

# Publications mathématiques de Besançon

## ALGÈBRE ET THÉORIE DES NOMBRES

Christian Wuthrich

**The sub-leading coefficient of the  $L$ -function of an elliptic curve**

2016, p. 95-96.

<[http://pmb.cedram.org/item?id=PMB\\_2016\\_\\_\\_\\_95\\_0](http://pmb.cedram.org/item?id=PMB_2016____95_0)>

© Presses universitaires de Franche-Comté, 2016, tous droits réservés.

L'accès aux articles de la revue « Publications mathématiques de Besançon » (<http://pmb.cedram.org/>), implique l'accord avec les conditions générales d'utilisation (<http://pmb.cedram.org/legal/>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

*Publication éditée par le laboratoire de mathématiques  
de Besançon, UMR 6623 CNRS/UFC*

cedram

*Article mis en ligne dans le cadre du  
Centre de diffusion des revues académiques de mathématiques  
<http://www.cedram.org/>*

---

## THE SUB-LEADING COEFFICIENT OF THE $L$ -FUNCTION OF AN ELLIPTIC CURVE

*by*

Christian Wuthrich

---

**Abstract.** — We show that there is a relation between the leading term at  $s = 1$  of an  $L$ -function of an elliptic curve defined over an number field and the term that follows.

**Résumé.** — On montre une relation entre le terme dominant de la série  $L$  en  $s = 1$  d'une courbe elliptique définie sur un corps de nombre et le terme suivant.

Let  $E$  be an elliptic curve defined over a number field  $K$ . We will assume that the  $L$ -function  $L(E, s)$  admits an analytic continuation to  $s = 1$  and that it satisfies the functional equation. By modularity [1], we know that this holds when  $K = \mathbb{Q}$ . The conjecture of Birch and Swinnerton-Dyer predicts that the behaviour at  $s = 1$  is linked to arithmetic information. More precisely, if

$$L(E, s) = a_r (s - 1)^r + a_{r+1} (s - 1)^{r+1} + \dots$$

is the Taylor expansion at  $s = 1$  with  $a_r \neq 0$ , then  $r$  should be the rank of the Mordell-Weil group  $E(K)$  and the leading term  $a_r$  is equal to a precise formula involving the Tate-Shafarevich group of  $E$ . It seems to have passed unnoticed that the sub-leading coefficient  $a_{r+1}$  is also determined by the following formula.

**Theorem 1.** — *With the above assumption, we have the equality*

$$(1) \quad a_{r+1} = \left( [K : \mathbb{Q}] \cdot (\gamma + \log(2\pi)) - \frac{1}{2} \log(N) - \log |\Delta_K| \right) \cdot a_r$$

where  $\gamma = 0.577216\dots$  is Euler's constant,  $N$  is the absolute norm of the conductor ideal of  $E/K$  and  $\Delta_K$  is the absolute discriminant of  $K/\mathbb{Q}$ .

In particular, the conjecture of Birch and Swinnerton-Dyer also predicts completely what the sub-leading coefficient  $a_{r+1}$  should be. One consequence for  $K = \mathbb{Q}$  is that for all curves with conductor  $N > 125$ , and this is all but 404 isomorphism classes of curves, the sign of  $a_{r+1}$  is the opposite of  $a_r$ . Of course, it is believed that  $a_r$  is positive for all  $E/\mathbb{Q}$ .

---

**Mathematical subject classification (2010).** — 11G40.

**Key words and phrases.** —  $L$ -functions of elliptic curves, Birch-Swinnerton-Dyer conjecture.

*Proof.* — Set  $f(s) = B^s \cdot \Gamma(s)^n$  with  $n = [K : \mathbb{Q}]$  and  $B = \sqrt{N} \cdot |\Delta_K| / (2\pi)^n$ . Then  $\Lambda(s) = f(s) \cdot L(E, s)$  is the completed  $L$ -function, which satisfies the functional equation  $\Lambda(s) = (-1)^r \cdot \Lambda(2 - s)$ , see [3]. For  $i \equiv r + 1 \pmod{2}$  it follows that  $\frac{d^i}{ds^i} \Lambda(s)|_{s=1} = 0$ . Hence for  $i = r + 1$ , we obtain that

$$(r + 1) \cdot f'(s) \cdot \frac{d^r}{ds^r} L(E, s) + f(s) \cdot \frac{d^{r+1}}{ds^{r+1}} L(E, s)$$

is zero at  $s = 1$ . Therefore  $(r + 1) f'(1) r! a_r + f(1) (r + 1)! a_{r+1} = 0$ . It remains to note that  $f(1) = B$  and  $f'(1) = B \cdot (\log(B) + n \cdot \Gamma'(1))$  together with  $\Gamma'(1) = -\gamma$ .  $\square$

Obviously a similar formula holds for the  $L$ -function of a modular form of weight 2 for  $\Gamma_0(N)$ . More generally, for any  $L$ -function with a functional equation there is a relation between the leading and the sub-leading coefficient of the Taylor expansion of the  $L$ -function at the central point.

Sub-leading coefficients of Dirichlet  $L$ -functions have been investigated; for instance Colmez [2] makes a conjecture, which is partially known. However these concern the much harder case when  $s$  is not at the centre but the boundary of the critical strip of the  $L$ -function.

## References

- [1] Christophe Breuil, Brian Conrad, Fred Diamond, and Richard Taylor. On the modularity of elliptic curves over  $\mathbf{Q}$ : wild 3-adic exercises. *J. Amer. Math. Soc.*, 14(4):843–939, 2001.
- [2] Pierre Colmez. Périodes des variétés abéliennes à multiplication complexe. *Ann. of Math. (2)*, 138(3):625–683, 1993.
- [3] Dale Husemöller. *Elliptic curves*, volume 111 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2004. With appendices by Otto Forster, Ruth Lawrence and Stefan Theisen.

---

January 18, 2016

CHRISTIAN WUTHRICH, Room C58 Mathematics Building, University Park, Nottingham, NG7 2RD, UK  
*E-mail* : christian.wuthrich@nottingham.ac.uk