4 *i.e.* $\mathbb{Q}(\zeta_f)^+ = \mathbb{Q}$.) Since $\mathrm{Gal}(\mathbb{Q}(\zeta_f)/\mathbb{Q}(\zeta_f)^+) = \{1, g_{-1}\}$ we have an isomorphism from $(\mathbb{Z}/f\mathbb{Z})^\times/\{\pm\bar{1}\}$ to $G = G_f^+ = \mathrm{Gal}(\mathbb{Q}(\zeta_f)^+/\mathbb{Q})$. Thus the map $\pi_{\mathbb{Q}(\zeta_f), \mathbb{Q}(\zeta_f)^+}$ of (12) sends both g_a and $g_{f-a} \neq g_a$ to the same element $\bar{g}_a$, say, of G_f^+. It follows easily from (13) that $\Theta_{S_f, \mathbb{Q}(\zeta_f)^+/\mathbb{Q}}(0) = 0$, so we can write

$$\Theta_{S_f, \mathbb{Q}(\zeta_f)^+/\mathbb{Q}}(s) = \Theta'_{S_f, \mathbb{Q}(\zeta_f)^+/\mathbb{Q}}(0)s + O(s^2) \quad \text{as } s \to 0 \tag{14}$$

where $\Theta'_{S_f, \mathbb{Q}(\zeta_f)^+/\mathbb{Q}}(0) \in \mathbb{R}[G]$. In fact, we have (see *e.g.* p. 203, paper IV of [**St**])

$$\begin{aligned} \Theta'_{S_f, \mathbb{Q}(\zeta_f)^+/\mathbb{Q}}(0) &= -\frac{1}{2} \sum_{\substack{1 \leq a < f/2 \\ (a,f)=1}} \log |((1 - \zeta_f^a)(1 - \zeta_f^{-a}))| \bar{g}_a^{-1} \\ &= -\frac{1}{2} \sum_{g \in G_f^+} \log |g(\varepsilon_f)| g^{-1} \end{aligned} \tag{15}$$

where $\varepsilon_f := (1 - \zeta_f)(1 - \zeta_f^{-1}) \in \mathbb{Q}(\zeta_f)^{+,\times}$. It is well known that ε_f is a local unit at finite places not dividing f (in fact at all finite places unless f is a prime power).

We remark that, thanks to (8) and (5), equations (13) and (15) can also be established character-by-character, using the corresponding formulae for $L(s, \hat{\chi})$ at $s = 0$, where $\hat{\chi}$ is an odd or even primitive Dirichlet character of conductor $\mathfrak{f}$ dividing $f\mathbb{Z}\infty$ (*cf.* Example 2.2). There are, however, complications when $\mathfrak{f}_0$ properly divides $f\mathbb{Z}$.

Example 4.3. — **The Case** $K = k$**.** In this case $\mathbb{C}[G]$ identifies with $\mathbb{C}$ and for any S containing S_∞, equation (9) gives

$$\Theta_{k/k,S}(s) = \prod_{\mathfrak{p} \in S \setminus S_\infty} (1 - N\mathfrak{p}^{-s}) \zeta_k(s)$$

Starting from the Analytic Class Number formula for $\zeta_k(s)$ at $s = 1$ and the functional equation, it follows (*cf* [**Ta**, Cor. I.2.2]) that

$$\Theta_{k/k,S}(s) = -\frac{h_{S,k}R_{S,k}}{W_k}s^{|S|-1} + O(s^{|S|}) \quad \text{as } s \to 0 \tag{16}$$

where $h_{S,k}$ is the cardinality of the S-class group $\mathrm{Cl}_S(k)$ of k and $R_{S,k}$ is the S-regulator of k, defined as follows. Let $U_S(k)$ be S-unit group of k (namely the elements of $k^\times$ which are local units at all finite places not in S). By Dirichlet's Theorem, we can choose a $\mathbb{Z}$-basis $\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_{|S|-1}$ of $U_S(k)/\mu(k)$ and if we choose also any $|S|-1$ places $v_1, \ldots, v_{|S|-1}$ in S, then $R_{S,k} := \left|\det\left(\log(||\varepsilon_i||_{v_j})\right)_{i,j=1}^{|S|-1}\right| \neq 0$, where $||\cdot||_{v_j}$ denotes the normalised absolute value at v_j. Moreover $R_{S,k}$ is easily seen to be independent of the choices and ordering of the ε_i and the v_j.

Notice that in each of the above examples, there is an integer $r \geq 0$ such that Θ_S vanishes to order at least r at $s = 0$ and, moreover, the coefficient of s^r in the Taylor series (denoted $\Theta_S^{(r)}(0)$ for simplicity) is a $\mathbb{Q}[G]$-multiple of an $r \times r$ determinant of 'G-equivariant logarithms' of S-units of K (a phrase to be made precise below). Indeed, we can take r to be the precise order of vanishing in each example – namely 0, 1 and $|S|-1$ respectively – provided we adopt the usual convention that the 0×0 determinant equals 1. Stark's Conjecture is a precise generalisation of this observation. To formulate it, we first need to calculate $r_S(\chi) := \mathrm{ord}_{s=0}(L_S(s,\chi))$ for each character $\chi \in \hat{G}$. Using (5), the functional equation (4), the definition of $\Lambda(s,\chi)$, properties of $\Gamma(s)$ and (3), we find

$$r_S(\chi) = |\{\mathfrak{p} \in S : \mathfrak{p} \nmid \mathfrak{f}_\chi, \hat{\chi}(\mathfrak{p}) = 1\}| + a_1(\hat{\chi}) + r_2 - \delta_{\chi,\chi_0} \tag{17}$$

This can be restated more elegantly as follows. For any place v of k, finite or infinite, we write D_v for the *decomposition subgroup* of G at v, thus

$$D_v := \begin{cases} D_{\mathfrak{p}}(K/k) & \text{if } v \text{ corresponds to a non-zero prime ideal } \mathfrak{p} \text{ of } \mathcal{O}_k, \\ \{1, c_v\} & \text{if } v|\mathfrak{f}_{\infty,K/k} \text{ is real with complex conjugation } c_v \in G, \text{ and} \\ \{1\} & \text{otherwise.} \end{cases}$$

We also say that v *splits* in K iff $D_v = \{1\}$. Using equation (17) one shows easily:

Proposition 4.4. — *If $\chi \in \hat{G}$ then $r_S(\chi) = |\{v \in S : D_v \subset \ker(\chi)\}| - \delta_{\chi,\chi_0}$. In particular $r_S(\chi) = r_S(\chi')$ whenever χ and χ' have the same kernel (*i.e. *they are Galois-conjugate over $\mathbb{Q}$)* e.g. *if $\chi' = \chi^{-1}$.*

4.2. The Conjecture. — Let K/k and S be as above and let $r \in \mathbb{Z}_{\geq 0}$. Consider

Hypothesis $H(K/k,S,r)$. — The following conditions are satisfied:

(i) There exist r distinct places $v_1, \ldots, v_r \in S$ which split in K.
(ii) $|S| \geq r + 1$.

It is clear from Proposition 4.4 that $H(K/k, S, r)$ implies $r_S(\chi) \geq r$ for every $\chi \in \hat{G}$ and hence, by equation (10), that there exists $\Theta^{(r)}_{S,K/k}(0) \in \mathbb{R}[G]$ (unique) such that

$$\Theta_{S,K/k}(s) = \Theta^{(r)}_{S,K/k}(0)s^r + O(s^{r+1}) \quad \text{as } s \to 0 \tag{18}$$

For any place w of K we define the above-mentioned G-equivariant logarithm:

$$\begin{array}{rccl} \mathrm{Log}_w & : & K^\times & \longrightarrow & \mathbb{R}[G] \\ & & x & \longmapsto & \sum_{g\in G} \log(||g(x)||_w)g^{-1} \end{array}$$

Let $S(K)$ denote the set of places of K lying above those in S. Because it is G-stable, the group $U_{S(K)}(K)$ – which by abuse of notation, we shall denote $U_S(K)$ – may be regarded as a finitely generated, multiplicative $\mathbb{Z}[G]$-module (sometimes written *additively*). Given $w_1, \ldots, w_r \in S(K)$ there is a unique $\mathbb{Q}[G]$-linear map

$$\mathrm{Reg}_S^{w_1,\ldots,w_r} : \mathbb{Q} \otimes_{\mathbb{Z}} \textstyle\bigwedge^r_{\mathbb{Z}[G]} U_S(K) \longrightarrow \mathbb{R}[G]$$

sending $a \otimes (\varepsilon_1 \wedge \ldots \wedge \varepsilon_r)$ to $a \det \left(\mathrm{Log}_{w_i}(\varepsilon_j)\right)^r_{i,j=1}$ for any $a \in \mathbb{Q}$ and $\varepsilon_1, \ldots, \varepsilon_r \in U_S(K)$. (Note that $\mathbb{Z}[G]$ is commutative as G is abelian, so $\bigwedge^r_{\mathbb{Z}[G]} U_S(K)$ is a well-defined $\mathbb{Z}[G]$-module, written additively. If $r = 0$ we interpret it as $\mathbb{Z}[G]$ and Reg_S as the natural injection $\mathbb{Q} \otimes \mathbb{Z}[G] \to \mathbb{R}[G]$ with image $\mathbb{Q}[G]$.) The following is essentially due to H. M. Stark, although the formulation given is Rubin's (see Remark 4.10 below).

Conjecture $SC(K/k, S, r)$. — **Basic Abelian Stark Conjecture at $s = 0$**
Let K/k, G, S and r be as above and suppose that Hypothesis $H(K/k, S, r)$ is satisfied. Thus (18) holds and we may choose r distinct places $v_1, \ldots, v_r \in S$ splitting in K and a place w_i of K above v_i for each i. Then

$$\Theta^{(r)}_{S,K/k}(0) = \mathrm{Reg}_S^{w_1,\ldots,w_r}(\eta) \quad \textit{for some } \eta \in \mathbb{Q} \otimes \textstyle\bigwedge^r_{\mathbb{Z}[G]} U_S(K)$$

We shall call any such η a 'solution of $SC(K/k, S, r)$ w.r.t. $w_1, \ldots, w_r$'.

Remark 4.5. — **More than r Split Places.** If $r + 1$ places in S split in K – and in particular if $K = k$ – then $SC(K/k, S, r)$ holds. Indeed, Proposition 4.4 implies that $\chi(\Theta^{(r)}_{S,K/k}(0)) = 0$ for all $\chi \neq \chi_0$ and hence by (12) that $\Theta^{(r)}_{S,K/k}(0) = \Theta^{(r)}_{S,k/k}(0)e_{\chi_0}$. It follows from equation (16) of Example 4.3 that 0 is a solution of $SC(K/k, S, r)$ unless $|S| = r + 1$ in which case a solution has the form $\eta := -(h_{S,k}/W_k)|G|^{-r} \otimes (\varepsilon_1 \wedge \ldots \wedge \varepsilon_r)$ where $\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_r$ is any $\mathbb{Z}$-basis of $U_S(k)/\mu(k)$ satisfying $\det \left(\log(||\varepsilon_i||_{v_j})\right)^{|S|-1}_{i,j=1} > 0$.

Remark 4.6. — **Dependence on the Places w_i.** The above shows that it suffices to consider the Basic Conjecture in the case where S contains *precisely* r splitting places. The places $w_1, \ldots, w_r$ are then determined up to replacing each w_i by g_iw_i for some $g_i \in G$ (which changes any putative solution by the action of $g_1 \ldots g_r$) and re-ordering (which affects only its sign). Because the dependence of the Conjecture on the choice of $w_1, \ldots, w_r$ is so simple, one often suppresses it and writes Reg_S instead of $\mathrm{Reg}_S^{w_1,\ldots,w_r}$.

Remark 4.7. — **Variation of S and K.** Suppose that K/k, S and r satisfy the conditions of $SC(K/k,S,r)$ and that η is a solution of the conjecture. If S' is a finite set of places containing S then, clearly, $H(K/k,S',r)$ is satisfied and it follows from (11) that $\eta' := \prod_{\substack{\mathfrak{p}\in S' \\ \mathfrak{p}\notin S}} \left(1-\sigma_{\mathfrak{p}}^{-1}\right)\eta$ is a solution of $SC(K/k,S',r)$. Similarly, if K' is an intermediate field with $K \supset K' \supset k$ and G' denotes $\mathrm{Gal}(K'/k)$, then $H(K'/k,S,r)$ is automatically satisfied. It is then a simple exercise to deduce from (12) that $N_{K/K'}\eta$ is a solution of $SC(K'/k,S,r)$ (w.r.t. the places of K' below $w_1,\ldots,w_r$) where $N_{K/K'}$ denotes the map from $\mathbb{Q}\otimes\bigwedge^r_{\mathbb{Z}[G]} U_S(K)$ to $\mathbb{Q}\otimes\bigwedge^r_{\mathbb{Z}[G']} U_S(K')$ induced by the norm from $U_S(K)$ to $U_S(K')$.

Remark 4.8. — It is clearly possible to have $r_S(\chi) \geq r$ for all $\chi \in \hat{G}$ even when $H(K/k,S,r)$ fails. In such a case equation (18) still holds and a corresponding variant of $SC(K/k,S,r)$ has been formulated and studied in **[EP]**

4.3. Uniqueness of the Solution. — Suppose that K/k, S and r satisfy the conditions of $SC(K/k,S,r)$, so in particular $r_S(\chi)\geq r$ for all $\chi\in\hat{G}$. The solution η (if one exists) is not in general unique but it may be rendered so by insisting that it lie in a certain 'eigenspace' of $\mathbb{Q}\otimes\bigwedge^r_{\mathbb{Z}[G]} U_S(K)$. To see this, consider a character $\chi\in\hat{G}$ such that $r_S(\chi)=r_S(\chi^{-1})>r$. Equations (18) and (10) and the definition of $r_S(\chi^{-1})$ then imply $e_\chi\Theta_S^{(r)}(0)=0$. Consequently,

$$\Theta_S^{(r)}(0) = 1.\Theta_S^{(r)}(0) = \left(\sum_{\chi\in\hat{G}} e_\chi\right)\Theta_S^{(r)}(0) = e_{S,r}\Theta_S^{(r)}(0) \tag{19}$$

where:

$$e_{S,r} := \sum_{\substack{\chi\in\hat{G} \\ r_S(\chi)\leq r}} e_\chi = \sum_{\substack{\chi\in\hat{G} \\ r_S(\chi)= r}} e_\chi$$

Although *a priori* an element of $\mathbb{C}[G]$, Prop. 4.4 shows that $e_{S,r}$ actually lies in $\mathbb{Q}[G]$ and, together with a little character theory, it even gives the formula

$$e_{S,r} = \begin{cases} \displaystyle\prod_{v\in S\setminus\{v_1,\ldots,v_r\}} \left(1-|D_v|^{-1}N_{D_v}\right) & \text{if } |S|>r+1 \\ \left(1-|D_v|^{-1}N_{D_v}\right)+e_{\chi_0,G} & \text{if } |S|=r+1 \text{ and } S\setminus\{v_1,\ldots,v_r\}=\{v\} \end{cases} \tag{20}$$

where $v_1,\ldots,v_r\in S$ split in K and for any finite group H we set $N_H=\sum_{h\in H} h\in\mathbb{Z}[H]$. Thus if η is a solution of $SC(K/k,S,r)$ and lies in $\mathbb{Q}\otimes\bigwedge^r_{\mathbb{Z}[G]} U_S(K)$ then so does $e_{S,r}\eta$ and equation (19) gives

$$\mathrm{Reg}_S(e_{S,r}\eta) = e_{S,r}\mathrm{Reg}_S(\eta) = e_{S,r}\Theta_S^{(r)}(0) = \Theta_S^{(r)}(0)$$

so, in fact, $e_{S,r}\eta$ is another solution *lying in the $e_{S,r}$-component (or 'eigenspace')* $e_{S,r}(\mathbb{Q}\otimes\bigwedge^r_{\mathbb{Z}[G]} U_S(K))$. Such a solution will be called *'canonical'*. On the other hand, one can use Dirichlet's Theorem to show that $e_{S,r}(\mathbb{Q}\otimes U_S(K))$ is free of rank r over $e_{S,r}\mathbb{Q}[G]$ and then deduce that Reg_S is *injective* on $e_{S,r}(\mathbb{Q}\otimes\bigwedge^r_{\mathbb{Z}[G]} U_S(K))$. We conclude:

Proposition 4.9. — *$SC(K/k,S,r)$ has a solution if and only if it has a* unique canonical solution.

The canonical solution of $SC(K/k, S, r)$ will be denoted $\eta_{S,K/k}$ or just η_S, if it exists.

Remark 4.10. — **Relation with Conjectures of Rubin and Stark.** Conjecture $SC(K/k, S, r)$ is equivalent Conjecture A' of [**Ru**] whose formulation is very similar but requires the choice of an auxiliary finite set T of finite places of k, subject to certain conditions. The choice of T does not affect its veracity, only the value of a putative solution. It becomes important only for Rubin's refined, integral (as opposed to 'basic') abelian Stark conjecture. This is his Conjecture B' which requires that the solution of Conjecture A' lie in a certain $\mathbb{Z}[G]$-lattice spanning the eigenspace over $\mathbb{Q}$ and depending on T. Some more details of the relationships between these conjectures can be found in [**So2**, Remark 2.8], in a special case.

Stark's original conjecture was formulated in terms of Artin L-functions of characters of a Galois extension which is not necessarily abelian. It appears as Conjecture I.5.1 in [**Ta**]. However, Propositions 2.3 and 2.4 of [**Ru**] show that in our set-up, Conjecture A' – and hence $SC(K/k, S, r)$ – are equivalent to Stark's conjecture for all characters $\chi \in \hat{G}$ such that $r_S(\chi) = r$.

5. Some Particular Cases of $SC(K/k, S, r)$

We briefly survey the known cases of the the basic Stark Conjecture and some of its integral refinements. These will be grouped according to the value of r: 0, 1 or ≥ 2. Let K/k be an abelian extension with group G and $S \supset S_{\min}$ be as above.

5.1. The Case $r = 0$. — For *any* such K/k, S, the Hypothesis $H(K/k, S, 0)$ is automatically satisfied and we shall see that $SC(K/k, S, 0)$ follows from results of Siegel-Klingen. First, the interpretation of Reg_S in the case $r = 0$ (explained above) means that $SC(K/k, S, 0)$ is equivalent to the statement that $\Theta_{S,K/k}(0)$ lies in $\mathbb{Q}[G]$ or, indeed, in $e_{S,0}\mathbb{Q}[G]$ by (19). We can assume $S = S_{\min}$ by (11). Now, if S_∞ contains a split place then the conjecture holds (see Remark 4.5). Thus we can assume K is totally complex and k totally real, which forces $|S| > 1$. Also, if $v \in S_\infty$ then $D_v = \{1, c_v\}$ where c_v is the complex conjugation associated to v. Equation (20) shows that $c_v e_{S,0} = -e_{S,0}$ so that $c_v\Theta_{S,K/k}(0) = -\Theta_{S,K/k}(0)$ by (19). Thus $\Theta_{S,K/k}(0)$ is fixed by the subgroup $H := \langle c_{v_1}c_{v_2} : v_1, v_2 \in S_\infty\rangle$ of G. Equation (12) therefore allows us to replace K by K^H, *i.e.* we can assume H is trivial and (still) that K is totally complex. This means that K *is of CM-type i.e.* c_v equals $c \in G$ (of order 2) independently of $v \in S_\infty$. In this set-up, it follows from work of Siegel [**Si**] and Klingen (or of Shintani [**Sh1**, Cor. to Thm. 1]) that $\zeta_{S_{\min}}(0, g) \in \mathbb{Q}$ for all $g \in G$, and the conjecture follows.

For the rest of this subsection we shall continue to assume that k is totally real, K is CM and $S = S_{\min}$. We have seen that c acts by -1 on $\Theta_{S,K/k}(0) \in \mathbb{Q}[G]$ and therefore $\Theta_{S,K/k}(0) \in (1-c)\mathbb{Q}[G]$. However, in the case of Example 4.1, it is evident from (13) that $\Theta_{S_f,\mathbb{Q}(\zeta_f)/\mathbb{Q}}(0)$ actually lies in $(1-c)W^{-1}_{\mathbb{Q}(\zeta_f)}\mathbb{Z}[G_f]$. (Note that $W_{\mathbb{Q}(\zeta_f)}$ is the l.c.m. of 2 and f.) We can take this further: let $\mathrm{Ann}_{\mathbb{Z}[G]}(\mu(K)) \lhd \mathbb{Z}[G]$ denote the annihilator of $\mu(K)$ as a $\mathbb{Z}[G]$-module (which clearly contains W_K). In Example 4.1, $\mathrm{Ann}_{\mathbb{Z}[G_f]}(\mu(\mathbb{Q}(\zeta_f)))$ is easily seen to be generated over $\mathbb{Z}$ by the elements $b - g_b$ for all integers b such that $(b, 2f) = 1$. Furthermore, it follows from (13) that $(b - g_b)\Theta_{S_f,\mathbb{Q}(\zeta_f)/\mathbb{Q}}(0)$ has coefficients in $\mathbb{Z}$, so:

$$\begin{aligned} \mathrm{Ann}_{\mathbb{Z}[G_f]}(\mu(\mathbb{Q}(\zeta_f)))\Theta_{S_f,\mathbb{Q}(\zeta_f)/\mathbb{Q}}(0) &= \langle (b-g_b)\Theta_{S_f,\mathbb{Q}(\zeta_f)/\mathbb{Q}}(0) : b \in \mathbb{Z},\ (b,2f)=1\rangle_{\mathbb{Z}} \\ &\subset \mathbb{Z}[G_f] \cap (1-c)\mathbb{Q}[G_f] = (1-c)\mathbb{Z}[G_f] \end{aligned} \tag{21}$$

The second member above is the *Stickelberger Ideal* of $\mathbb{Z}[G_f]$. (It differs very slightly from that of [**Wa**, §6.2], for example, which is not quite contained in $(1-c)\mathbb{Z}[G_f]$.) For any K/k as above, we define the *Generalised Stickelberger Ideal*

$$\mathrm{Stick}(K/k) := \mathrm{Ann}_{\mathbb{Z}[G]}(\mu(K))\Theta_{S_{\min},K/k}(0) \tag{22}$$

From what we already know, this is a $\mathbb{Z}[G]$-submodule of $(1-c)\mathbb{Q}[G]$. However a result of Deligne-Ribet [**DR**] (and, independently, of Pi. Cassou-Noguès [**C-N**]) gives the following generalisation of (21) which may be seen as an 'integral' refinement of $SC(K/k, S_{\min}, 0)$.

Theorem 5.1. —
With assumptions and notations as above, $\mathrm{Stick}(K/k)$ *is contained in* $(1-c)\mathbb{Z}[G]$.

5.2. The Case $r = 1$. — Assume Hypothesis $H(K/k, S, 1)$ is satisfied *i.e.* $|S| \geq 2$ and there exists $v_1 \in S$ splitting in K. Fix w_1 above v_1 in $S(K)$. Any element $\eta \in \mathbb{Q} \otimes \Lambda^1_{\mathbb{Z}[G]} U_S(K) = \mathbb{Q} \otimes U_S(k)$ may be written $\frac{1}{m} \otimes \varepsilon$ with $\varepsilon \in U_S(K)$ and clearly,

$$\begin{aligned} \eta \text{ is a solution of } SC(K/k,S,1) &\iff \Theta'_{S,K/k}(0) = \frac{1}{m}\mathrm{Log}_{w_1}(\varepsilon) \\ &\iff \zeta'_{S,K/k}(0,g) = \frac{1}{m}\log\|g(\varepsilon)\|_{w_1}\ \forall\, g \in G \end{aligned} \tag{23}$$

Remark 5.2. — **Criterion for the Canonical Solution when $r = 1$**
If $|S| > 2$ and $\eta = \frac{1}{m} \otimes \varepsilon$ is a solution of $SC(K/k, S, 1)$ then it is the *canonical* solution η_S iff $\|\varepsilon\|_w = 1$ *for all* $w \in S(K)$ *not above* v_1. If $|S| = 2$ the italicised condition must be replaced by '$\|\varepsilon\|_w$ *is independent of* $w \in S(K)$ *above* v' where $S \setminus \{v_1\} = \{v\}$. We leave the proofs of these statements as an exercise.

Stark gave an integral refinement of $SC(K/k, S, 1)$ as follows.

Conjecture $RSC(K/k, S)$. — **Refined Abelian Stark Conjecture for $r = 1$**
Suppose K/k, and S satisfy Hypothesis $H(K/k, S, 1)$. Then $SC(K/k, S, 1)$ holds with canonical solution η_S such that

(i) $\eta_S = \frac{1}{W_K} \otimes \varepsilon_S$ *for some $\varepsilon_S \in U_S(K)$ (depending on choice of w_1) and*
(ii) *the extension $K(\varepsilon_S^{1/W_K})/k$ is abelian.*

The arguments of Remark 4.5 extend to prove the $RSC(K/k, S)$ if S contains a split place other than v_1 *e.g.* if $K = k$. This is shown in Prop. 3.1 of [**Ta**, Ch. IV]. If v_1 is an *infinite* place, the remaining proven cases of the Refined Conjecture – and indeed of $SC(K/k, S, 1)$ –

are as follows. For v_1 finite, more will be said at the beginning of Section 6 and in both cases, more details can be found in *loc. cit.*

(i) $\underline{k=\mathbb{Q}.}$ Here $v_1=\infty$ and K is a real, absolutely abelian field (w.l.o.g. different from $\mathbb{Q}$) so that $\mathfrak{f}_{K/\mathbb{Q}}=f\mathbb{Z}$ for some $f>4$, $f\not\equiv 2 \pmod 4$. The Kronecker-Weber Theorem implies $K\subset\mathbb{Q}(\zeta_f)^+=\mathbb{Q}(\zeta_f+\zeta_f^{-1})$. For $SC(K/\mathbb{Q},S,1)$, we use Remark 4.7 to reduce to the special case $K=\mathbb{Q}(\zeta_f)^+$, $S=S_{\min}(\mathbb{Q}(\zeta_f)^+/\mathbb{Q})=S_f$ for which equation (15) of Example 4.2 shows that $\frac{1}{2}\otimes\varepsilon_f^{-1}$ is a solution of $SC(\mathbb{Q}(\zeta_f)^+/\mathbb{Q},S_f,1)$ with w_1 given by the inclusion $\mathbb{Q}(\zeta_f)^+\hookrightarrow\mathbb{R}$. The Refined Conjecture $RSC(K/\mathbb{Q},S)$ can also be reduced to this special case (use Prop. 3.4 and 3.5 of [**Ta**, Ch. IV]). Remark 5.2 implies that $\frac{1}{2}\otimes\varepsilon_f^{-1}$ is the *canonical* solution. Moreover, $W_{\mathbb{Q}(\zeta_f)^+}=2$ and one checks that $\mathbb{Q}(\sqrt{\varepsilon_f})$ is contained in $\mathbb{Q}(\zeta_{4f})$ (and even in $\mathbb{Q}(\zeta_{2f})$ if $2|f$) so $RSC(\mathbb{Q}(\zeta_f)^+,S_f)$ holds.

(ii) <u>*k imaginary quadratic.*</u> In this case v_1 is the unique (complex) infinite place and $RSC(K/k,S)$ is proven in the paper IV of [**St**]. The reader can also refer to the abridged account given in [**Ta**, IV.3.9]. Again one reduces to the case where K is a certain ray-class field for which the canonical solution is given in terms of an elliptic unit and (23) is proven via the Second Kronecker Limit Formula.

(iii) <u>*G is 2-elementary.*</u> If $G\cong(\mathbb{Z}/2\mathbb{Z})^t$ for some t then $SC(K/k,S,r)$ holds quite generally for any admissible S and r (see below). Taking $r=1$, the Refined Conjecture $RSC(K/k,S)$ is proven in [**Ta**, IV.5.5] under the additional assumption that G is generated by the subgroups D_v for $v\in S$. (This does not require v_1 to be infinite.)

As far as the author is aware, the only other cases of $SC(K/k,S,1)$ proven to date (with v_1 infinite) are due to Shintani. In [**Sh2**], he proves a version of $SC(K/k,S,1)$ in certain cases where k is real quadratic and K is a quadratic extension of an absolutely abelian field such that only one real place v_1 of k splits in K.

Remark 5.3. — **Construction of Abelian Extensions.**
Notice that condition (i) of $RSC(K/k,S)$ determines ε_S up to an element of $U_S(K)_{\mathrm{tor}}=\mu(K)$. Furthermore, equation (23) now predicts

$$||\varepsilon_S||_{g^{-1}w_1}=\exp(W_K\zeta'_{S,K/k}(0,g))\ \forall\, g\in G \tag{24}$$

Suppose v_1 is real. Then so is $g^{-1}w_1\ \forall\, g$ and $W_K=2$. If we could prove $RSC(K/k,S)$ in this case then (24) would give a transcendental formula for $\pm\varepsilon_S\in K^\times$. This would thus lead to a solution of Hilbert's 12th Problem – the construction of abelian extensions of k – using special values of derivatives of partial-zeta functions *that are intrinsic to k.* (Or, indeed, using ray-class L-functions, via (7) and (10).) If we only *assume* $RSC(K/k,S)$, one can still use (24) and the other facts about ε_S to identify it precisely on a computer. This leads to a method for the algorithmic construction of certain ray-class fields which has been implemented in PARI/GP [**PARI**].

5.3. The Case $r \geq 2$. — For any r, Conjecture $SC(K/k, S, r)$ can be proven under the following hypothesis extending that of Remark 4.5:

$$\chi \in \hat{G},\ r_S(\chi) = r \Rightarrow \mathrm{ord}(\chi)|2 \tag{25}$$

Indeed, every character of order 1 or 2 satisfies Conjecture I.5.1 of [**Ta**] for the same K/k and S. (This follows from properties of the latter under inflation, induction and addition of characters and the case $K = k$. See for example [**Ta**, I.7.1 and II.1.1]). As noted in Remark 4.10 it follows that $SC(K/k, S, r)$ holds whenever (25) does, and in particular whenever G is 2-elementary (see above). If $r \geq 2$, the only proven cases of $SC(K/k, S, r)$ without (25) come by using induction of characters to 'raise the base field' from known cases with $r = 1$ (see *e.g.* [**Po**]). This applies, for instance, in some cases where when K is abelian over $\mathbb{Q}$ and $[k : \mathbb{Q}] = r$.
There exist two integral refinements of $SC(K/k, S, r)$ for $r \geq 2$, both generalising $RSC(K/k, S)$. These are 'Conjecture B'' of Rubin mentioned in Remark 4.10 and a version due to Popescu in [**Po**] which dispenses with Rubin's auxiliary sets T. We shall not give full statements here but note that both imply the following, rather crude generalisation of condition (i) of $RSC(K/k, S)$: *η_S is of the form $\frac{1}{mW_K} \otimes \tilde{\eta}$ for some $\tilde{\eta} \in \bigwedge^r_{\mathbb{Z}[G]} U_S(K)$ and some $m \in \mathbb{N}$ whose prime factors all divide $|G|$.* (Cases are known in which $m \neq 1$ is forced.) Rubin's and/or Popescu's refinements were established for $|G| = 2$ in [**Ru**], by Sands in some other cases where G is 2-elementary (see his article in [**BPSS**]) and by Popescu [**Po**], Cooper [**Co**] and Burns (see below) in other cases by base-raising.

Two other types of evidence support $SC(K/k, S, r)$ and its refinements. Firstly, both Rubin's and Popescu's Conjectures are shown in [**Bu**] to follow from a particular case of the very general Equivariant Tamagawa Number Conjecture of Burns and Flach. This was proven for $k = \mathbb{Q}$ and $K/\mathbb{Q}$ abelian in [**BG**] (see also Flach's article in [**BPSS**] for the case $p = 2$). Since it behaves well under raising the base field, one can even establish Rubin's and Popescu's conjectures for such K, any $k \subset K$ and any admissible S and r. Secondly, there is considerable computational evidence in support of $SC(K/k, S, r)$. We refer to Dummitt's article in [**BPSS**] for a survey concentrating on the case $r = 1$ (with v_1 finite or infinite). Numerical confirmation of some cases with $r = 2$, k real quadratic and v_1, v_2 real is given in [**RS1**] (along with an analogous p-adic conjecture) and in [**RS2**].

6. The Brumer-Stark Conjecture and the Annihilation of Class Groups

For the rest of this article we shall assume that k is totally real, K is of CM-type and $G = \mathrm{Gal}(K/k)$ is abelian with unique complex conjugation denoted c and maximal real subfield $K^+ = K^{\langle c \rangle}$. This forces $|S_{\min}| > 1$. To simplify, we shall take $S = S_{\min}$ until further notice and write Θ for Θ_S. Theorem 5.1 implies that $W_K\Theta(0)$ lies in $(1 - c)\mathbb{Z}[G]$. With these assumptions, and temporarily using a multiplicative notation for $\mathbb{Z}[G]$-actions, we can state the

Conjecture $BSC(K/k)$. — **Brumer-Stark Conjecture (with $S = S_{\min}$)**
For every fractional ideal $\mathfrak{a} \in I(K)$ there exists $\gamma_{\mathfrak{a}} \in K^\times$ such that

(i) $\mathfrak{a}^{W_K\Theta(0)}$ *equals the principal ideal* $(\gamma_{\mathfrak{a}})$,
(ii) $||\gamma_{\mathfrak{a}}||_w = 1$ *for all* $w \in S_\infty(K)$ *and*
(iii) *the extension* $K(\gamma_{\mathfrak{a}}^{1/W_K})/k$ *is abelian.*

Note that conditions (i) and (ii) determine $\gamma_{\mathfrak{a}}$ up to an element of $\mu(K)$. Also, property (i) implies that $||\gamma_{\mathfrak{a}}||_w = 1$ for any place w of K above $v \in S_{\mathrm{ram}}$, since $|S_{\min}| > 1$ implies $N_{D_v}\Theta(0) = 0$ (by (19) and (20) with $r = 0$). In our set-up, $BSC(K/k)$ is therefore equivalent to the case of Conjecture IV.6.2 of [**Ta**] with 'T' equal to $S_{\min}$ and this implies all other cases by Cor. 6.6 *ibid.*.
The explanation of the name 'Brumer-Stark' is as follows. Firstly, one can show that $BSC(K/k)$ holds if and only if it holds with $\mathfrak{a} = \mathfrak{P}$ for any prime ideal $\mathfrak{P}$ above any prime $\mathfrak{p}$ of k that splits in K. (This follows from [**Ta**, Prop. IV.6.4] and the fact that the classes of such $\mathfrak{P}$ generate $\mathrm{Cl}(K)$.) But Remark 5.2 and the relation $\Theta'_{S_{\min}\cup\{\mathfrak{p}\}}(0) = \log(N\mathfrak{p})\Theta(0)$ coming from (11) show that conditions (i)–(iii) with $\mathfrak{a} = \mathfrak{P}$ are in fact equivalent to the statement that $\frac{1}{W_K} \otimes \gamma_{\mathfrak{P}}$ is the (canonical) solution of $RSC(K/k, S_{\min} \cup \{\mathfrak{p}\})$ with $w_1 = \mathfrak{P}$. Hence $BSC(K/k)$ is equivalent to these cases of the Refined Stark Conjecture for $r = 1$.
Secondly, $BSC(K/k)$ clearly implies that $W_K\Theta(0)$ annihilates $\mathrm{Cl}(K)$ as a $\mathbb{Z}[G]$-module, which had previously been conjectured by Armand Brumer. We can go further. Given any $\mathfrak{a} \in I(K)$, it follows easily from condition (iii) of $BSC(K/k)$ that for every $x \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu(K))$ there exists $y_{\mathfrak{a},x} \in K^\times$ with $\gamma_{\mathfrak{a}}^x = y_{\mathfrak{a},x}^{W_K}$. It then follows from condition (i) that $\mathfrak{a}^{xW_K\Theta(0)} = (y_{\mathfrak{a},x})^{W_K}$ and since $x\Theta(0) \in \mathbb{Z}[G]$ by Thm. 5.1, we must have $\mathfrak{a}^{x\Theta(0)} = (y_{\mathfrak{a},x})$. In particular, $BSC(K/k)$ implies the

Conjecture $BC(K/k)$. — **Brumer Conjecture (with $S = S_{\min}$)**
In the above situation $\mathrm{Cl}(K)$ *is annihilated by the ideal* $\mathrm{Stick}(K/k) \subset (1-c)\mathbb{Z}[G]$ *defined in (22).*

Example 6.1. — Brumer and Brumer-Stark Conjectures for $k = \mathbb{Q}$.
In this case, K is an imaginary abelian field and $BC(K/\mathbb{Q})$ is simply Stickelberger's Theorem. The traditional proof of the latter (see *e.g.* [**Wa**]) uses the factorisation of (norms of) cyclotomic Gauss sums attached to prime ideals $\mathfrak{P}$ of $\mathcal{O}_K$. In fact, if $\mathfrak{P}$ divides a rational prime $p \in \mathbb{Q}$ split in K and the latter is the full cyclotomic field $\mathbb{Q}(\zeta_f)$ (notations as above, so $f|(p-1)$) then exactly the same factorisations establish that $W_{\mathbb{Q}(\zeta_f)}$th power of the Gauss sum is (essentially) the element $\gamma_{\mathfrak{P}}$ appearing in $BSC(\mathbb{Q}(\zeta_f)/\mathbb{Q})$. As explained above, this means that $\gamma_{\mathfrak{P}}$ also gives rise to the solution of $RSC(\mathbb{Q}(\zeta_f)/\mathbb{Q}, S_f \cup \{p\})$ with $v_1 = p$. In this way one also establishes $BSC(K/\mathbb{Q})$ for arbitrary imaginary abelian K and also $RSC(K/\mathbb{Q}, S)$ for such K and arbitrary $S \supset S_{\min}$ containing a *finite* split place v_1.

For arbitrary $S' \supset S_{\min}$ one can make an 'S'-Brumer Conjecture' by replacing $\Theta_{S_{\min}}(0)$ with $\Theta_{S'}(0)$ in (22). This is the viewpoint of the excellent article by Greither in [**BPSS**]. (Of

course, the S'-version of $BC(K/k)$ is weaker: it follows from our version because of (11).) One or other version is now known in many cases, thanks to work of Greither and Wiles (see below) and many others. For a survey, for higher analogues involving $\Theta(-n)$ with $n \geq 1$ (the Coates-Sinnott Conjecture) and for precise connections between $BSC(K/k)$ and Rubin's and Stark's Conjectures, we refer to *ibid.* Our focus now will be on strengthening the annihilation statement of $BC(K/k)$ in a different direction to that of $BSC(K/k)$.
First, we localise: let p be a prime number and write $\mathrm{Cl}(K)_p$ for the p-Sylow subgroup of $\mathrm{Cl}(K)$ considered as a module for $\mathcal{R}_p := \mathbb{Z}_p[G]$. If $\mathrm{Stick}(K/k)_p$ denotes the $\mathbb{Z}_p$-span of $\mathrm{Stick}(K/k)$ inside $(1-c)\mathcal{R}_p$ then, clearly, $BC(K/k)$ is equivalent to the following local statement for all p, which we denote $BC(K/k)_p$:

$$\mathrm{Cl}(K)_p \text{ is annihilated by } \mathrm{Stick}(K/k)_p$$

We *assume henceforth that* $p \neq 2$ which implies that $\mathcal{R}_p$ is a product of rings $\mathcal{R}_p^+ \times \mathcal{R}_p^- =: (1+c)\mathcal{R}_p \times (1-c)\mathcal{R}_p$ and, correspondingly, $\mathrm{Cl}(K)_p = \mathrm{Cl}(K)_p^+ \oplus \mathrm{Cl}(K)_p^- := (1+c)\mathrm{Cl}(K)_p \oplus (1-c)\mathrm{Cl}(K)_p$. Since $\mathrm{Stick}(K/k)_p$ is an ideal of $\mathcal{R}_p^-$, it automatically annihilates $\mathrm{Cl}(K)_p^+$ which is isomorphic to $\mathrm{Cl}(K^+)_p$ (since $p \neq 2$). This means firstly that $BC(K/k)_p$ tells us nothing interesting about the class groups of totally real fields (at least, not directly). Secondly,

$$BC(K/k)_p \Longleftrightarrow \mathrm{Stick}(K/k)_p \subset \mathrm{Ann}_{\mathcal{R}_p^-}(\mathrm{Cl}(K)_p^-) \tag{26}$$

7. Exact Annihilators and Fitting Ideals

We shall examine first the obvious question of whether one should actually expect equality on the R.H.S. of (26). There are at least two reasons why this cannot hold in general, the first being that $\mathrm{Ann}_{\mathcal{R}_p^-}$ contains $|\mathrm{Cl}(K)_p^-|$ so is of finite index in $\mathcal{R}_p^-$ but $\mathrm{Stick}(K/k)_p$ may not be, essentially because of 'trivial zeroes' of $S_{\min}$-truncated L-functions at $s = 0$. Indeed, $\Theta(0)$ and hence $\mathrm{Stick}(K/k)_p$ are killed by $(1-c)N_{D_v}$ for any $v \in S_{\min}$ (see above) which is a non-zero element of $\mathcal{R}_p^-$ whenever $c \notin D_v$. To get around this, we need to enlarge $\mathrm{Stick}(K/k)_p$. One way to do this is to define

$$\widetilde{\mathrm{Stick}}(K/k)_p := \sum_{K \supset K' \supset k} \mathrm{cores}_{K'}^{K}(\mathrm{Stick}(K'/k)_p) \tag{27}$$

Here, K' runs through intermediate (CM) sub-extensions of K/k and $\mathrm{cores}_{K'}^{K} : \mathbb{Z}_p[\mathrm{Gal}(K'/k)] \to \mathbb{Z}_p[G] = \mathcal{R}_p$ is the $\mathbb{Z}_p$-linear map sending $g' \in \mathrm{Gal}(K'/k)$ to the sum of its pre-images under $\pi_{K/K'}$. It is not hard to show that $\widetilde{\mathrm{Stick}}(K/k)_p$ is an ideal of finite index in $\mathcal{R}_p^-$ containing $\mathrm{Stick}(K/k)_p$ and, moreover, that if $BC(K'/k)_p$ held for every K' then one would have $\widetilde{\mathrm{Stick}}(K/k)_p \subset \mathrm{Ann}_{\mathcal{R}_p^-}(\mathrm{Cl}(K)_p^-)$. Other 'enlargements' of $\mathrm{Stick}(K/k)_p$ appear in the literature, mostly variants of (27) which agree with $\widetilde{\mathrm{Stick}}(K/k)_p$ when $p \nmid |G|$.
However, one still cannot always have equality for another reason which becomes particularly clear for $[K:k] = 2$, when all enlargements coincide with the basic $\mathrm{Stick}(K/k)_p$. We explain briefly, leaving details to the reader. In this case, we can identify $\mathcal{R}_p^- = \mathbb{Z}_p(1-c)$ with $\mathbb{Z}_p$

and evaluate $\Theta(0)$ precisely using the Analytic Class Number Formula for $\zeta_K(s)$ and $\zeta_k(s)$. We get:

(28) $$\text{If } [K:k]=2 \text{ then } \widetilde{\mathrm{Stick}}(K/k)_p = \mathrm{Stick}(K/k)_p = (h_K/h_k)\mathbb{Z}_p = |\mathrm{Cl}(K)_p^-|\mathbb{Z}_p$$

This proves that $\widetilde{\mathrm{Stick}}(K/k)_p \subset \mathrm{Ann}_{\mathcal{R}_p^-}(\mathrm{Cl}(K)_p^-)$ (and hence $BC(K/k)_p$) in this case but equality clearly holds if and only if $\mathrm{Cl}(K)_p^-$ is a *cyclic* abelian group. This fails, for example, when $p=3$ and $K/k=\mathbb{Q}(\sqrt{-974})/\mathbb{Q}$ so that $\mathrm{Cl}(K)_3^- = \mathrm{Cl}(K)_3 \cong (\mathbb{Z}/3\mathbb{Z})^2$.
Instead of comparing $\widetilde{\mathrm{Stick}}(K/k)_p$ with $\mathrm{Ann}_{\mathcal{R}_p^-}(\mathrm{Cl}(K)_p^-)$ – which is generated by the *exponent* of $\mathrm{Cl}(K)_p^-$ as an abelian group when $[K:k]=2$ – equation (28) suggests that one should in general compare it with an ideal of $\mathcal{R}_p^-$ which, in some sense, measures the *'size'* of $\mathrm{Cl}(K)_p^-$ as an $\mathcal{R}_p^-$-module. The most obvious candidate is the *(initial) Fitting ideal* $\mathrm{Fitt}_{\mathcal{R}_p^-}(\mathrm{Cl}(K)_p^-)$. We briefly recall its definition and make some general remarks, referring to **[No]**, the appendix of **[MW]** or Greither's article in **[BPSS]** for more details. For any finitely generated module A over a commutative (noetherian) ring $\mathcal{R}$ one can choose a presentation

$$\mathcal{R}^s \xrightarrow{M} R^t \longrightarrow A \to 0$$

where $s,t\in\mathbb{N}$ and M is a $t\times s$ matrix with coefficients in $\mathcal{R}$. One then defines $\mathrm{Fitt}_{\mathcal{R}}(A)$ to be the ideal of $\mathcal{R}$ generated by all $t\times t$ minor-determinants of M (which is zero if $s<t$). It turns out that this is independent of the choice of s,t and M and that $\mathrm{Ann}_{\mathcal{R}}(A)^t \subset \mathrm{Fitt}_{\mathcal{R}}(A) \subset \mathrm{Ann}_{\mathcal{R}}(A)$ for any possible t. If A is cyclic over R we can take $t=1$ so $\mathrm{Fitt}_{\mathcal{R}}(A)=\mathrm{Ann}_{\mathcal{R}}(A)$. Now suppose A is *finite*. If $\mathcal{R}=\mathbb{Z}$ or $\mathbb{Z}_p$ one can take $s=t$ and the theory of elementary divisors gives $\mathrm{Fitt}_{\mathcal{R}}(A)=|A|\mathcal{R}$. If $\mathcal{R}=\mathbb{Z}_p[H]$ for a finite abelian group H, then $s\geq t$ is forced and $\mathrm{Fitt}_{\mathcal{R}}(A)$ contains $|A|^t$ so is an ideal of finite index in $\mathcal{R}$. Moreover, one can show that $s=t$ is possible if and only if A is a cohomologically trivial H-module. In this case, $\mathrm{Fitt}_{\mathcal{R}}(A)$ is clearly principal, generated by $\det(M)$. (For a converse, see Prop. 2.2.2 of Greither's article in **[BPSS]**). If $p\nmid |H|$ then A is always cohomologically trivial and $\mathbb{Z}_p[H]$ is a product of unramified extensions of $\mathbb{Z}_p$ corresponding to $\bar{\mathbb{Q}}_p$-valued characters of H. One can decompose everything using such characters so that the Fitting ideal behaves much like the case $\mathcal{R}=\mathbb{Z}_p$. If, however, $p||H|$ then the Fitting ideal can be non-principal and is, in general, a far more subtle invariant.
We are interested in the case $H=G$ where all the above remarks remain true with $\mathcal{R}=\mathcal{R}_p^-$, a direct factor of $\mathbb{Z}_p[G]$. In particular, $\mathrm{Fitt}_{\mathcal{R}_p^-}(\mathrm{Cl}(K)_p^-)$ is an ideal of finite index in $\mathcal{R}_P^-$, contained in $\mathrm{Ann}_{\mathcal{R}_p^-}(\mathrm{Cl}(K)_p^-)$ and one can ask:

(29) *is* $\widetilde{\mathrm{Stick}}(K/k)_p$ *equal to* $\mathrm{Fitt}_{\mathcal{R}_p^-}(\mathrm{Cl}(K)_p^-)$*?*

Positive answers to this question were obtained by various authors under different hypotheses. For example, when $p\nmid |G|$, characters are used to treat the case $k=\mathbb{Q}$ in **[MW]** and $k\neq\mathbb{Q}$ (totally real) in **[Wi]** (under an additional hypothesis on characters). *Without* the assumption $p\nmid|G|$, three other large classes with $k=\mathbb{Q}$ are treated in **[Ku1]** (using a different enlargement of $\mathrm{Stick}(K/k)_p$) and **[Gr1]** treats the case in which K/k is 'nice' (a

condition which implies $\widetilde{\mathrm{Stick}}(K/k)_p = \mathrm{Stick}(K/k)_p$ among other things). In each case we can deduce $BC(K/k)_p$ using (26). One should also mention certain 'higher Stickelberger ideals' defined by M. Kurihara in the case $p \nmid |G|$ and k totally real. Using characters and Euler Systems, he shows in **[Ku2]** how these can give more information on the $\mathcal{R}_p^-$-structure of $\mathrm{Cl}(K)_p^-$, essentially determining the latter under certain hypotheses.
Nevertheless, in 2006 Greither showed that the Equivariant Tamagawa Number Conjecture (ETNC) mentioned in Section 5 predicts a result that runs somewhat counter to (29): If A is a finite $\mathcal{R}_p$-module let us write $\hat{A}$ for the Pontrjagin dual $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}_p}(A, \mathbb{Q}_p/\mathbb{Z}_p)$ of A made into an $\mathcal{R}_p$-module by defining $(gf)(a)$ to be $f(ga)$ for any $f \in \hat{A}$, $g \in G$. (This makes sense because G is abelian and clearly $\hat{A}^- \cong \widehat{A^-}$ as $\mathcal{R}_p^-$-modules.) Theorem 8.8 of **[Gr2]** can now be stated as

Theorem 7.1. — *Suppose K/k and p satisfy our current assumptions, that $G = \mathrm{Gal}(K/k)$ and in addition that*

(i) $\mu_{p^\infty}(K)$ *is G-cohomologically trivial and*
(ii) *the ETNC holds for the pair* $(K/k, h^0(K))$

Then $\mathrm{SKu}(K/k)_p = \mathrm{Fitt}_{\mathcal{R}_p^-}(\widehat{\mathrm{Cl}(K)_p^-})$.

Here, $\mathrm{SKu}(K/k)_p$ is a variant of $\widetilde{\mathrm{Stick}}(K/k)_p$ satisfying

$$\mathrm{Stick}(K/k)_p \subset \mathrm{SKu}(K/k)_p \subset \widetilde{\mathrm{Stick}}(K/k)_p$$

If $p \nmid |G|$ then on the one hand $\mathrm{SKu}(K/k)_p = \widetilde{\mathrm{Stick}}(K/k)_p$ and on the other, any finite $\mathcal{R}_p$-module A is isomorphic to $\hat{A}$ as an $\mathcal{R}_p$-module. So Thm. 7.1 predicts a positive answer to question (29) in this case, agreeing with Wiles' results. However, if $p||G|$ things get decidedly more complicated. On the one hand one can have $\mathrm{SKu}(K/k)_p \neq \widetilde{\mathrm{Stick}}(K/k)_p$. On the other, one can easily construct finite $\mathcal{R}_p^-$-modules A with $A \ncong \hat{A}$. In this case, one still has $\mathrm{Fitt}_{\mathcal{R}_p^-}(\hat{A}) \subset \mathrm{Ann}_{\mathcal{R}_p^-}(\hat{A}) = \mathrm{Ann}_{\mathcal{R}_p^-}(A)$ so, in particular, Thm. 7.1 supports $BC(K/k)_p$. However, $\mathrm{Fitt}_{\mathcal{R}_p^-}(A)$ and $\mathrm{Fitt}_{\mathcal{R}_p^-}(\hat{A})$ will differ unless a special condition holds, *e.g.* A is cohomologically trivial or G is p-cyclic. (The sufficiency of the latter follows from [**MW**, Appendix, Prop. 1].)
An explicit counter-example to (29) was finally given by Greither and Kurihara. Taking $p = 3$ they found an extension K/k with $k = \mathbb{Q}(\sqrt{29})$, $\mu_{p^\infty}(K) = \{1\}$ and $G \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})^2$ for which $\mathrm{Stick}(K/k)_p$ – and *a fortiori* any enlargement of it – is *not contained in* $\mathrm{Fitt}_{\mathcal{R}_p^-}(\mathrm{Cl}(K)_p^-)$. (See [**GK**, §3.2] for more details, noting that our K is their K_1.) In the function-field case, counter-examples had previously been given by Popescu. We also mention the recent, unconditional results of **[GP]** concerning Fitting ideals of duals in the function-field case. Analogues for number fields may be forthcoming.
In view of Thm. 7.1, it seems reasonable to ask whether one can use Θ_S to construct an ideal related to $\mathrm{Fitt}_{\mathcal{R}_p^-}(\mathrm{Cl}(K)_p^-)$ rather than $\mathrm{Fitt}_{\mathcal{R}_p^-}(\widehat{\mathrm{Cl}(K)_p^-})$. A result of this type is given in **[KM]** for $k = \mathbb{Q}$ but doesn't seem to generalise. We shall return to this question in §9.1.

8. Θ_S at $s = 1$

8.1. Plus- and Minus-Parts. — We maintain the above notations and assumptions on K/k but once more allow S to contain $S_{\min}$ properly. A character $\chi \in \hat{G}$ is called (totally) *even* or *odd* according as $\chi(c) = 1$ or -1. If $\mathcal{R}$ is a commutative ring in which 2 is invertible, we write e^+ and e^- respectively for the idempotents $\frac{1}{2}(1 + c)$ and $\frac{1}{2}(1 - c)$ of $\mathcal{R}[G]$ and $A^\pm$ for $e^\pm A$ where A is any $\mathcal{R}[G]$ module, so $A = A^+ \oplus A^-$. The meromorphic functions $\Theta_S^+(s) := e^+\Theta_S(s)$ and $\Theta_S^-(s) := e^-\Theta_S(s)$ take values in $\mathbb{C}[G]^+$ and $\mathbb{C}[G]^-$ respectively and

$$\Theta_S(s) = \Theta_S^+(s) + \Theta_S^-(s) = \sum_{\substack{\chi\in\hat{G}\\ \chi \text{ even}}} L_S(s, \chi^{-1})e_\chi + \sum_{\substack{\chi\in\hat{G}\\ \chi\text{odd}}} L_S(s, \chi^{-1})e_\chi$$

by (10). Now, if χ is even, then $D_v \subset \ker(\chi^{-1})$ for all $v \in S_\infty$ so that $\mathrm{ord}_{s=0}(L_S(s, \chi^{-1})) = r_S(\chi^{-1}) \geq 1$ by Prop. 4.4 (for $\chi = \chi_0$, use $|S| \geq 2$). Thus $\Theta_S^+(0) = 0$ so that the value $\Theta_S(0)$ studied in Sections 6 and 7 is naturally equal to $\Theta_S^-(0)$. In contrast, equations (3) and (2) show that $\mathrm{ord}_{s=1}(L_S(s, \chi^{-1})) = -\delta_{\chi,\chi_0}$ for both odd and even χ. Thus, even if we subtracted the pole due to the trivial character, Θ_S^+ would still make a non-zero contribution to the value of Θ_S at $s = 1$ and one, moreover, which we can expect to be of a very different nature from that of Θ_S^-. Indeed, the functional equation (4) for L-functions and the case $r \geq 1$ of $SC(K/k, S, r)$ mean that the former should – in a vague sense – 'contain non-trivial, transcendental regulators'. On the other hand, since $\tau(\chi)$ is always algebraic, one can use the case $r = 0$ of $SC(K/k, S, r)$ – *i.e.* $\Theta_S^-(0) \in \mathbb{Q}[G]^-$ – and the functional equation to show that $\Theta_S^-(1) \in \pi^n\bar{\mathbb{Q}}[G]^-$, where we recall that $n = [k : \mathbb{Q}]$.
In the rest of this article we shall therefore consider only the *minus-part* $\Theta_S^-(1)$, following [**So1**], [**So2**] and [**RS2**]. More precisely, we shall study

$$b_S^- = b_{S,K/k}^- := (i/\pi)^n\Theta_S^-(1)^* = (i/\pi)^n \lim_{s\to1} e^-\Theta_S(s)^* \in \bar{\mathbb{Q}}[G]^-$$

where $* : \mathbb{C}[G] \to \mathbb{C}[G]$ is the $\mathbb{C}$-linear involution sending $g \in G$ to g^{-1}. The non-vanishing of the L-functions at $s = 1$ shows that b_S^- lies in $(\bar{\mathbb{Q}}[G]^-)^\times$ and the limit (with $s \in \mathbb{R}_{>0}$) shows that it has coefficients in $i^n\mathbb{R}$. We also have have the following more precise algebraicity result.

Proposition 8.1. — *If $\mathfrak{f}_{K/k} \cap \mathbb{Z} = f\mathbb{Z}$ with $f \in \mathbb{Z}_{>0}$ then the coefficients of $\sqrt{d_k}b_S^-$ lie in both $\mathbb{Q}(\zeta_f)$ and the normal closure of K over $\mathbb{Q}$.*

The proof is essentially that of [**RS2**, Prop. 2]. The latter also contains an integrality result for the coefficients and assumes that $n = 2$ and that S is a particular set of places. However, for the properties we want, the proof adapts immediately to our more general situation since $\sqrt{d_k}b_S^-$ equals $\sqrt{d_k}a_{K/k,S}^{-,*} = d_k e^- \prod_{\mathfrak{q}\in S\setminus S_{\min}}(1 - N\mathfrak{q}^{-1}\sigma_\mathfrak{q})\Phi_{K/k}(0)^*$, where undefined notations are as in *ibid.* and [**So2**, eq. (9)].

8.2. A p-adic Logarithmic Map. — Let $p \neq 2$ be a prime number as before, and henceforth take $S = S_p(k) \cup S_{\min}$ where $S_p(F)$ denotes the set of places dividing p in a number field F. We sometimes drop S from the notation. Fixing an embedding $j : \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$, we may apply it to the coefficients of b^- to get $j(b^-) \in \bar{\mathbb{Q}}_p[G]^-$. We shall use this to construct a p-adic map $\mathfrak{s}_{K/k,p}$ on (an exterior power of) the p-semilocal units of K.

For each $\mathfrak{P} \in S_p(K)$, let $K_{\mathfrak{P}}$ denote the completion of K at $\mathfrak{P}$ (containing K). Let $\mathcal{O}_{K_{\mathfrak{P}}}$ denote its ring of valuation integers and let K_p denote the product $\prod_{\mathfrak{P} \in S_p(K)} K_{\mathfrak{P}}$ with the product topology. The diagonal map $K \to K_p$ extends $\mathbb{Q}_p$-linearly to a ring isomorphism $K \otimes \mathbb{Q}_p \cong K_p$ which we regard as an identification. The natural action of G on K therefore gives rise to an action (by continuous ring automorphisms) on K_p which 'mixes up' the factors $K_{\mathfrak{P}}$. We write $U^1(K_p)$ for $\prod_{\mathfrak{P} \in S_p(K)} U^1(K_{\mathfrak{P}}) \subset K_p^\times$ where $U^1(K_{\mathfrak{P}})$ denotes the group of principal units of $K_{\mathfrak{P}}$, *i.e.* $1 + \mathfrak{P}\mathcal{O}_{K_{\mathfrak{P}}}$. The multiplicative $\mathbb{Z}$-action on $U^1(K_p)$ extends by continuity to $\mathbb{Z}_p$ so we may regard $U^1(K_p)$ as a module for $\mathcal{R}_p = \mathbb{Z}_p[G]$. Now, every $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ gives rise to a ring homomorphism $(j \circ \tau) \otimes 1 : K_p \to \bar{\mathbb{Q}}_p$. We write simply $j\tau$ for its restriction to $U^1(K_p)$. This map factors through the projection onto $U^1(K_{\mathfrak{P}})$ (where $\mathfrak{P}$ is determined by $j \circ \tau$) and takes values in the disc $\{x \in \bar{\mathbb{Q}}_p : |x-1|_p < 1\}$ on which the p-adic logarithm $\log_p$, defined by the usual power-series, converges. We may therefore define a 'p-semilocal, G-equivariant logarithm', $\mathrm{Log}_{\tau,p}$, by

$$\begin{array}{rcccl} \mathrm{Log}_{\tau,p} & : & U^1(K_p) & \longrightarrow & \bar{\mathbb{Q}}_p[G] \\ & & u & \longmapsto & \sum_{g \in G} \log_p(j\tau(gu))g^{-1} \end{array}$$

(compare with Log_w of Section 4). It is continuous and $\mathbb{Z}[G]$-linear, so $\mathcal{R}_p$-linear. Now *fix a choice* $\tau_1, \dots, \tau_n$ *of left-coset representatives of* $\mathrm{Gal}(\bar{\mathbb{Q}}/k)$ *in* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. This gives a unique $\mathcal{R}_p$-linear p-semilocal regulator

$$\mathrm{Reg}_p = \mathrm{Reg}_p^{\tau_1,\dots,\tau_n} : \textstyle\bigwedge_{\mathcal{R}_p}^n U^1(K_p) \longrightarrow \bar{\mathbb{Q}}_p[G]$$

sending $u_1 \wedge \ldots \wedge u_n$ to $\det\left(\mathrm{Log}_{\tau_i,p}(u_j)\right)_{i,j=1}^n$. Finally, we make the

Definition 8.2. — Let $\mathfrak{s}_p = \mathfrak{s}_{K/k,p}^{\tau_1,\dots,\tau_n}$ be the $\mathcal{R}_p$-linear map

$$\begin{array}{rcccl} \mathfrak{s}_p & : & \bigwedge_{\mathcal{R}_p}^n U^1(K_p) & \longrightarrow & \bar{\mathbb{Q}}_p[G]^- \\ & & \theta & \longmapsto & j(b^-)\mathrm{Reg}_p(\theta) \end{array}$$

and let $\mathfrak{S}_p = \mathfrak{S}_{K/k,p}$ be its image in $\bar{\mathbb{Q}}_p[G]^-$.

Of course, $\mathfrak{s}_p$ factors through the projection e^- onto $\bigwedge_{\mathcal{R}_p}^n U^1(K_p)^-$. In particular, it vanishes on $\bigwedge_{\mathcal{R}_p}^n U^1(K_p)^+$. Also, changing the choice and ordering of $\tau_1, \dots, \tau_n$ only multiplies $\mathrm{Reg}_p^{\tau_1,\dots,\tau_n}$, and hence $\mathfrak{s}_{K/k,p}$, by $\pm g$ for some $g \in G$, so has no effect on $\mathfrak{S}_{K/k,p}$. The following is proved in [**So2**, Props. 2.16, 2.17] using results of [**So1**].

Proposition 8.3. — *With notations as above,*

(i) $\mathfrak{s}_p$ *is independent of* j *and takes values in* $\mathbb{Q}_p[G]^-$.

(ii) $\mathfrak{S}_p$ *is a fractional ideal of* $\mathbb{Q}_p[G]^-$, *that is, a finitely generated* $\mathbb{Z}_p$*-submodule spanning* $\mathbb{Q}_p[G]^-$ *over* $\mathbb{Q}_p$.

(iii) $\ker(\mathfrak{s}_p) \cap \bigwedge_{\mathcal{R}_p}^n U^1(K_p)^-$ *is precisely the* $\mathbb{Z}_p$*-torsion submodule of* $\bigwedge_{\mathcal{R}_p}^n U^1(K_p)^-$.

Example 8.4. — **The Case $K/k = \mathbb{Q}(\zeta_{p^t})/\mathbb{Q}$ for $t \geq 1$.** In this case $S = \{\infty, p\}$ and $S_p(K) = \{\mathfrak{P}\}$ where $\mathfrak{P} := (1 - \zeta_{p^t})$ is totally ramified over p. Thus j induces an isomorphism $K_p = K_{\mathfrak{P}} \to \mathbb{Q}_p(j(\zeta_{p^t}))$ which we treat as an identification, allowing us also to identify $G = \mathrm{Gal}(K_{\mathfrak{P}}/\mathbb{Q}_p)$ with $\mathrm{Gal}(\mathbb{Q}_p(j(\zeta_{p^t}))/\mathbb{Q}_p)$. From now on, we suppress j and write ζ for ζ_{p^t}, identified with $j(\zeta_{p^t})$. Computing $\Theta^-(1)$ directly (see *e.g.* [**So2**, Lemma 7.1]) we find

$$b^- = e^- \frac{1}{p^t} \sum_{g \in G} g\left(\frac{\zeta}{1-\zeta}\right) g$$

Now take τ_1 to be $1 \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ so that $\mathrm{Log}_{\tau_1,p}(u)$ becomes simply $\sum_{h \in G} \log_p(hu)h^{-1}$ for any $u \in U^1(K_p) = U^1(\mathbb{Q}_p(\zeta))$. Since $n = 1$, this coincides with $\mathrm{Reg}_p(u)$. Therefore, assuming w.l.o.g. that $u \in U^1(K_p)^-$ and multiplying out, we get

$$\mathfrak{s}_p(u) = b^- \mathrm{Reg}_p(u) = \frac{1}{p^t}\Big(\sum_{h' \in G} h'\Big(\frac{\zeta}{1-\zeta}\Big)h'\Big)\Big(\sum_{h \in G} \log_p(hu)h^{-1}\Big) = \sum_{g \in G} a(g(u))g^{-1} \tag{30}$$

where, for any $v \in U^1(\mathbb{Q}_p(\zeta))$ we have set

$$a(v) := \frac{1}{p^t}\mathrm{Tr}_{\mathbb{Q}_p(\zeta)/\mathbb{Q}_p}\left(\frac{\zeta}{1-\zeta}\log_p(v)\right)$$

To take this example further, we observe that the coefficient $a(v)$ appears in the explicit reciprocity law of Artin and Hasse (see [**AH**]). More precisely, the reciprocity map of local class field theory sends any $\alpha \in K_{\mathfrak{P}}^\times = \mathbb{Q}_p(\zeta)^\times$ to an element $s_\alpha = s_{\alpha, K_{\mathfrak{P}}/\mathbb{Q}_p}$, say, of $\mathrm{Gal}(K_{\mathfrak{P}}^{\mathrm{ab}}/K_{\mathfrak{P}})$ where $K_{\mathfrak{P}}^{\mathrm{ab}}$ denotes a given abelian closure. If β also lies in $K_{\mathfrak{P}}^\times$ then any p^tth root β^{1/p^t} lies in $K_{\mathfrak{P}}^{\mathrm{ab}}$ because K, hence $K_{\mathfrak{P}}$, contains the p^tth roots of unity. For our purposes the *Hilbert symbol* $(\alpha, \beta)_{K_{\mathfrak{P}}, p^t}$ can therefore be defined as $s_\alpha(\beta^{1/p^t})/\beta^{1/p^t}$, a p^tth root of unity in K depending only on α and β. Now if v lies in $U^1(\mathbb{Q}_p(\zeta))$ and $a(v)$ is as above, the Artin-Hasse law states firstly that $a(v) \in \mathbb{Z}_p$ and secondly that

$$(1-\zeta, v)_{K_{\mathfrak{P}}, p^t} = \zeta^{-a(v)} \tag{31}$$

The first fact implies $\mathfrak{s}_p(u) \in \mathbb{Z}_p[G]$ for all $u \in U^1(K_p)^-$, in other words

$$\mathfrak{S}_{\mathbb{Q}(\zeta_{p^t})/\mathbb{Q},p} \subset \mathcal{R}_p^- \tag{32}$$

Finally, we note that in this example, results of [**Iw**] allow one to calculate $\mathfrak{S}_p$ exactly. Indeed, it is shown in [**So3**] that

$$\mathfrak{S}_{\mathbb{Q}(\zeta_{p^t})/\mathbb{Q},p} = \mathrm{Stick}(\mathbb{Q}(\zeta_{p^t})/\mathbb{Q})_p$$

where $\mathrm{Stick}(\mathbb{Q}(\zeta_{p^t})/\mathbb{Q})_p$ is as in Section 6. One can also show $\mathrm{Stick}(\mathbb{Q}(\zeta_{p^t})/\mathbb{Q})_p = \mathrm{Fitt}_{\mathcal{R}_p^-}(\mathrm{Cl}(\mathbb{Q}(\zeta_{p^t}))_p^-)$, using [**Ku1**, Thm. 0.5] for example. Thus we get

$$\mathfrak{S}_{\mathbb{Q}(\zeta_{p^t})/\mathbb{Q},p} = \mathrm{Fitt}_{\mathcal{R}_p^-}(\mathrm{Cl}(\mathbb{Q}(\zeta_{p^t}))_p^-) \tag{33}$$

We now consider some possible generalisations of equations (31), (32) and (33).

9. Conjectures at $s = 1$

We maintain the notations and assumptions on K/k, p and S introduced in §8.2.

9.1. The Ideal $\mathfrak{S}_p$: Integrality and Fitting Ideals. — The following generalisation of (32) was conjectured in [**So1**], [**So2**]

Conjecture *$IC(K/k,p)$*. — **Integrality Conjecture (with $S = S_p(k) \cup S_{\min}$)**
In the above situation, $\mathfrak{S}_{K/k,p}$ is contained in $\mathcal{R}_p^-$.

We summarise the current evidence for this conjecture. First, it is proven in [**So1**] whenever p is unramified in K and also whenever p splits completely in k. The latter case is considerably harder than the former and requires a minor auxiliary condition. Next, the conjecture is established in [**So2**] whenever $p \nmid |G|$. Results of A. Jones in [**Jo**] imply that a somewhat stronger statement than $IC(K/k,p)$ follows from a certain case of the ETNC (see below). Since the latter is known whenever K is abelian over $\mathbb{Q}$, the conjecture is then proven unconditionally. (For a direct proof in this case, not using the ETNC but imposing a mild condition if $k \neq \mathbb{Q}$, see [**So2**, Sec. 8].) Finally, as a by-product of the computations in [**RS2**] (see below) one gets actual proofs of $IC(K/k,p)$ in a dozen cases not otherwise covered, all with k real quadratic and $p = 3$ or 5.
We now assume that $IC(K/k,p)$ holds and ask what might replace the finer statement (33) in the general case. The results of Jones mentioned above show that if the ETNC holds (so, in particular, if K is absolutely abelian) then $\mathfrak{S}(K/k)_p$ is contained in $\mathrm{Fitt}_{\mathcal{R}_p^-}(\mathrm{Cl}_{\mathfrak{m}}(K)_p^-)$ where $\mathrm{Cl}_{\mathfrak{m}}(K)$ is a certain ray-class group. For more details and an unconditional result when $p \nmid |G|$, we refer to [**So2**], §4.2, §4.3 and Remark 6.2. The latter hints that the appearance of $\mathrm{Cl}_{\mathfrak{m}}(K)$ – rather than its quotient $\mathrm{Cl}(K)$ – is explained by the imprimitivity of the L-functions making up $\Theta_S^-(1)$. This does not lead to trivial zeroes as it does at $s = 0$ but still suggests enlarging $\mathfrak{S}(K/k)_p$ to $\tilde{\mathfrak{S}}(K/k)_p$, say, using intermediate fields as in (27). The above-mentioned results then prompt the

Question. — *Does one have $\tilde{\mathfrak{S}}(K/k)_p = \mathrm{Fitt}_{\mathcal{R}_p^-}(\mathrm{Cl}(K)_p^-)$ whenever $\mu_{p^\infty}(K) = \mu_{p^\infty}(K_p)^-$?*

Here, $\mu_{p^\infty}(K_p)$ denotes the $\mathcal{R}_p^-$-module $\prod_{\mathfrak{P}} \mu_{p^\infty}(K_{\mathfrak{P}})$. (The given condition therefore fails, for instance, whenever $\mu_p \subset K$ and $|S_p(K^+)| > 1$ and one would like to relax it.) An obvious first test is the extension $K/\mathbb{Q}(\sqrt{29})$ mentioned in Section 7, for which Greither and Kurihara showed $\mathrm{Stick}(K/k)_p \not\subset \mathrm{Fitt}_{\mathcal{R}_p^-}(\mathrm{Cl}(K)_p^-)$ when $p = 3$. Unpublished computations of X. Roblot and the author show that indeed $\tilde{\mathfrak{S}}(K/k)_p = \mathfrak{S}(K/k)_p = \mathrm{Fitt}_{\mathcal{R}_p^-}(\mathrm{Cl}(K)_p^-)$ in this case (so $\mathrm{Stick}(K/k)_p \not\subset \tilde{\mathfrak{S}}(K/k)_p$). However the above question is still open even for general abelian K and $k = \mathbb{Q}$. It is equally possible that some other enlargement of $\mathfrak{S}(K/k)_p$, perhaps analogous to $\mathrm{SKu}(K/k)_p$, should replace $\tilde{\mathfrak{S}}(K/k)_p$.

9.2. The Map $\mathfrak{s}_p$: a Conjectural Explicit Reciprocity Law for η_S. — *Continuing to assume that $IC(K/k,p)$ holds*, we now discuss a refinement in a different direction that makes a connection between $\mathfrak{s}_{K/k,p}$ in the *minus-part* and the conjectural solution of $SC(K^+/k,S,n)$ in the *plus-part* which was discussed in §4 and §5. As motivation, we first take $K/k=\mathbb{Q}(\zeta_{p^t})/\mathbb{Q}$ and reformulate equations (30) and (31) using the notations and conventions of that example. For any $\xi\in\mu_{p^t}$ we write $\mathrm{Ind}_{p^t}(\xi)$ for the unique element $x\in\mathbb{Z}/p^t\mathbb{Z}$ such that $\zeta_{p^t}^x=\xi$. Thus the Hilbert symbol gives rise to a bilinear pairing

$$\begin{array}{rcrcl} & [\cdot,\cdot]_{p^t} & : \; U_S(K^+)\times U^1(K_p) & \longrightarrow & (\mathbb{Z}/p^t\mathbb{Z})[G] \\ (34) & & (\alpha,\beta) & \longmapsto & \displaystyle\sum_{g\in G}\mathrm{Ind}_{p^t}\Big((\alpha,g\beta)_{K_{\mathfrak{P}},p^t}\Big)\,g^{-1} \end{array}$$

This 'extends' naturally so that the first (global) variable may lie in $\mathbb{Z}_{(p)}\otimes U_S(K^+)$ where $\mathbb{Z}_{(p)}$ denotes the local subring $\{a/b: a,b\in\mathbb{Z}, p\nmid b\}$ of $\mathbb{Q}$. Since $U_S(K^+)$ has no p-torsion, $\mathbb{Z}_{(p)}\otimes U_S(K^+)$ injects into $\mathbb{Q}\otimes U_S(K^+)$. Recall from Example 4.2 and Subsection 5.2 that the canonical solution of $SC(K^+/\mathbb{Q},S,1)$ in $\mathbb{Q}\otimes U_S(K^+)$ is

$$\eta_{S,K^+/\mathbb{Q}}=\frac{1}{2}\otimes((1-\zeta_{p^t})(1-\zeta_{p^t}^{-1}))^{-1}$$

where w_1 is given by the inclusion $K^+\hookrightarrow\mathbb{R}$. (The reader can check that this holds even if $p^t=3$ when $K^+=\mathbb{Q}$.) Since $p\neq 2$ we can therefore regard $\eta_{S,K^+/\mathbb{Q}}$ as an element of $\mathbb{Z}_{(p)}\otimes U_S(K^+)$ and one checks easily that equations (30) and (31) amount to the congruence

$$\overline{\mathfrak{s}_p(u)}=[\eta_{S,K^+/\mathbb{Q}},u]_{p^t}\quad\text{in } (\mathbb{Z}/p^t\mathbb{Z})[G],\text{ for all } u\in U^1(K_p) \tag{35}$$

(Use the fact that $(\alpha,\beta)_{K_{\mathfrak{P}},p^t}=1$ if α,β both lie in $U^1(K_p)^-$ or in $U^1(K_p)^+$ so, in particular, both sides of (35) vanish if $u\in U^1(K_p)^+$.)
The Congruence Conjecture of **[So2]** generalises (35) for K/k, p and S as considered in this section with the *additional assumption that* $\mu_{p^t}\subset K$ for some given $t\geq 1$. Thus if $\alpha=(\alpha_{\mathfrak{P}})_{\mathfrak{P}}$ and $\beta=(\beta_{\mathfrak{P}})_{\mathfrak{P}}$ lie in $K_p^\times$ we can regard $(\alpha_{\mathfrak{P}},\beta_{\mathfrak{P}})_{K_{\mathfrak{P}},p^t}\in\mu_{p^t}(K_{\mathfrak{P}})$ as an element of μ_{p^t} for each $\mathfrak{P}$ and define

$$(\alpha,\beta)_{K_p,p^t}:=\prod_{\mathfrak{P}\in S_p(K)}(\alpha_{\mathfrak{P}},\beta_{\mathfrak{P}})_{K_{\mathfrak{P}},p^t}=\prod_{\mathfrak{P}\in S_p(K)}(\beta_{\mathfrak{P}},\alpha_{\mathfrak{P}})^{-1}_{K_{\mathfrak{P}},p^t}\in\mu_{p^t}$$

Remark 9.1. — The second equality comes from the skew-symmetry of the Hilbert symbol. It shows that if α lies in $K^\times$ (regarded as a subgroup of $K_p^\times$ by the diagonal embedding) then $(\alpha,\beta)_{K_p,p^t}=(\psi_\beta(\alpha^{1/p^t})/\alpha^{1/p^t})^{-1}$ where $\alpha^{1/p^t}\in K^{\mathrm{ab}}$ and $\psi_\beta\in\mathrm{Gal}(K^{\mathrm{ab}}/K)$ is the image of β under the composition of the natural embedding $K_p^\times\hookrightarrow\mathrm{Id}(K)$ with the *global* reciprocity map $\mathrm{Id}(K)\to\mathrm{Gal}(K^{\mathrm{ab}}/K)$.

We now take $\alpha\in U_S(K^+)$ and $\beta\in U^1(K_p)$ and define $[\alpha,\beta]_{p^t}\in(\mathbb{Z}/p^t\mathbb{Z})[G]$ just as in (34) but replacing $(\alpha,g\beta)_{K_{\mathfrak{P}},p^t}$ by $(\alpha,g\beta)_{K_p,p^t}$ for each $g\in G$. Let $\bar\kappa$ denote the homomorphism $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})\to(\mathbb{Z}/p^t\mathbb{Z})^\times$ given by $\gamma(\xi)=\xi^{\bar\kappa(\gamma)}$ for all $\xi\in\mu_{p^t}$. The restriction of $\bar\kappa$ to $\mathrm{Gal}(\bar{\mathbb{Q}}/k)$

clearly factors through a homomorphism $G \to (\mathbb{Z}/p^t\mathbb{Z})^\times$ also denoted $\bar\kappa$. One checks that $[\alpha,\beta]_{p^t}$ is ($\mathbb{Z}$-)bilinear and also G-semi-bilinear in the sense that

$$[g\alpha, h\beta]_{p^t} = \bar\kappa(g)g^{-1}h[\alpha,\beta]_{p^t} \quad \text{for all } \alpha \in U_S(K^+), \beta \in U^1(K_p) \text{ and } g,h \in G \tag{36}$$

Taking $g = h = c$, one deduces $[\alpha, e^-\beta]_{p^t} = [\alpha,\beta]_{p^t}$. Writing G^+ for $\mathrm{Gal}(K^+/k) = G/\langle c\rangle$, it also follows from (36) that there exists a unique pairing

$$\mathcal{H}_{K/k,p^t} : \bigwedge\nolimits^n_{\mathbb{Z}[G^+]} U_S(K^+) \times \bigwedge\nolimits^n_{\mathcal{R}_p} U^1(K_p) \longrightarrow (\mathbb{Z}/p^t\mathbb{Z})[G]^-$$

satisfying $\mathcal{H}_{K/k,p^t}(\varepsilon_1\wedge\ldots\wedge\varepsilon_n, u_1\wedge\ldots\wedge u_n) = \det\big([\varepsilon_i, u_l]_{p^t}\big)^n_{i,l=1}$ for any $\varepsilon_1,\ldots,\varepsilon_n \in U_S(K^+)$ and $u_1,\ldots,u_n \in U^1(K_p)$. Note also that $\mathcal{H}_{K/k,p^t}$ is $\mathcal{R}_p$-linear in the second variable and factors through the projection e^- onto $\bigwedge^n_{\mathcal{R}_p} U^1(K_p)^-$, just like $\mathfrak{s}_{K/k,p}$. As before, we extend it naturally so that the first variable may lie in $\mathbb{Z}_{(p)} \otimes \bigwedge^n_{\mathbb{Z}[G^+]} U_S(K^+)$.

For each $i = 1,\ldots,n$ the restriction of τ_i to K^+ corresponds to real place w_i lying above a distinct place v_i of k which splits in K^+. Since $S \supset S_p(k)\cup S_\infty(k) = S_p(k)\cup\{v_1,\ldots,v_n\}$, the hypotheses of Conjecture $SC(K^+/k,S,n)$ are satisfied. *We shall assume that it holds and has canonical solution $\eta_S \in \mathbb{Q}\otimes\bigwedge^n_{\mathbb{Z}[G^+]} U_S(K^+)$ w.r.t. the choice of places $w_1,\ldots,w_n$.* We would like to apply $\mathcal{H}_{K/k,p^t}$ to η_S but for $n \geq 2$ we cannot simply treat the latter as an element of $\mathbb{Z}_{(p)}\otimes\bigwedge^n_{\mathbb{Z}[G^+]} U_S(K^+)$. Indeed, if also $p \| G^+|$ then the map $\nu_S : \mathbb{Z}_{(p)}\otimes\bigwedge^n_{\mathbb{Z}[G^+]} U_S(K^+) \to \mathbb{Q}\otimes\bigwedge^n_{\mathbb{Z}[G^+]} U_S(K^+)$ may not be injective. Furthermore, in these circumstances Rubin's Conjecture B' does not imply $\eta_S \in \mathrm{im}(\nu_S)$ (see §5.3). To get round this, we define in [**So2**, § 2.2] a certain lattice $\Lambda_{0,S} = \Lambda_{0,S}(K^+/k) \subset \mathbb{Q}\otimes\bigwedge^n_{\mathbb{Q}[G^+]} U_S(K^+)$ such that $\mathbb{Z}_{(p)}\Lambda_{0,S}$ contains $\mathrm{im}(\nu_S)$, and also a natural 'extension' $H_{K/k,p^t}$ (there denoted '$H_{K/k,S,n}$') of the pairing $\mathcal{H}_{K/k,p^t}$ to $\mathbb{Z}_{(p)}\Lambda_{0,S}\times\bigwedge^n_{\mathcal{R}_p} U^1(K_p)$ such that for each $\theta \in \bigwedge^n_{\mathcal{R}_p} U^1(K_p)$ the diagram

(37)
$$\begin{tikzcd}
\mathbb{Z}_{(p)}\otimes\bigwedge^n_{\mathbb{Z}[G^+]} U_S(K^+) \arrow[d, "\nu_S"'] \arrow[dr, "\mathcal{H}_{K/k,p^t}(\cdot,\theta)"] & \\
\mathbb{Z}_{(p)}\Lambda_{0,S} \arrow[r, "H_{K/k,p^t}(\cdot,\theta)"'] & (\mathbb{Z}/p^t\mathbb{Z})[G]^-
\end{tikzcd}$$

commutes. (This follows from [**So2**, eq. (20)]. Note that the vertical map is an isomorphism whenever $p \nmid |G|$.) Finally, we show in [**So2**, Rem. 2.8] that the full version of Rubin's Conjecture B' (for varying auxiliary sets T) implies that η_S *lies in* $\mathbb{Z}_{(p)}\Lambda_{0,S}$ (in fact, in $\frac{1}{2}\Lambda_{0,S}$). We can at last state the

Conjecture $CC(K/k,p^t)$. — **Congruence Conjecture (with $S = S_p(k)\cup S_{\min}$)**
Suppose that K/k, p and S are as in §8.2 and in addition that

(i) *$IC(K/k,p)$ holds,*
(ii) *$\mu_{p^t} \subset K$ for some $t \geq 1$,*

(iii) *$SC(K^+/k, S, n)$ holds and the canonical solution η_S w.r.t. the choice of places $w_1, \ldots, w_n$ lies in $\mathbb{Z}_{(p)}\Lambda_{0,S}$.*
Then, for all $\theta \in \bigwedge^n_{\mathcal{R}_p} U^1(K_p)$, we have the congruence

$$\overline{\mathfrak{s}_p(\theta)} = \bar{\kappa}(\tau_1 \ldots \tau_n) H_{K/k,p^t}(\eta_S, \theta) \quad \textit{in } (\mathbb{Z}/p^t\mathbb{Z})[G]^- \tag{38}$$

We recall that the choice of the coset representatives $\tau_1, \ldots, \tau_n$ affects both the definition of $\mathfrak{s}_p$ and of the places $w_1, \ldots, w_n$, hence of η_S. However, one checks easily that the factor $\bar{\kappa}(\tau_1 \ldots \tau_n)$ in (38) makes the Conjecture independent of this choice.
The motivation of both the Integrality and Congruence Conjectures came from the author's article in [**BPSS**] although neither is explicitly mentioned there. Their statements appear first in [**So1**] but that of the latter conjecture is rather awkward. An improved formulation appears in [**So2**] for any S *containing* $S_p(k) \cup S_{\min}$ but it is shown in Prop. 5.4 *loc. cit.* that this is implied by the special case $S = S_p(k) \cup S_{\min}$ which is all we have given above. We summarise the current evidence for $CC(K/k, p^t)$. Firstly, no connection with the ETNC is known but in [**So2**] a generalisation of the Artin-Hasse law due to Coleman is shown to imply the CC in the case $k = \mathbb{Q}$, and also for K absolutely abelian (but with the same mild condition as for the IC if $k \neq \mathbb{Q}$). It is also shown that the congruence (38) holds as '$0 = 0$' whenever $p \nmid |G|$ and θ is a $\mathbb{Z}_p$-torsion element. M. Bovey considered the case $k = K^+$, *i.e.* $|G| = 2$. The CC is then trivial unless $|S_p(k)| = 1$. In this case conditions (i) and (iii) hold and both the map $\mathfrak{s}_p$ and the element η_S can be written down in terms of certain S-class-numbers and S-units of K and k *etc.* (See Rem. 4.5 for η_S.) Nevertheless, the congruence (38) seems to be new and unknown. A weakening of it is proven in [**Bo**]. A variant with $p = 2$ is also stated and one congruence or the other is fully numerically verified in over 100 cases. Finally, in the case where k is real quadratic but $K^+/\mathbb{Q}$ is not abelian, one cannot usually prove $SC(K^+/k, 2)$ but high-precision computation allows one to identify the solution η_S with virtual certainty. This was done in [**RS2**] allowing the verification of $IC(K/k, p)$ and $CC(K/k, p^t)$ in nearly 50 such cases with varying p and $t = 1$ or 2.

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