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THE NUMBER OF LARGE PRIME FACTORS OF INTEGERS AND NORMAL NUMBERS

by

Jean-Marie De Koninck and Imre Kátai

Abstract. — In a series of papers, we constructed large families of normal numbers using the concatenation of the values of the largest prime factor $P(n)$, as n runs through particular sequences of positive integers. A similar approach using the smallest prime factor function also allowed for the construction of normal numbers. Letting $\omega(n)$ stand for the number of distinct prime factors of the positive integer n, we then showed that the concatenation of the successive values of $|\omega(n) - \lfloor \log\log n \rfloor|$ in a fixed base $q \geq 2$, as n runs through the integers $n \geq 3$, yields a normal number. Here we prove the following. Let $q \geq 2$ be a fixed integer. Given an integer $n \geq n_0 = \max(q, 3)$, let N be the unique positive integer satisfying $q^N \leq n < q^{N+1}$ and let $h(n, q)$ stand for the residue modulo q of the number of distinct prime factors of n located in the interval $[\log N, N]$. Setting $x_N := e^N$, we then create a normal number in base q using the concatenation of the numbers $h(n, q)$, as n runs through the integers $\geq x_{n_0}$.

Résumé. — Dans une série d'articles, nous avons construit de grandes familles de nombres normaux en utilisant la concaténation des valeurs successives du plus grand facteur premier $P(n)$, où n parcourt certaines suites d'entiers positifs. Une approche similaire en utilisant la fonction plus petit facteur premier nous a aussi permis de construire d'autres familles de nombres normaux. En désignant par $\omega(n)$ le nombre de nombres premiers distincts de n, nous avons montré que la concaténation des valeurs successives de $|\omega(n) - \lfloor \log\log n \rfloor|$ dans une base fixe $q \geq 2$, où n parcourt les entiers $n \geq 3$, donne place à un nombre normal. Ici, nous démontrons le résultat suivant. Soit $q \geq 2$ un entier fixe. Étant donné un entier $n \geq n_0 = \max(q, 3)$, soit N l'unique entier positif satisfaisant $q^N \leq n < q^{N+1}$ et désignons par $h(n, q)$ le résidu modulo q du nombre de facteurs premiers distincts de n situés dans l'intervalle $[\log N, N]$. En posant $x_N := e^N$, nous créons alors un nombre normal dans la base q en utilisant la concaténation des nombres $h(n, q)$, où n parcourt les entiers $\geq x_{n_0}$.

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1. Introduction

Given an integer $q \geq 2$, we say that an irrational number η is a *q-normal number* if the q-ary expansion of η is such that any preassigned sequence of length $k \geq 1$, taken within this expansion, occurs with the expected limiting frequency, namely $1/q^k$.

Even though constructing specific normal numbers is a very difficult problem, several authors picked up this challenge. One of the first was Champernowne [2] who, in 1933, showed that the number made up of the concatenation of the natural numbers, namely the number

$$0.1234567891011121314151617181920 21\ldots,$$

is normal in base 10. In 1946, Copeland and Erdős [4] proved that the same is true if one replaces the sequence of natural numbers by the sequence of primes, namely for the number

$$0.23571113171923293137\ldots$$

In the same paper, they conjectured that if $f(x)$ is any nonconstant polynomial whose values at $x = 1, 2, 3, \ldots$ are positive integers, then the decimal $0.f(1)f(2)f(3)\ldots$, where $f(n)$ is written in base 10, is a normal number. Six years later, Davenport and Erdős [5] proved this conjecture.

Since then, many others have constructed various families of normal numbers. To name only a few, let us mention Nakai and Shiokawa [15], Madritsch, Thuswaldner and Tichy [14] and finally Vandehey [17]. More examples of normal numbers as well as numerous references can be found in the recent book of Bugeaud [1].

In a series of papers, we also constructed large families of normal numbers using the distribution of the values of $P(n)$, the largest prime factor function (see [6], [7], [8] and [9]). Recently in [10], we showed how the concatenation of the successive values of the smallest prime factor $p(n)$, as n runs through the positive integers, can also yield a normal number. Letting $\omega(n)$ stand for the number of distinct prime factors of the positive integer n, we then showed that the concatenation of the successive values of $|\omega(n) - \lfloor \log\log n \rfloor|$ in a fixed base $q \geq 2$, as n runs through the integers $n \geq 3$, yields a normal number.

Given an integer $N \geq 1$, for each integer $n \in J_N := (e^N, e^{N+1})$, let $q_N(n)$ be the smallest prime factor of n which is larger than N; if no such prime factor exists, set $q_N(n) = 1$. Fix an integer $Q \geq 3$ and consider the function $f(n) = f_Q(n)$ defined by $f(n) = \ell$ if $n \equiv \ell \pmod{Q}$ with $(\ell, Q) = 1$ and by $f(n) = \epsilon$ otherwise, where ϵ stands for the empty word. Then consider the sequence $(\kappa(n))_{n\geq 3} = (\kappa_Q(n))_{n\geq 3}$ defined by $\kappa(n) = f(q_N(n))$ if $n \in J_N$ with $q_N(n) > 1$ and by $\kappa(n) = \epsilon$ if $n \in J_N$ with $q_N(n) = 1$. Then, given an integer $N \geq 1$ and writing $J_N = \{j_1, j_2, j_3, \ldots\}$, consider the concatenation of the numbers $\kappa(j_1), \kappa(j_2), \kappa(j_3), \ldots$, that is define

$$\theta_N := \text{Concat}(\kappa(n) : n \in J_N) = 0.\kappa(j_1)\kappa(j_2)\kappa(j_3)\ldots.$$

Then, set $\alpha_Q := \text{Concat}(\theta_N : N = 1, 2, 3, \ldots)$ and let $B_Q = \{\ell_1, \ell_2, \ldots, \ell_{\varphi(Q)}\}$ be the set of reduced residues modulo Q, where φ stands for the Euler function. In [11], we showed that α_Q is a normal sequence over B_Q, that is, the real number $0.\alpha_Q$ is a normal number over B_Q.

Here we prove the following. Let $q \geq 2$ be a fixed integer. Given an integer $n \geq n_0 = \max(q, 3)$, let N be the unique positive integer satisfying $q^N \leq n < q^{N+1}$ and let $h(n, q)$ stand for the residue modulo q of the number of distinct prime factors of n located in the interval $[\log N, N]$. Setting $x_N := e^N$, we then create a normal number in base q using the concatenation of the numbers $h(n, q)$, as n runs through the integers $\geq x_{n_0}$.

2. The main result

Theorem 2.1. — *Let $q \geq 2$ be a fixed integer. Given an integer $n \geq n_0 = \max(q,3)$, let N be the unique positive integer satisfying $q^N \leq n < q^{N+1}$ and let $h(n,q)$ stand for the residue modulo q of the number of distinct prime factors of n located in the interval $[\log N, N]$. For each integer $N \geq 1$, set $x_N := e^N$. Then, Concat$(h(n,q) : x_{n_0} \leq n \in \mathbb{N})$ is a q-ary normal sequence.*

Proof. — For each integer $N \geq 1$, let $J_N = (x_N, x_{N+1})$. Further let S_N stand for the set of primes located in the interval $[\log N, N]$ and T_N for the product of the primes in S_N. Let $n_0 = \max(q,3)$. Given a large integer N, consider the function

$$f(n) = f_N(n) = \sum_{\substack{p \mid n \\ \log N \leq p \leq N}} 1. \tag{1}$$

Let us further introduce the following sequences:

$$\begin{aligned} U_N &= \text{Concat}\left(h(n,q) : n \in J_N\right), \\ V_\infty &= \text{Concat}\left(U_N : N \geq n_0\right) = \text{Concat}\left(h(n,q) : n \geq x_{n_0}\right), \\ V_x &= \text{Concat}\left(h(n,q) : x_{n_0} \leq n \leq x\right). \end{aligned}$$

Let us set $A_q := \{0, 1, \ldots, q-1\}$. If we fix an arbitrary integer r, it is sufficient to prove that given any particular word $w \in A_q^r$, the number of occurrences $F_w(V_x)$ of w in V_x satisfies

$$F_w(V_x) = (1 + o(1))\frac{x}{q^r} \qquad (x \to \infty). \tag{2}$$

For each integer $r \geq 1$, considering the polynomial

$$Q_r(u) = u(u+1)\cdots(u+r-1).$$

and letting

$$\rho_r(d) = \#\{u \bmod d : Q_r(u) \equiv 0 \bmod d\},$$

it is clear that, since N is large,

$$\rho_r(p) = r \qquad \text{if } p \in S_N. \tag{3}$$

Observe that it follows from the Turán-Kubilius Inequality (see for instance Theorem 7.1 in the book of De Koninck and Luca [12]), that for some positive constant C,

$$\sum_{n \in J_N} (f(n) - \log\log N)^2 \leq Ce^N \log\log N. \tag{4}$$

Letting $\varepsilon_N = 1/\log\log\log N$, it follows from (4) that

$$\frac{1}{x_N}\#\{n \in J_N : |f(n) - \log\log N| > \frac{1}{\varepsilon_N}\sqrt{\log\log N}\} \to 0 \qquad (N \to \infty). \tag{5}$$

This means that in the estimation of $F_w(V_x)$, we may ignore those integers n appearing in the concatenation $h(2,q)h(3,q)\ldots h(\lfloor x \rfloor, q)$ for which the corresponding $f(n)$ is "far" from $\log\log N$ in the sense described in (5).

Let X be a large number. Then there exists a large integer N such that $\dfrac{X}{e} < x_N \leq X$. Letting $\mathscr{L} = \left]\dfrac{X}{e}, X\right]$, we write

$$\mathscr{L} = \left]\frac{X}{e}, x_N\right] \cup]x_N, X] = \mathscr{L}_1 \cup \mathscr{L}_2,$$

say, and $\lambda(\mathscr{L}_i)$ for the length of the interval $\mathscr{L}_i$ for $i = 1, 2$.
Given an arbitrary function δ_N which tends to 0 arbitrarily slowly, it is sufficient to consider those $\mathscr{L}_1$ and $\mathscr{L}_2$ such that

$$(6) \qquad \lambda(\mathscr{L}_1) \geq \delta_N X \quad \text{and} \quad \lambda(\mathscr{L}_2) \geq \delta_N X.$$

The reason for this is that those $n \in \mathscr{L}_1$ (resp. $n \in \mathscr{L}_2$) for which $\lambda(\mathscr{L}_1) < \delta_N X$ (resp. $\lambda(\mathscr{L}_2) < \delta_N X$) are $o(x)$ in number and can therefore be ignored in the proof of (2).

Let us first consider the set $\mathscr{L}_2$. We start by observing that any subword taken in the concatenation $h(n,q)h(n+1,q)\dots h(n+r-1,q)$ is made of co-prime divisors of T_N (since no two members of the sequence $h(n,q), h(n+1,q), \dots, h(n+r-1,q)$ of r elements may have a common prime divisor $p > \log N$). So, let $d_0, d_1, \dots, d_{r-1}$ be co-prime divisors of T_N and let $B_N(\mathscr{L}_2; d_0, d_1, \dots, d_{r-1})$ stand for the number of those $n \in \mathscr{L}_2$ for which $d_j \mid n + j$ for $j = 0, 1, \dots, r-1$ and such that $\left(Q_r(n), \frac{T_N}{d_0 d_1 \cdots d_{r-1}}\right) = 1$. We can assume that each of the d_j's is squarefree, since the number of those $n+j \leq X$ for which $p^2 \mid n+j$ for some $p > \log N$ is $\ll X \sum_{p>\log N} \frac{1}{p^2} = o(X)$.
In light of (4), we may assume that

$$(7) \qquad \omega(d_j) \leq 2 \log\log N \qquad \text{for } j = 0, 1, \dots, r-1.$$

By using the Eratosthenian sieve (see for instance the book of De Koninck and Luca [12]) and recalling that condition (6) ensures that $X - x_N$ is large, we obtain that, as $N \to \infty$,

$$\begin{aligned} B_N(\mathscr{L}_2; d_0, d_1, \dots, d_{r-1}) &= \frac{X - x_N}{d_0 d_1 \cdots d_{r-1}} \prod_{p \mid T_N/(d_0 d_1 \cdots d_{r-1})} \left(1 - \frac{r}{p}\right) \\ (8) \qquad &\quad + o\left(\frac{x_N}{d_0 d_1 \cdots d_{r-1}} \prod_{p \mid T_N/(d_0 d_1 \cdots d_{r-1})} \left(1 - \frac{r}{p}\right)\right). \end{aligned}$$

Letting $\theta_N := \prod_{p \mid T_N} \left(1 - \frac{r}{p}\right)$, one can easily see that

$$(9) \qquad \theta_N = (1 + o(1)) \frac{(\log\log N)^r}{(\log N)^r} \qquad (N \to \infty).$$

Let us also introduce the strongly multiplicative function κ defined on primes p by $\kappa(p) = p - r$. Then, (8) can be written as

$$(10) \quad B_N(\mathscr{L}_2; d_0, d_1, \dots, d_{r-1}) = \frac{X - x_N}{\kappa(d_0)\kappa(d_1)\cdots\kappa(d_{r-1})} \theta_N + o\left(\frac{x_N}{\kappa(d_0)\kappa(d_1)\cdots\kappa(d_{r-1})} \theta_N\right)$$

as $N \to \infty$. For each integer $N > e^e$, let

$$R_N := \left[\log\log N - \frac{\sqrt{\log\log N}}{\varepsilon_N}, \log\log N + \frac{\sqrt{\log\log N}}{\varepsilon_N}\right].$$

Let $\ell_0, \ell_1, \ldots, \ell_{r-1}$ be an arbitrary collection of non negative integers $< q$. Note that there are q^r such collections. Our goal is to count how many times, amongst the integers $n \in \mathscr{L}_2$, we have $f(n+j) \equiv \ell_j \pmod{q}$ for $j = 0, 1, \ldots, r-1$. In light of (5), we only need to consider those $n \in \mathscr{L}_2$ for which

$$f(n+j) \in R_N \qquad (j = 0, 1, \ldots, r-1).$$

Let

$$\mathscr{S}(\ell_0, \ell_1, \ldots, \ell_{r-1}) := \sideset{}{^*}\sum_{\substack{f(d_j)\equiv \ell_j \bmod q \\ d_j | T_N \\ j=0,1,\ldots,r-1}} \frac{1}{\kappa(d_0)\kappa(d_1)\cdots\kappa(d_{r-1})}, \tag{11}$$

where the star over the sum indicates that the summation runs only on those d_j satisfying $f(d_j) \in R_N$ for $j = 0, 1, \ldots, r-1$.
From (10), we therefore obtain that

$$\begin{aligned} &\#\{n \in \mathscr{L}_2 : f(n+j) \equiv \ell_j \bmod q,\ j = 0, 1, \ldots, r-1\} \\ &\qquad = (X - x_N)\theta_N \mathscr{S}(\ell_0, \ell_1, \ldots, \ell_{r-1}) + o\left(x_N \theta_N \mathscr{S}(\ell_0, \ell_1, \ldots, \ell_{r-1})\right) \end{aligned} \tag{12}$$

as $N \to \infty$. Let us now introduce the function

$$\eta = \eta_N = \sum_{p | T_N} \frac{1}{\kappa(p)}.$$

Observe that, as $N \to \infty$,

$$\begin{aligned} \eta &= \sum_{\log N \le p \le N} \frac{1}{p(1 - r/p)} = \sum_{\log N \le p \le N} \frac{1}{p} + O\left(\sum_{\log N \le p \le N} \frac{1}{p^2}\right) \\ &= \log\log N - \log\log\log N + o(1) + O\left(\frac{1}{\log N}\right) \\ &= \log\log N - \log\log\log N + o(1). \end{aligned} \tag{13}$$

From the definition (11), one easily sees that

$$\mathscr{S}(\ell_0, \ell_1, \ldots, \ell_{r-1}) = (1 + o(1)) \sum_{\substack{t_j \equiv \ell_j \bmod q \\ t_j \in R_N}} \frac{\eta^{t_0 + t_1 + \cdots + t_{r-1}}}{t_0! t_1! \cdots t_{r-1}!} \qquad (N \to \infty), \tag{14}$$

where we ignore in the denominator of the summands the factors $\kappa(p)^a$ (with $a \ge 2$) since their contribution is negligible.
Moreover, for $t \in R_N$, one can easily establish that

$$\frac{\eta^{t+1}}{(t+1)!} = (1 + o(1)) \frac{\eta^t}{t!} \qquad (N \to \infty)$$

and consequently that, for each $j \in \{0, 1, \ldots, r-1\}$,

$$\sum_{\substack{t_j \equiv \ell_j \bmod q \\ t_j \in R_N}} \frac{\eta^{t_j}}{t_j!} = (1 + o(1)) \frac{1}{q} \sum_{t \in R_N} \frac{\eta^t}{t!} = (1 + o(1)) \frac{e^\eta}{q} \qquad (N \to \infty). \tag{15}$$

Using (15) in (14), we obtain that

$$\mathscr{S}(\ell_0, \ell_1, \ldots, \ell_{r-1}) = (1 + o(1))\frac{e^{\eta r}}{q^r} \qquad (N \to \infty). \tag{16}$$

Combining (12) and (16), we obtain that

$$\begin{aligned} \#\{n \in \mathscr{L}_2 : f(n+j) \equiv \ell_j \bmod q,\ j = 0, 1, \ldots, r-1\} \\ &= (X - x_N)\theta_N \frac{e^{\eta r}}{q^r} + o\left(x_N \theta_N \frac{e^{\eta r}}{q^r}\right) \\ &= \frac{X - x_N}{q^r} + o\left(x_N \frac{1}{q^r}\right) \qquad (N \to \infty), \end{aligned} \tag{17}$$

where we used (9) and (13).
Since the first term on the right hand side of (17) does not depend on the particular collection $\ell_0, \ell_1, \ldots, \ell_{r-1}$, we may conclude that the frequency of those integers $n \in \mathscr{L}_2$ for which $f(n+j) \equiv \ell_j \pmod q$ for $j = 0, 1, \ldots, r-1$ is the same independently of the choice of $\ell_0, \ell_1, \ldots, \ell_{r-1}$.

The case of those $n \in \mathscr{L}_1$ can be handled in a similar way.
We have thus shown that the number of occurrences of any word $w \in A_q^r$ in $h(n,q)h(n+1,q)\ldots h(n+r-1,q)$ as n runs over the $\lfloor X - X/e \rfloor$ elements of $\mathscr{L}$ is $(1+o(1))\dfrac{(X - X/e)}{q^r}$.
Repeating this for each of the intervals

$$\left]\frac{X}{e^{j+1}}, \frac{X}{e^j}\right] \qquad (j = 0, 1, \ldots, \lfloor \log x \rfloor),$$

we obtain that the number of occurrences of w for $n \le x$ is $(1+o(1))\dfrac{x}{q^r}$, as claimed.

The proof of (2) is thus complete and the Theorem is proved.

□

3. Final remarks

First of all, let us first mention that our main result can most likely be generalized in order that the following statement will be true:

> Let $a(n)$ and $b(n)$ be two monotonically increasing sequences of n for $n = 1, 2, \ldots$ such that $n/b(n)$, $b(n)/a(n)$ and $a(n)$ all tend to infinity monotonically as $n \to \infty$. Let $f(n)$ stand for the number of prime divisors of n located in the interval $[a(n), b(n)]$ and let $h(n,q)$ be the residue of $f(n)$ modulo q; then, the sequence $h(n,q)$, $n = 1, 2, ...,$ is a q-ary normal sequence.

Secondly, let us first recall that it was proven by Pillai [16] (with a more general result by Delange [13]) that the values of $\omega(n)$ are equally distributed over the residue classes modulo q for every integer $q \ge 2$, and that the same holds for the function $\Omega(n)$, where $\Omega(n) := \sum_{p^\alpha \| n} \alpha$. We believe that each of the sequences $\mathrm{Concat}(\omega(n) \pmod q : n \in \mathbb{N})$ and $\mathrm{Concat}(\Omega(n) \pmod q : n \in \mathbb{N})$ represents a normal sequence for each base $q = 2, 3, \ldots$. However, the proof of these statements could be very difficult to obtain. Indeed, in the particular case $q = 2$, such a result would imply the famous Chowla conjecture

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} \lambda(n)\lambda(n+a_1)\cdots\lambda(n+a_k) = 0,$$

where $\lambda(n) := (-1)^{\Omega(n)}$ is known as the Liouville function and where $a_1, a_2, \ldots, a_k$ are k distinct positive integers (see Chowla [3]).

Thirdly, we had previously conjectured that, given any integer $q \ge 2$ and letting $\mathrm{res}_q(n)$ stand for the residue of n modulo q, it may not be possible to create an infinite sequence of positive integers $n_1 < n_2 < \cdots$ such that

$$0.\mathrm{Concat}(\mathrm{res}_q(n_j) : j = 1, 2, \ldots)$$

is a q-normal number. However, we now have succeeded in creating such a monotonic sequence. It goes as follows. Let us define the sequence $(m_k)_{k\ge1}$ by

$$m_k = f(k) + k!,$$

where f is the function defined in (1). In this case, we obtain that

$$m_{k+1} - m_k = k! \cdot k + f(k+1) - f(k),$$

a quantity which is positive for all integers $k \ge 1$ provided

$$f(k+1) - f(k) > -k! \cdot k, \tag{18}$$

that is if

$$f(k) < k! \cdot k. \tag{19}$$

But since we trivially have

$$f(k) \le \omega(k) \le 2\log k \le k! \cdot k,$$

then (19) follows and therefore (18) as well.
Hence, in light of Theorem 2.1, if we choose $n_k = m_k$, our conjecture is disproved.

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References

[1] Y. Bugeaud, *Distribution modulo one and Diophantine approximation*. Cambridge Tracts in Mathematics, 193. Cambridge University Press, Cambridge, 2012. xvi+300 pp.

[2] D. G. Champernowne, *The construction of decimals normal in the scale of ten*, J. London Math. Soc. **8** (1933), 254-260.

[3] S. Chowla, *The Riemann Hypothesis and Hilbert's Tenth Problem*, Gordon and Breach, New York, 1965.

[4] A. H. Copeland and P. Erdős, *Note on normal numbers*, Bull. Amer. Math. Soc. **52** (1946), 857-860.

[5] H. Davenport and P. Erdős, *Note on normal decimals*, Can. J. Math. **4** (1952), 58-63.

[6] J. M. De Koninck and I. Kátai, *On a problem on normal numbers raised by Igor Shparlinski*, Bulletin of the Australian Mathematical Society **84** (2011), 337–349.

[7] J. M. De Koninck and I. Kátai, *Some new methods for constructing normal numbers*, Annales des Sciences Mathématiques du Québec **36** (2012), no.2, 349–359.

[8] J. M. De Koninck and I. Kátai, *Construction of normal numbers using the distribution of the k-th largest prime factor*, Bull. Australian Mathematical Society **88** (2013), 158-168.

[9] J. M. De Koninck and I. Kátai, *Using large prime divisors to construct normal numbers*, Annales Univ. Sci. Budapest, Sect. Comput. **39** (2013), 45-62.

[10] J. M. De Koninck and I. Kátai, *Normal numbers generated using the smallest prime factor function*, Annales mathématiques du Québec (to appear).

[11] J. M. De Koninck and I. Kátai, *Constructing normal numbers using residues of selective prime factors of integers*, Ann. Univ. Sci. Budapest, Sect. Comp., Volume 42, 2014 (to appear).

[12] J. M. De Koninck and F. Luca, *Analytic Number Theory: Exploring the Anatomy of Integers*, Graduate Studies in Mathematics, Vol. 134, American Mathematical Society, Providence, Rhode Island, 2012.

[13] H. Delange, *Sur la distribution des valeurs de certaines fonctions arithmétiques*, Colloque sur la Théorie des Nombres, Bruxelles, 1955, pp. 147–161. Georges Thone, Liège; Masson and Cie, Paris, 1956.

[14] M. G. Madritsch, J. M. Thuswaldner and R. F. Tichy, *Normality of numbers generated by the values of entire functions*, J. of Number Theory **128** (2008), 1127-1145.

[15] Y. Nakai and I. Shiokawa, *Normality of numbers generated by the values of polynomials at primes*, Acta Arith. **81** (1997), no. 4, 345-356.

[16] S. S. Pillai, *Generalisation of a theorem of Mangoldt*, Proc. Indian Acad. Sci., Sect. A. **11** (1940), 13–20.

[17] J. Vandehey, *The normality of digits in almost constant additive functions*, Monatshefte für Mathematik, 2012, 1-17.

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