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A NOTE ON KAWASHIMA FUNCTIONS

by

Shuji Yamamoto

Abstract. —This note is a survey of results on the function $F_{\mathbf{k}}(z)$ introduced by G. Kawashima, and its applications to the study of multiple zeta values. We stress the viewpoint that the Kawashima function is a generalization of the digamma function $\psi(z)$, and explain how various formulas for $\psi(z)$ are generalized. We also discuss briefly the relationship of the results on the Kawashima functions with the recent work on Kawashima's MZV relation by M. Kaneko and the author.

Résumé. — (*Une note sur les fonctions de Kawashima*)L'objet de cette note est de faire une revue des résultats sur la fonction $F_{\mathbf{k}}(z)$ définie par G. Kawashima et des applications à l'étude des valeurs de fonctions zêtas multiples. Nous mettons l'accent sur le fait que cette fonction de Kawashima est une généralisation de la fonction digamma $\psi(z)$ et nous expliquons comment des formules valables pour $\psi(z)$ se généralisent. Nous survolons également les liens entre les résultats sur les fonctions de G. Kawashima avec les travaux récents des relations MZV de Kawashima de M. Kaneko et de l'auteur.

1. Introduction

In [3], G. Kawashima introduced a family of special functions $F_{\mathbf{k}}(z)$, where $\mathbf{k} = (k_1, \dots, k_r)$ is a sequence of positive integers, and proved some remarkable properties of them. As an application, he obtained a large class of algebraic relations among the multiple zeta values (MZVs), called *Kawashima's relation*. Kawashima's relation can be used to derive some of other classes of relations (duality, Ohno's relation, quasi-derivation relation and cyclic sum formula; see [3, 6, 7]), and is expected to imply all algebraic relations.

In this note, we survey results on these functions $F_{\mathbf{k}}(z)$, which we call the *Kawashima functions*, and their connections with MZVs. We stress the viewpoint that *the Kawashima function is a multiple version of the digamma function*. Recall that the digamma function $\psi(z)$ is defined as the logarithmic derivative of the gamma function: $\psi(z) = \frac{\mathrm{d}}{\mathrm{d}z} \log \Gamma(z)$. This is one of

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the well-studied functions in classical analysis. Here we list some formulas on $\psi(z)$ (γ denotes the Euler–Mascheroni constant):

- Newton series:

$$\psi(z+1) = -\gamma + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \binom{z}{n}. \tag{1.1}$$

- Interpolation property: For an integer $N \geq 0$,

$$\psi(N+1) = -\gamma + \sum_{n=1}^{N} \frac{1}{n}. \tag{1.2}$$

- Integral representation:

$$\psi(z+1) = -\gamma + \int_0^1 \frac{1-t^z}{1-t} \mathrm{d}t. \tag{1.3}$$

- Partial fraction series:

$$\psi(z+1) = -\gamma + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+z} \right). \tag{1.4}$$

- Taylor series:

$$\psi(z+1) = -\gamma + \sum_{m=1}^{\infty} (-1)^{m-1} \zeta(m+1) z^m. \tag{1.5}$$

In Section 2.2, we define the Kawashima function $F_{\mathbf{k}}(z)$ by a Newton series generalizing (1.1). Then we explain how the formulas (1.2), (1.3) and (1.4) are extended to $F_{\mathbf{k}}(z)$, in Sections 2.3, 2.4 and 2.5 respectively.

The Taylor expansion of $F_{\mathbf{k}}(z)$ at $z = 0$, which generalizes (1.5), is descirbed in Section 3.2. In fact, there are three methods to compute the Taylor coefficients, each of which expresses the coefficients in terms of MZVs (Proposition 3.1, Proposition 3.2 and Corollary 3.5). In Section 3.3, we treat another important property of Kawashima functions, the *harmonic relation* (Theorem 3.7). Then by combining it with the Taylor series (3.3), we deduce Kawashima's algebraic relation for MZVs (Corollary 3.8).

At the Lyon Conference, the author talked on a new proof of Kawashima's MZV relation based on the double shuffle relation and the regularization theorem, which is a part of the work with M. Kaneko [2]. In Section 3.4, we briefly discuss the relationship between this proof and the results on Kawashima functions presented in Sections 3.2 and 3.3.

Though this is basically an expository article on known results (largely due to Kawashima), it includes some results which appear in print for the first time; Proposition 2.8, Proposition 2.11 and Corollary 2.12. On the other hand, we should also note that we leave out some important works related with Kawashima functions and Kawashima's MZV relation; particularly, their q-analogue studied by Takeyama [5], and the generalization of Kawashima's relation to 'interpolated' MZVs by Tanaka and Wakabayashi [8]. For details, we refer the reader to their original articles.

2. Definition and Formulas of the Kawashima function

In this section, we define the Kawashima function by generalizing the Newton series in (1.1), and present generalizations of (1.2), (1.3) and (1.4).

2.1. Multiple harmonic sums. — Let $\mathbf{k} = (k_1, \dots, k_r)$ be an *index*, i.e., a sequence of positive integers of finite length r. We call $|\mathbf{k}| := k_1 + \cdots + k_r$ the *weight* of $\mathbf{k}$. We regard the sequence of length 0 as an index, the *empty index* denoted by ?, though we mainly consider nonempty indices.

For a nonempty index $\mathbf{k} = (k_1, \dots, k_r)$ and an integer $N \geq 0$, we put

$$s(\mathbf{k}, N) = \sum_{0<m_1<\cdots<m_{r-1}<m_r=N} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}},$$

$$s^{\star}(\mathbf{k}, N) = \sum_{0<m_1\leq\cdots\leq m_{r-1}\leq m_r=N} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}},$$

$$S(\mathbf{k}, N) = \sum_{0<m_1<\cdots<m_{r-1}<m_r\leq N} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} = \sum_{n=1}^{N} s(\mathbf{k}, n),$$

$$S^{\star}(\mathbf{k}, N) = \sum_{0<m_1\leq\cdots\leq m_{r-1}\leq m_r\leq N} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} = \sum_{n=1}^{N} s^{\star}(\mathbf{k}, n).$$

In [9], integral representations of $s^{\star}(\mathbf{k}, N)$ and $S^{\star}(\mathbf{k}, N)$ are given:

Theorem 2.1. — For a nonempty index $\mathbf{k} = (k_1, \dots, k_r)$, *put* $k = |\mathbf{k}|$ *and*

$$A(\mathbf{k}) = \{k_1, k_1 + k_2, \dots, k_1 + \cdots + k_{r-1}\},$$

$$\Delta(\mathbf{k}) = \left\{ (t_1, \dots, t_k) \in (0,1)^k \;\middle|\; \begin{array}{l} t_j > t_{j+1} \text{ if } j \notin A(\mathbf{k}), \\ t_j < t_{j+1} \text{ if } j \in A(\mathbf{k}) \end{array} \right\}.$$

Then we have

$$s^{\star}(\mathbf{k}, N) = \int_{\Delta(\mathbf{k})} \omega_{\delta(1)}(t_1) \cdots \omega_{\delta(k-1)}(t_{k-1})\, t_k^{N-1} \mathrm{d}t_k, \tag{2.1}$$

$$S^{\star}(\mathbf{k}, N) = \int_{\Delta(\mathbf{k})} \omega_{\delta(1)}(t_1) \cdots \omega_{\delta(k-1)}(t_{k-1}) \frac{1 - t_k^N}{1 - t_k} \mathrm{d}t_k, \tag{2.2}$$

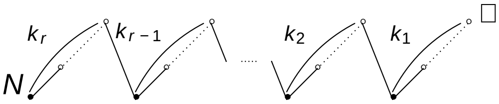
where $\omega_0(t) = \frac{\mathrm{d}t}{t}$, $\omega_1(t) = \frac{\mathrm{d}t}{1-t}$ *and*

$$\delta(j) = \begin{cases} 0 & \text{if } j \notin A(\mathbf{k}), \\ 1 & \text{if } j \in A(\mathbf{k}). \end{cases}$$

Proof. — The first formula (2.1) is [9, Theorem 1.2], stated in different symbols (in [9], the inverse order is adopted for the index). The second (2.2) is an immediate consequence of the first, since $\sum_{n=1}^{N} t_k^{n-1} = \frac{1-t_k^N}{1-t_k}$.

Definition 2.2. — We represent the integral in (2.2) by a labeled Hasse diagram as follows:

$$S^{\star}(\mathbf{k}, N) = I\left(\quad \right).$$



In general, the symbol I(diagram) means an integral determined by the following rule:

- Each vertex (◦ or •) corresponds to a variable t between 0 and 1.
- Each edge connecting two vertices expresses an inequality $t < t$ of corresponding variables, where the higher vertex in the diagram corresponds to the larger variable.
- For a vertex represented by ◦ (resp. •), we integrate $\omega_0(t)$ (resp. $\omega_1(t)$). The • with the label N (leftmost in the above diagram) expresses $\frac{1-t^N}{1-t}\mathrm{d}t$ instead of $\omega_1(t)$.

Moreover, we abbreviate the above diagram as

$$S\ (\mathbf{k},N) = I\left(\overset{N}{\bullet}\!-\!\boxed{\mathbf{k}}\right).$$

As noted in [9], the integral representations (2.1) and (2.2) imply the following identities, known as *Hoffman's duality*:

Theorem 2.3 ([1, 3]). — *Let* $\mathbf{k}$ *be the* Hoffman dual *of* $\mathbf{k}$, *i.e., the index characterized by*

$$|\mathbf{k}| = |\mathbf{k}\ |, \quad A(\mathbf{k}) \quad A(\mathbf{k}\ ) = \{1,2,\ldots,|\mathbf{k}|-1\}.$$

Then we have

$$s\ (\mathbf{k},N) = \sum_{n=1}^{N} (-1)^{n-1} s\ (\mathbf{k}\ ,n) \binom{N-1}{n-1}, \tag{2.3}$$

$$S\ (\mathbf{k},N) = \sum_{n=1}^{N} (-1)^{n-1} s\ (\mathbf{k}\ ,n) \binom{N}{n}. \tag{2.4}$$

Proof. — Under the change of variables $t_i \to 1-t_i$, $\omega_0(t_i)$ and $\omega_1(t_i)$ are interchanged and $\Delta(\mathbf{k})$ maps onto $\Delta(\mathbf{k}\ )$. Hence the identities follow from

$$(1-t_k)^{N-1} = \sum_{n=1}^{N} (-t_k)^{n-1} \binom{N-1}{n-1},$$

$$\frac{1-(1-t_k)^N}{1-(1-t_k)} = \sum_{n=1}^{N} (-t_k)^{n-1} \binom{N}{n}.$$

2.2. Newton series (definition). — Following Kawashima [3], we define the Kawashima function by a Newton series:

Definition 2.4. — For a nonempty index $\mathbf{k}$, we define the *Kawashima function* $F_{\mathbf{k}}(z)$ as

$$F_{\mathbf{k}}(z) = \sum_{n=1}^{\infty} (-1)^{n-1} s\ (\mathbf{k}\ ,n) \binom{z}{n}. \tag{2.5}$$

As a convention, we put $F_{?}(z) = 1$.

From the Newton series formula for the digamma function (1.1), we see that $F_1(z) = \psi(z+1) + \gamma$. Hence the Kawashima function may be viewed as a generalization of (a slight modification of) the digamma function.

With regard to the convergence of the series (2.5), Kawashima proved:

Proposition 2.5 ([3, Proposition 5.1]). — *Let* $\mathbf{k}$ *be a nonempty index and* ρ *the last component of the Hoffman dual of* $\mathbf{k}$*. Then the Newton series* $F_{\mathbf{k}}(z)$ *has the abscissa of convergence* $-\rho$*, i.e., converges uniformly on compact sets in the half plane* $\mathrm{Re}(z) > -\rho$*, and diverges on* $\mathrm{Re}(z) < -\rho$*.*

In particular, all Kawashima functions are defined and holomorphic on $\mathrm{Re}(z) > -1$. Hence, at least, it makes sense to consider the Taylor expansion at $z = 0$. We present explicit results in Section 3.2.

Remark 2.6. — If we write $\mathbf{k} = (k_1, \dots, k_q, \underbrace{1, \dots, 1}_{l})$, where $k_q > 1$ or $q = 0$, then ρ is given by

$$\rho = \begin{cases} l + 1 & \text{if } q \geq 1, \\ l & \text{if } q = 0. \end{cases}$$

In [3, Proposition 5.1], the latter case seems to be missed.

2.3. Interpolation property. —

Proposition 2.7. — *For any integer* $N \geq 0$*, we have*

$$F_{\mathbf{k}}(N) = S(\mathbf{k}, N). \tag{2.6}$$

Conversely, if a Newton series $f(z) = \sum_{n=0}^{\infty} a_n \binom{z}{n}$ *satisfies* $f(N) = S(\mathbf{k}, N)$ *for all* $N \geq 0$*, then* $f(z)$ *coincides with* $F_{\mathbf{k}}(z)$ *coefficientwise (i.e.,* $a_n = (-1)^{n-1} s(\mathbf{k}, n)$ *hold for all* n*).*

Proof. — The identity (2.6) follows from (2.4). For the second assertion, note the fact that the identity

$$f(N) = \sum_{n=0}^{N} a_n \binom{N}{n}$$

determines inductively the coefficients a_n by the values $f(N)$.

This characterization of the Kawashima function by its values at non-negative integers plays an essential role in Kawashima's proofs of the fraction series expansion (Theorem 2.14) and the harmonic relation (Theorem 3.7).

2.4. Integral representation. —

Proposition 2.8. — *With the same notation as in Theorem 2.1, we have*

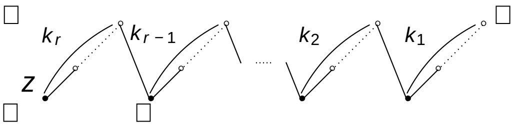
$$F_{\mathbf{k}}(z) = \int_{\Delta(\mathbf{k})} \omega_{\delta(1)}(t_1) \cdots \omega_{\delta(k-1)}(t_{k-1}) \frac{1 - t_k^z}{1 - t_k} \mathrm{d}t_k. \tag{2.7}$$

Proof. — Just as in the proof of (2.4), make the change of variables $t_i \to 1 - t_i$ and use the identity

$$\frac{1 - (1 - t_k)^z}{1 - (1 - t_k)} = \sum_{n=1}^{\infty} (-t_k)^{n-1} \binom{z}{n}.$$

Remark 2.9. — By the diagram introduced in Definition 2.2, the formula (2.7) is written as

$$F_{\mathbf{k}}(z) = I\left(z \; \cdots \right)$$



$$= I\left(z \bullet\!\!-\!\boxed{\mathbf{k}} \right).$$

Example 2.10. — Let us describe the relation between the polygamma function $\psi^{(m)}(z) = \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^m \psi(z)$ and the Kawashima function. For $m = 0$, we already know that $F_1(z) = \psi^{(0)}(z+1) + \gamma$. For $m > 0$, we have

$$\psi^{(m)}(z+1) = \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^m \int_0^1 \frac{1-t^z}{1-t}\mathrm{d}t = -\int_0^1 (\log t)^m \frac{t^z}{1-t}\mathrm{d}t.$$

Since

$$(\log t)^m = \left(-\int_t^1 \frac{\mathrm{d}u}{u} \right)^m = (-1)^m m! \int_{1>u_1>\cdots>u_m>t} \frac{\mathrm{d}u_1}{u_1} \cdots \frac{\mathrm{d}u_m}{u_m},$$

we have

$$\begin{aligned}\psi^{(m)}(z+1) &= (-1)^{m-1} m! \int_{1>u_1>\cdots>u_m>t>0} \frac{\mathrm{d}u_1}{u_1} \cdots \frac{\mathrm{d}u_m}{u_m} \frac{t^z}{1-t}\mathrm{d}t \\ &= (-1)^m m! \left(F_{m+1}(z) - \zeta(m+1) \right).\end{aligned}$$

Here we use the integral representation (2.7) for $F_{m+1}(z)$ together with the iterated integral expression

$$\zeta(m+1) = \int_{1>u_1>\cdots>u_m>t>0} \frac{\mathrm{d}u_1}{u_1} \cdots \frac{\mathrm{d}u_m}{u_m} \frac{1}{1-t}\mathrm{d}t. \tag{2.8}$$

Hence we get

$$F_{m+1}(z) = \frac{(-1)^m}{m!} \psi^{(m)}(z+1) + \zeta(m+1) \tag{2.9}$$

for integers $m > 0$. Note that this also holds for $m = 0$ if we interpret $\zeta(1)$ as γ.

2.5. Fraction series.

— Here we give two generalizations of (1.4). The first is an inductive formula:

Proposition 2.11. — *Let $\mathbf{k} = (k_1, \ldots, k_r)$ be a nonempty index and write $\mathbf{k}_- = (k_1, \ldots, k_{r-1})$ (when $r = 1$, $\mathbf{k}_-$ is the empty index ?). Then we have*

$$F_{\mathbf{k}}(z) = \sum_{n=1}^{\infty} \left(s(\mathbf{k}, n) - \frac{F_{\mathbf{k}_-}(n+z)}{(n+z)^{k_r}} \right). \tag{2.10}$$

Proof. — Put $k = |\mathbf{k}|$ and $k' = |\mathbf{k}_-|$. Then the tail of the multiple integral (2.7) is written as

$$\begin{aligned}
&\frac{dt_{k}}{1-t_{k}}\int_{t_{k}}^{1}\frac{dt_{k'+1}}{t_{k'+1}}\int_0^{t_{k'+1}}\frac{dt_{k'+2}}{t_{k'+2}}\cdots\int_0^{t_{k-2}}\frac{dt_{k-1}}{t_{k-1}}\int_0^{t_{k-1}}\frac{1-t_k^z}{1-t_k}dt_k\\
&\qquad=\sum_{n=1}^{\infty}\frac{dt_{k}}{1-t_{k}}\int_{t_{k}}^{1}\frac{dt_{k'+1}}{t_{k'+1}}\int_0^{t_{k'+1}}\frac{dt_{k'+2}}{t_{k'+2}}\cdots\int_0^{t_{k-2}}\frac{dt_{k-1}}{t_{k-1}}\int_0^{t_{k-1}}(t_k^{n-1}-t_k^{n+z-1})dt_k\\
&\qquad=\sum_{n=1}^{\infty}\frac{dt_{k}}{1-t_{k}}\left(\frac{1-t_k^n}{n^{k_r}}-\frac{1-t_k^{n+z}}{(n+z)^{k_r}}\right).
\end{aligned}$$

Hence the whole integral is equal to

$$\begin{aligned}
&\sum_{n=1}^{\infty}\int_{\Delta(\mathbf{k}_-)}\prod_{j=1}^{k'-1}\omega_{\delta(j)}(t_j)\left(\frac{1}{n^{k_r}}\frac{1-t_k^n}{1-t_k}-\frac{1}{(n+z)^{k_r}}\frac{1-t_k^{n+z}}{1-t_k}\right)dt_k\\
&\qquad=\sum_{n=1}^{\infty}\left(\frac{F_{\mathbf{k}_-}(n)}{n^{k_r}}-\frac{F_{\mathbf{k}_-}(n+z)}{(n+z)^{k_r}}\right)=\sum_{n=1}^{\infty}\left(s(\mathbf{k},n)-\frac{F_{\mathbf{k}_-}(n+z)}{(n+z)^{k_r}}\right).
\end{aligned}$$

For $\mathbf{k} = (1)$, the above formula (2.10) is the same as the formula (1.4) for the digamma function. See Example 2.15 below.

Corollary 2.12. — *With the same notation as in Proposition 2.11, Kawashima functions satisfy the difference equation*

$$F_{\mathbf{k}}(z)-F_{\mathbf{k}}(z-1)=\frac{F_{\mathbf{k}_-}(z)}{z^{k_r}}. \tag{2.11}$$

Proof. — Since both sides are analytic, we may assume that z is real. From Proposition 2.11, we obtain

$$F_{\mathbf{k}}(z)-F_{\mathbf{k}}(z-1)=\frac{F_{\mathbf{k}_-}(z)}{z^{k_r}}-\lim_{n\to\infty}\frac{F_{\mathbf{k}_-}(n+z)}{(n+z)^{k_r}},$$

hence the proposition follows from that

$$\frac{F_{\mathbf{k}_-}(z)}{z^{k_r}}\to 0\quad(z\to\infty).$$

Moreover, from Proposition 2.8, we see that $F_{\mathbf{k}}(z)$ is monotone increasing for $z \geq 0$. Therefore, it suffices to show that

$$\frac{F_{\mathbf{k}_-}(N)}{N^{k_r}}=s(\mathbf{k},N)\to 0\quad(N\to\infty).$$

Now we have an estimate

$$0\leq s(\mathbf{k},N)\leq s\Bigl(\underbrace{(1,\dots,1)}_{r},N\Bigr)=\frac{1}{N}\sum_{n=1}^{N}s\Bigl(\underbrace{(1,\dots,1)}_{r-1},n\Bigr),$$

and the statement is proven by induction on r.

Note that, from (2.11), it follows that $F_{\mathbf{k}}(z)$ is meromorphically continued to the whole complex plane.

The second generalization of (1.4), which seems more nontrivial than the first, was given by Kawashima [4]. To present it, we make some definitions.

For a nonempty index $\mathbf{k} = (k_1, \dots, k_r)$, write $\overleftarrow{\mathbf{k}} = (k_r, \dots, k_1)$.

Definition 2.13. — For integers $r > 0$ and $n_1, \dots, n_r > 0$, we put

$$P_r(n_1, \dots, n_r; z) = \frac{1}{n_1 \cdots n_{r-1}(n_r + z)},$$
$$\tilde{P}_r(n_1, \dots, n_r; z) = \frac{1}{n_1 \cdots n_{r-1}} \left(\frac{1}{n_r} - \frac{1}{n_r + z} \right).$$

Then, for a nonempty index $\mathbf{k} = (k_1, \dots, k_r)$ of weight $k = |\mathbf{k}|$, we define

$$G_{\mathbf{k}}(z) = \sum P_{k_1}(n_1, \dots, n_{k_1}; z) P_{k_2}(n_{k_1+1}, \dots, n_{k_1+k_2}; z) \cdots \cdot P_{k_{r-1}}(n_{k_1+\dots+k_{r-2}+1}, \dots, n_{k_1+\dots+k_{r-1}}; z) \cdot \tilde{P}_{k_r}(n_{k_1+\dots+k_{r-1}+1}, \dots, n_k; z), \tag{2.12}$$

where the sum is taken over all sequences of positive integers $n_1, \dots, n_k$ satisfying

$$\begin{cases} n_j < n_{j+1} & \text{if } j \notin A(\mathbf{k}), \\ n_j \leq n_{j+1} & \text{if } j \in A(\mathbf{k}) \end{cases} \tag{2.13}$$

(recall that $A(\mathbf{k})$ denotes the set $\{k_1, k_1 + k_2, \dots, k_1 + \cdots + k_{r-1}\}$).

For example,

$$G_{1,3}(z) = \sum_{0<n_1 \leq n_2 < n_3 < n_4} \frac{1}{(n_1 + z) n_2 n_3} \left(\frac{1}{n_4} - \frac{1}{n_4 + z} \right).$$

By the following theorem, this is equal to $F_{1,1,2}(z)$.

Theorem 2.14 ([4, Theorem 4.4]). — *For a nonempty index* $\mathbf{k}$*, we have*

$$F_{\mathbf{k}}(z) = G_{\overleftarrow{\mathbf{k}}}(z). \tag{2.14}$$

Example 2.15. — Let us consider an index $\mathbf{k} = (k)$ of length 1. By (2.10) and (2.14), we have

$$\begin{aligned} F_k(z) &= \sum_{n=1}^{\infty} \left(\frac{1}{n^k} - \frac{1}{(n + z)^k} \right) \\ &= G_{\underbrace{1, \dots, 1}_{k}}(z) = \sum_{0<n_1 \leq \cdots \leq n_k} \frac{1}{(n_1 + z) \cdots (n_{k-1} + z)} \left(\frac{1}{n_k} - \frac{1}{n_k + z} \right). \end{aligned}$$

In particular, when $k = 1$, both expressions coincide with the formula (1.4) for the digamma function. For $k > 1$, in contrast, it seems not easy to see that these two expressions are equal.

3. Kawashima's relation of multiple zeta values

In this section, we discuss the connections of Kawashima functions with multiple zeta values.

3.1. Notation related to multiple zeta values. — A nonempty index $\mathbf{k} = (k_1, \dots, k_r)$ is said *admissible* if $k_r > 1$. For such $\mathbf{k}$, we define the multiple zeta value (MZV) and the multiple zeta-star value (MZSV) by

$$\zeta(\mathbf{k}) = \sum_{0<m_1<\cdots<m_r} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} = \sum_{n=1}^{\infty} s(\mathbf{k}, n), \tag{3.1}$$

$$\zeta\ (\mathbf{k}) = \sum_{0<m_1\le\cdots\le m_r} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} = \sum_{n=1}^{\infty} s\ (\mathbf{k}, n). \tag{3.2}$$

We also regard the empty index ? as admissible, and put $\zeta(?) = \zeta\ (?) = 1$.

Let $\mathfrak{H}^1 = \bigoplus_{\mathbf{k}} \mathbb{Q}\cdot\mathbf{k}$ be the $\mathbb{Q}$-vector space freely generated by all indices $\mathbf{k}$, and $\mathfrak{H}^0$ the subspace generated by the admissible indices. There are two $\mathbb{Q}$-bilinear products $\ $ and $\bar{}$, called the *harmonic products*, for which ? is the unit element and which satisfies

$$\begin{aligned}
\mathbf{k}\ \ \mathbf{l} &= (\mathbf{k}_-\ \ \mathbf{l}, k_r) + (\mathbf{k}\ \ \mathbf{l}_-, l_s) + (\mathbf{k}_-\ \ \mathbf{l}_-, k_r + l_s),\\
\mathbf{k}\,\bar{}\,\mathbf{l} &= (\mathbf{k}_-\,\bar{}\,\mathbf{l}, k_r) + (\mathbf{k}\,\bar{}\,\mathbf{l}_-, l_s) - (\mathbf{k}_-\,\bar{}\,\mathbf{l}_-, k_r + l_s),
\end{aligned}$$

where $\mathbf{k} = (k_1, \dots, k_r)$ and $\mathbf{l} = (l_1, \dots, l_s)$ are any nonempty indices and $\mathbf{k}_- = (k_1, \dots, k_{r-1})$, $\mathbf{l}_- = (l_1, \dots, l_{s-1})$. In the following, we also need another product

$$\mathbf{k} \sim \mathbf{l} = \left(\mathbf{k}_-\ \ \mathbf{l}_-, k_r + l_s\right),$$

defined on the subspace of $\mathfrak{H}^1$ generated by all nonempty indices.

We extend the map $\mathbf{k} \to s(\mathbf{k}, z)$ to a linear map on $\mathfrak{H}^1$. That is, for $v = \sum_{\mathbf{k}} a_{\mathbf{k}} \cdot \mathbf{k}\ \ \mathfrak{H}^1$, we put

$$s(v, z) = \sum_{\mathbf{k}} a_{\mathbf{k}} s(\mathbf{k}, z).$$

The same rule also applies to $S(\mathbf{k}, N)$, $F_{\mathbf{k}}(z)$, $\zeta(\mathbf{k})$ and so on. Then one can see that

$$\begin{aligned}
s(v, N)s(w, N) &= s(v \sim w, N),\\
S(v, N)S(w, N) &= S(v\ \ w, N), &\qquad S\ (v, N)S\ (w, N) &= S\ (v\,\bar{}\,w, N),\\
\zeta(v)\zeta(w) &= \zeta(v\ \ w), &\qquad \zeta\ (v)\zeta\ (w) &= \zeta\ (v\,\bar{}\,w).
\end{aligned}$$

Moreover, we define a linear operator $v \to v\ $ on $\mathfrak{H}^1$ by

$$(k_1, \dots, k_r)\ \ = \sum_{0<j_1<\cdots<j_q=r} \left(k_1 + \cdots + k_{j_1}, k_{j_1+1} + \cdots + k_{j_2}, \dots, k_{j_{q-1}+1} + \cdots + k_{j_q}\right),$$

so that $s\ (v, N) = s(v\ , N)$, $S\ (v, N) = S(v\ , N)$ and $\zeta\ (v) = \zeta(v\ )$.

3.2. Taylor series. — We give three ways to express the Taylor coefficients of $F_{\mathbf{k}}(z)$ at $z = 0$ in terms of MZVs. The first is to substitute

$$\begin{aligned}\binom{z}{n} &= \frac{z(z-1)(z-2)\cdots(z-n+1)}{n!}\\ &= \left(\frac{z}{1}-1\right)\left(\frac{z}{2}-1\right)\cdots\left(\frac{z}{n-1}-1\right)\frac{z}{n}\\ &= \sum_{m=1}^{n}\sum_{0<a_1<\cdots<a_m=n}\frac{(-1)^{n-m}z^m}{a_1\cdots a_m}\\ &= \sum_{m=1}^{n}(-1)^{n-m}s^{\star}\Big((\underbrace{1,\ldots,1}_{m}), n\Big)z^m\end{aligned}$$

into the definition (2.5) of $F_{\mathbf{k}}(z)$. The result is:

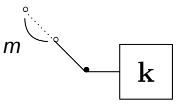
Proposition 3.1 ([3, Proposition 5.2]). — *For any nonempty index* $\mathbf{k}$, *the Taylor expansion of* $F_{\mathbf{k}}(z)$ *at* $z = 0$ *is given by*

$$F_{\mathbf{k}}(z) = \sum_{m=1}^{\infty}(-1)^{m-1}\zeta^{\star}\Big((\underbrace{1,\ldots,1}_{m}) \sim (\mathbf{k}^{\vee})\Big)z^m. \tag{3.3}$$

The second method is to differentiate repeatedly the integral representation (2.7) as in the proof of (2.9). By this method, we obtain the following formula.

Proposition 3.2. — *With the same notation as in Theorem 2.1, we put*

$$\begin{aligned}A_m(\mathbf{k}) &= \int_{\Delta(\mathbf{k},\underbrace{1,\ldots,1}_{m})}\omega_{\delta(1)}(t_1)\cdots\omega_{\delta(k-1)}(t_{k-1})\frac{\mathrm{d}t_k}{1-t_k}\frac{\mathrm{d}t_{k+1}}{t_{k+1}}\cdots\frac{\mathrm{d}t_{k+m}}{t_{k+m}}\\ &= I\Bigg(\qquad\qquad\qquad\Bigg).\end{aligned}$$



Then we have

$$F_{\mathbf{k}}(z) = \sum_{m=1}^{\infty}(-1)^{m-1}A_m(\mathbf{k})z^m. \tag{3.4}$$

The third method is based on Theorem 2.14 and a computation of the derivatives of $G_{\mathbf{k}}(z)$ at $z = 0$.

Definition 3.3. — Let $\mathbf{k} = (k_1, \ldots, k_r)$ be a nonempty index of weight $k = |\mathbf{k}|$. For an index $\mathbf{l} = (l_1, \ldots, l_k)$ of length k, define

$$\zeta_{\mathbf{k}}(\mathbf{l}) = \sum \frac{1}{n_1^{l_1}\cdots n_k^{l_k}} \tag{3.5}$$

where the sum is taken just as in the definition of $G_{\mathbf{k}}(z)$, i.e., over all sequences of positive integers $n_1, \ldots, n_k$ satisfying (2.13).

Proposition 3.4 ([4, Proposition 5.2]). — *For a nonempty index* $\mathbf{k} = (k_1, \ldots, k_r)$ *and an integer* $m \geq 1$*, put*

$$C_m(\mathbf{k}) = \sum_{\substack{l_1,\ldots,l_{r-1} \geq 0, l_r \geq 1 \\ l_1+\cdots+l_r = m}} \zeta_{\mathbf{k}}(\underbrace{1, \ldots, 1}_{k_1-1}, l_1 + 1, \ldots, \underbrace{1, \ldots, 1}_{k_r-1}, l_r + 1).$$

Then we have

$$\frac{G_{\mathbf{k}}^{(m)}(0)}{m!} = (-1)^{m-1} C_m(\mathbf{k}). \tag{3.6}$$

Corollary 3.5. — *For a nonempty index* $\mathbf{k}$*, we have*

$$F_{\mathbf{k}}(z) = \sum_{m=1}^{\infty} (-1)^{m-1} C_m(\overleftarrow{\mathbf{k}^{\vee}}) z^m. \tag{3.7}$$

By comparing the above three expressions of the Taylor expansion of $F_{\mathbf{k}}(z)$, we get

$$\zeta^{\star}(\underbrace{1, \ldots, 1}_{m}) \sim (\mathbf{k}^{\vee})^{\star} = A_m(\mathbf{k}) = C_m(\overleftarrow{\mathbf{k}^{\vee}}). \tag{3.8}$$

Since each of these expressions can be written as a sum of finitely many MZVs, this identity gives linear relations among MZVs. The relation

$$\zeta^{\star}(\underbrace{1, \ldots, 1}_{m}) \sim (\mathbf{k}^{\vee})^{\star} = A_m(\mathbf{k}) \tag{3.9}$$

appears in [2] with a different proof (see Section 3.4 below), while

$$\zeta^{\star}(\underbrace{1, \ldots, 1}_{m}) \sim \mathbf{k}^{\star} = C_m(\overleftarrow{\mathbf{k}}) \tag{3.10}$$

($\mathbf{k}^{\vee}$ is replaced by $\mathbf{k}$) is given in [4, Proposition 5.3]. Kawashima also proved the equivalence of (3.10) for $m = 1$ and the duality relation.

Example 3.6. — Let us consider the case of $\mathbf{k} = (1)$. Then the formula (3.3) says that

$$F_1(z) = \sum_{m=1}^{\infty} (-1)^{m-1} \zeta(\underbrace{1, \ldots, 1}_{m-1}, 2) z^m.$$

On the other hand, (3.4) and (3.7) give

$$F_1(z) = \sum_{m=1}^{\infty} (-1)^{m-1} \zeta(m+1) z^m,$$

which is exactly the classical formula (1.5) in the introduction. Hence we obtain

$$\zeta(\underbrace{1, \ldots, 1}_{m-1}, 2) = \zeta(m+1),$$

which is a special case of the duality.

3.3. Harmonic relation. —

Theorem 3.7 ([3, Theorem 5.3]). — *For any indices* $\mathbf{k}$ *and* $\mathbf{l}$*, we have*

$$F_{\mathbf{k}}(z)F_{\mathbf{l}}(z) = F_{\mathbf{k}\,\overline{*}\,\mathbf{l}}(z). \tag{3.11}$$

By substituting the Taylor expansion (3.3) into this relation (3.11), we obtain algebraic relations among MZVs.

Corollary 3.8 (**Kawashima's relation**). — *For any indices* $\mathbf{k}$, $\mathbf{l}$ *and any integer* $m \geq 1$*, we have*

$$\sum_{\substack{p,q\geq 1\\ p+q=m}} \zeta\Big((\underbrace{1,\ldots,1}_{p}) \circledast (\mathbf{k}^{\vee})\Big)\,\zeta\Big((\underbrace{1,\ldots,1}_{q}) \circledast (\mathbf{l}^{\vee})\Big) = -\zeta\Big((\underbrace{1,\ldots,1}_{m}) \circledast ((\mathbf{k}\,\overline{*}\,\mathbf{l})^{\vee})\Big). \tag{3.12}$$

3.4. Remark on the work of Kaneko–Yamamoto.

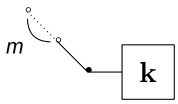
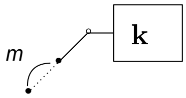
— Here we briefly discuss the relationship of some formulas presented in this section with the results of [2].

Note that we may substitute the Taylor expansion (3.4) instead of (3.3) into the harmonic relation (3.11). This implies

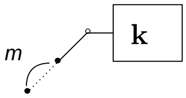
$$\sum_{\substack{p,q\geq 1\\ p+q=m}} A_p(\mathbf{k})A_q(\mathbf{l}) = -A_m(\mathbf{k}\,\overline{*}\,\mathbf{l}). \tag{3.13}$$

In other words, we may rewrite Kawashima's relation (3.12) in terms of the values $A_m(\mathbf{k})$, using the identity (3.9).

In [2, Section 6], we go in reverse. That is, we directly deduce (3.13) from the regularized double shuffle relation for MZVs. On the other hand, by making the change of variables $t \to 1 - t$ (in other words, by using the duality relation for MZVs), we have

(3.14) $A_m(\mathbf{k}) = I\Bigg($  $\Bigg) = I\Bigg($  $\Bigg).$

Moreover, the integral-series identity [2, Theorem 4.1] implies that

(3.15) $\zeta\Big((\underbrace{1,\ldots,1}_{m}) \circledast (\mathbf{k}^{\vee})\Big) = I\Bigg($  $\Bigg).$

Hence the relation (3.13) is equivalent to Kawashima's relation (3.12).

Thus we can prove Kawashima's relation without using the Kawashima function. An advantage of this proof is its algebraic nature. In fact, [2, Theorem 6.7] states that Kawashima's relation holds for any $\mathbb{Q}$-linear map $Z : \mathfrak{H}^0 \to \mathbb{R}$ satisfying the regularized double shuffle relation and the duality relation. In this algebraic setting, transcendental objects such as $F_{\mathbf{k}}(z)$ are not available.

On the other hand, it seems very hard to find relations such as (3.12) without considering the Kawashima function. It remains still fruitful to investigate applications of the Kawashima function to the study of MZVs.

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