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David Goss†, Bruno Anglès, Tuan Ngo Dac, Federico Pellarin, and Floric Tavares Ribeiro

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THE DIGIT PRINCIPLE AND DERIVATIVES OF CERTAIN L-SERIES

by

David Goss†, Bruno Anglès, Tuan Ngo Dac, Federico Pellarin and Floric Tavares Ribeiro

Abstract. —We discuss a digit principle for derivatives of certain ζ-values in Tate algebras of positive characteristic discovered by David Goss.

Résumé. — (*Principe des chi res en base q et dérivées de certaines séries L*)Dans cet article nous discutons d'un principe des chi□res (« digit principle ») en base q pour les dérivées de certaines valeurs zêta dans les algèbres de Tate en caractéristique non nulle.

1. Introduction

The present paper was initially conceived as an appendix of the paper of [4], and the main result, essentially due to David Goss, is Theorem 3.1, a new kind of digit principle for certain derivatives of ζ-values in Tate algebras, generalizing the so-called Carlitz zeta values. Later, David Goss and us, the other authors, decided to make it into an independent article, but this plan was interrupted because David Goss suddenly died on April, 4, 2017. In the present newer version, the paper also reflects the mathematical exchanges between us and him. We would like to dedicate it to his memory.

1.1. Derivatives of Riemann's zeta function and Goss' zeta function. — The functional equation of Riemann's zeta function $\zeta : \mathbb{C} \to \mathbb{P}_1(\mathbb{C})$ induces, as it is well known, trivial zeroes at the negative even integers. These zeroes are simple, and we have the following identities for the first derivatives:

$$\zeta'(-2n) = (-1)^n \frac{(2n)!}{2(2\pi)^{2n}} \zeta(2n+1), \quad n > 0. \tag{1.1}$$

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Moreover, the function $\zeta(z)$ has no zero at $z=0$, but we have the classical formula

$$\zeta'(0) = -\frac{1}{2}\ln(2\pi), \tag{1.2}$$

which is again a consequence of the functional equation.
Let now $\mathbb{F}_q$ be the finite field having q elements and let θ be an indeterminate over $\mathbb{F}_q$. We consider the local field $K_\infty = \mathbb{F}_q((\theta^{-1}))$, which is the completion of the field $K = \mathbb{F}_q(\theta)$ for the valuation at infinity v_∞ (with $v_\infty(\theta) = -1$), as an analogue of the real line. We observe indeed that $A = \mathbb{F}_q[\theta]$ is discrete and co-compact in K_∞.
In the years 1980, David Goss introduced a theory of global zeta functions in the setting of function fields of positive characteristic. His program was strongly motivated also by several signs going toward the possible existence of a functional equation, and one among them was the phenomenon of trivial zeroes. Indeed, in the above setting, defining, following Goss:

$$\zeta_A(-n,z) = \prod_P \left(1 - z^{\deg_\theta(P)} P^n\right)^{-1} = \sum_{d\geq 0} z^d \left(\sum_{a\in A_{+,d}} a^n\right) \in 1 + zA[[z]], \quad n \geq 0$$

where A_+ (resp. $A_{+,d}$) denotes the multiplicative monoid of monic polynomials (resp. monic polynomials of degree d) and with the product running over the irreducible polynomials of A_+, one sees that $\zeta_A(-n,z) \in A[z]$. It is also quite easy to show that $\zeta_A(-n,1) = 0$ if and only if $n > 0$ and $n \equiv 0 \pmod{q-1}$. Moreover, in this case, the first derivative $\zeta_A(-n,z)'$ in z does not vanish at $z = 1$, so the trivial zeroes are in this way simple, just as those of Riemann's zeta function. The polynomials $\zeta_A(-n,z)$ and certain natural generalizations, have been the object of extensive investigations by several authors. Nevertheless, no analytic reason has been found, such as the poles of a gamma factor, to justify the above properties, and no relationship connecting these first derivatives to the positive values of Goss' zeta functions has been clearly recognized.

1.2. Zeta values in Tate algebras. — Let us introduce new variables $t_1,\ldots,t_s$. For notational convenience, we shall denote by $\underline{t}_s$ the set of $\{t_1,\ldots,t_s\}$. We set $\underline{t}_0 := \emptyset$ by convention. The ring $K_\infty[\underline{t}_s]$ carries the Gauss valuation (infimum of the valuations of the coefficients of a polynomial), again denoted by v_∞. Its completion $\mathbb{T}_s(K_\infty) = \widehat{K_\infty[\underline{t}_s]}$ is an ultrametric Banach algebra, the *standard s-dimensional Tate algebra over K_∞*.
In [7, 24], the following functions

$$\zeta_A(n;s)(\underline{t}_s) = \sum_{d\geq 0}\sum_{a\in A_{+,d}} \frac{a(t_1)\cdots a(t_s)}{a^n} \in \mathbb{T}_s(K_\infty)^\times, \quad n>0, s\geq 0$$

have been introduced and studied. By [7, Proposition 6] we know that, for all $n > 0$ and $s \geq 0$, $\zeta_A(n;s)$ defines an entire function in s variables. By [7, Theorem 1], if $n \equiv s \pmod{q-1}$, there exists $\lambda_{n,s} \in K(\underline{t}_s) \cap \mathbb{T}_s(K_\infty)^\times$ such that:

$$\zeta_A(n;s) = \lambda_{n,s}\frac{\widetilde{\pi}^n}{\omega(t_1)\cdots\omega(t_s)}, \tag{1.3}$$

where

$$\widetilde{\pi} := \theta(-\theta)^{\frac{1}{q-1}} \prod_{i>0}\left(1 - \frac{\theta}{\theta^{q^i}}\right)^{-1} \in \theta(-\theta)^{\frac{1}{q-1}}\left(1 + \theta^{-1}\mathbb{F}_q[[\theta^{-1}]]\right)$$

is a fundamental period of the *Carlitz exponential* (see [21, Section 3.1]) and

$$\omega(t) := (-\theta)^{\frac{1}{q-1}} \prod_{i\geq 0}\left(1-\frac{t}{\theta^{q^i}}\right)^{-1} \in (-\theta)^{\frac{1}{q-1}}\left(1+\theta^{-1}\mathbb{F}_q[t][[\theta^{-1}]]\right)$$

is *Anderson–Thakur's* function (see Section 4). The above definitions depend on a common choice of $q-1$-th root of $-\theta$ (see [3]), but the ratio $\frac{\widetilde{\pi}^n}{\omega(t_1)\cdots\omega(t_s)}$ does not.
In fact, ω is the inverse of an entire function in the variable t, and its poles determine analytically, trivial zeroes of the functions $\zeta_A(n;s)$, from which arises naturally the idea of studying the Taylor expansion of the functions $\zeta_A(n;s)$ in the neighborhood of these trivial zeroes. In particular, if $s>1$ and $s\equiv 1 \pmod{q-1}$, the function $\zeta_A(1;s)$ vanishes at the point $\underline{t}_s=(t_1,\dots,t_s)=(\theta,\dots,\theta)$. In this paper, we will study the values

$$\delta_s := \frac{\mathrm{d}}{\mathrm{d}t_1}\cdots\frac{\mathrm{d}}{\mathrm{d}t_s}\left(\zeta_A(1;s)\right)_{t_1=\dots=t_s=\theta} \in K_\infty, \tag{1.4}$$

and we will show, in Theorem 3.1 that a sort of digit principle holds for them, first highlighted by David Goss.

2. Notation

In this paper, we will use the following notation.

- $\mathbb{N}$: the set of non-negative integers.
- $\mathbb{N}^*=\mathbb{N}\setminus\{0\}$: the set of positive integers.
- $\mathbb{Z}$: the set of integers.
- $\mathbb{F}_q$: a finite field having q elements.
- p: the characteristic of $\mathbb{F}_q$.
- θ: an indeterminate over $\mathbb{F}_q$.
- A: the polynomial ring $\mathbb{F}_q[\theta]$.
- A_+: the set of monic elements in A.
- For $d\in\mathbb{N}$, $A_{+,d}$ denotes the set of monic elements in A of degree d.
- $K=\mathbb{F}_q(\theta)$: the fraction field of A.
- ∞: the unique place of K which is a pole of θ.
- v_∞: the discrete valuation on K corresponding to the place ∞ normalized such that $v_\infty(\theta)=-1$.
- $K_\infty=\mathbb{F}_q((\frac{1}{\theta}))$: the completion of K at ∞.
- $\mathbb{C}_\infty$: the completion of a fixed algebraic closure of K_∞. The unique valuation of $\mathbb{C}_\infty$ which extends v_∞ will still be denoted by v_∞.

– $\lambda_\theta = (-\theta)^{\frac{1}{q-1}}$ a fixed $(q-1)$-th-root of $-\theta$ in $\mathbb{C}_\infty$.

– For $s \in \mathbb{N}$, $\{t_1, t_2, \ldots, t_s\}$ denotes a set of s variables and we will also denote it by $\underline{t}_s$.

3. The digit Principle

Let N be a positive integer. We consider its base-q expansion

$$N = \sum_{i=0}^{k} n_i q^i, \tag{3.1}$$

so that $n_i \in \{0, \ldots, q-1\}$ for all i. We recall that $\ell_q(N) = \sum_{i=0}^{k} n_i$ and the definition of the Carlitz factorial:

$$\Pi(N) = \prod_{i \geq 0} D_i^{n_i} \in A_+,$$

where $[i] = \theta^{q^i} - \theta$ if $i > 0$ and $D_j = [j][j-1]^q \cdots [1]^{q^{j-1}}$ for $j > 0$, while we set $D_0 = 1$.
It is easy to see (the details are in Sections 4, 5 and 6) that, if we denote by a' the derivative $\frac{\mathrm{d}}{\mathrm{d}\theta} a$ of $a \in A$ with respect to θ, the series

$$\sum_{d \geq 1} \sum_{a \in A_{+,d}} \frac{a'^N}{a}$$

converges in K_∞ to a limit that we denote by δ_N. This limit is easily seen to be equal to the evaluation of entire function of the variables $\underline{t}_N$

$$\delta_N := \frac{\mathrm{d}}{\mathrm{d}t_1} \cdots \frac{\mathrm{d}}{\mathrm{d}t_N} (\zeta_A(1; N))_{t_1 = \cdots = t_N = \theta}$$

in compatibility with (1.4).
In particular, if $n = q^j$ with $j > 0$, we will see (Proposition 5.1) that

$$\delta_1 = -\sum_{k \geq 1} \frac{1}{[k]} \text{ and } \delta_{q^j} = \frac{D_j}{[j]} \tilde{\pi}^{1-q^j}.$$

Let $N \geq 1$, $\ell_q(N) \geq 2$ and $N \equiv 1 \pmod{q-1}$. We set:

$$B_N(t, \theta) = (-1)^{\frac{\ell_q(N)-1}{q-1}} L_N(t) \left(\prod_{i=0}^{k} \omega(t^{q^i})^{n_i} \right) \tilde{\pi}^{-1}, \tag{3.2}$$

where N has base q expansion (3.1), $L_N(t) = \sum_{a \in A_+} \frac{a(t)^N}{a}$, and $\omega(t)$ is the Anderson–Thakur special function (see Section 4). By [9, Lemma 7.6], we have:

$$B_N(t, \theta) \in A[t].$$

We will prove the following:

Theorem 3.1. — *If $N \geq q$ is such that $N \equiv 1 \pmod{q-1}$ and $\ell_q(N) \geq q$, then*

$$\frac{\delta_N}{\tilde{\pi}} = \beta_N \frac{\Pi(N)}{\Pi([\frac{N}{q}])^q} \prod_{i=1}^{k} \left(\frac{\delta_{q^i}}{\tilde{\pi}} \right)^{n_i},$$

where for $x \in \mathbb{R}$, $[x]$ denotes the integer part of x, and where

$$\beta_N = (-1)^{\frac{\ell_q(N)-1}{q-1}} B_N(\theta,\theta).$$

Theorem 3.1 can be viewed as a kind of *digit principle* for the values δ_j in the sense of [14]. In Section 4, using a log-algebraic result which was originally discovered by Leonard Carlitz in 1942, we give the first properties of Anderson and Thakur function $\omega(t)$. In Section 5 we discuss the one-digit case of our Theorem, while the general case is discussed in Section 6. In Section 8 we also give some complements on these problems.

4. Carlitz log-algebraic result and its ramifications

This section is an elementary introduction to some of the recent developments on the arithmetic of special values of certain L-functions introduced by David Goss in 1979 ([20]). We have tried to keep this paragraph as self-contained as possible. All the results contained in this section are well-known but some of their proofs are new.

Lemma 4.1. — *Let $X_1,\ldots,X_m$ be $m \geq 1$ indeterminates over $\mathbb{C}_\infty$. Let $d \in \mathbb{N}$ be an integer such that $(q-1)d > m$. Then:*

$$\sum_{a\in A_{+,d}} a(X_1)\cdots a(X_m) = 0.$$

Proof. — This Lemma is a special case of [21, Lemma 8.8.1]. We have:

$$\sum_{a\in A_{+,d}} a(X_1)\cdots a(X_m) = \sum_{\zeta_1,\ldots,\zeta_d\in\mathbb{F}_q} \prod_{k=1}^{m}\left(X_k^d + \sum_{l=1}^{d}\zeta_l X_k^{l-1}\right).$$

If we develop the right hand side of the above equality and we use that $\sum_{\zeta\in\mathbb{F}_q}\zeta^n = 0$ if $n \not\equiv 0 \pmod{q-1}$, we get the assertion of the Lemma.

Lemma 4.2. — *Let $d \geq 1$ be an integer. Then:*

$$\sum_{a\in A_{+,d}} \frac{1}{a} = \frac{1}{l_d},$$

where $l_d = \prod_{k=1}^{d}(\theta - \theta^{q^k})$.

Proof. — Let us set:

$$e_d(X) = \prod_{a\in A,\deg_\theta a<d} (X-a).$$

Then one can show by induction on d (see [21, p. 46–47]) the following identity due to Leonard Carlitz:

$$e_d(X) = \sum_{k=0}^{d} \frac{D_d}{D_k \ell_{d-k}} X^{q^k},$$

where $\ell_0 = 1$. Taking the logarithmic derivative, we get:

$$\frac{D_d}{l_d e_d(X)} = \sum_{a\in A,\deg_\theta a<d} \frac{1}{X-a}.$$

Evaluating the above equality at θ^d and using the fact that $e_d(\theta^d) = D_d$ ([21, Proposition 3.1.6]), we get the desired result.

Let t be an indeterminate over $\mathbb{C}_\infty$. Leonard Carlitz also obtained the following remarkable result ([13, (5.8)]):

Proposition 4.3. — *Let $d \in \mathbb{N}$, $d \geq 1$. Then:*

$$\sum_{a\in A_{+,d}} \frac{a(t)}{a} = \frac{1}{l_d}\prod_{k=0}^{d-1}(t-\theta^{q^k}).$$

Proof. — Let us set:

$$F(t) = \sum_{a\in A_{+,d}} \frac{a(t)}{a} \in K[t].$$

Then for $k \in \{1,\dots,d-1\}$, we have by Lemma 4.1:

$$F(\theta^{q^k}) = \sum_{a\in A_{+,d}} a^{q^k-1} = \sum_{a\in A_{+,d}} a(\theta)^{q-1}a(\theta^q)^{q-1}\cdots a(\theta^{q^{k-1}})^{q-1} = 0.$$

One also observes that $F(\theta) = 0$ since $d \geq 1$. Therefore:

$$F(t) = \left(\sum_{a\in A_{+,d}} \frac{1}{a}\right)\prod_{k=0}^{d-1}(t-\theta^{q^k}).$$

It remains to apply Lemma 4.2.

Let $\tau : \mathbb{C}_\infty[[t]] \to \mathbb{C}_\infty[[t]]$ be the homomorphism of $\mathbb{F}_q[[t]]$-algebras such that:

$$\tau\left(\sum_{n\geq 0}\alpha_n t^n\right) = \sum_{n\geq 0}\alpha_n^q t^n, \quad \alpha_n \in \mathbb{C}_\infty.$$

We denote by $\mathbb{T}_t \subset \mathbb{C}_\infty[[t]]$ the Tate algebra in the variable t with coefficients in $\mathbb{C}_\infty$, which is the completion $\widehat{\mathbb{C}_\infty[t]}_{v_\infty}$ for the Gauss valuation at infinity v_∞. Observe that:

$$\{f \in \mathbb{C}_\infty[[t]], \tau(f) = f\} = \mathbb{F}_q[[t]],$$

from which one deduces easily that

$$\{f \in \mathbb{T}_t, \tau(f) = f\} = \mathbb{F}_q[t].$$

Let $\varphi : A \to A[t]\{\tau\}$ be the homomorphism of $\mathbb{F}_q$-algebras such that:

$$\varphi_\theta = \theta + (t-\theta)\tau.$$

We refer the reader to [9] for a detailed study of such objects that we may call Drinfeld modules over Tate algebras. Let $\log_\varphi$ be the unique element in $1 + \tau\mathbb{C}_\infty[[t]]\{\{\tau\}\}$ such that:

$$\log_\varphi \varphi_\theta = \theta \log_\varphi.$$

Lemma 4.4. — *We have:*

$$\log_\varphi = 1 + \sum_{d\geq 1}\frac{1}{l_d}\prod_{k=0}^{d-1}(t-\theta^{q^k})\tau^d.$$

Proof. — Write $\log_\varphi = \sum_{n\geq 0} l_n(\varphi)\tau^n$, $l_n(\varphi) \in \mathbb{C}_\infty[[t]]$, with $l_0(\varphi) = 1$. From the equation $\log_\varphi \varphi_\theta = \theta \log_\varphi$, we get for $n \geq 1$:

$$(\theta - \theta^{q^n}) l_n(\varphi) = l_{n-1}(\varphi)(t - \theta^{q^{n-1}}).$$

The Lemma follows.

Let us observe that $\log_\varphi$ converges on $\{f \in \mathbb{T}_t, v_\infty(f) > -1\}$ since for all $d \geq 0$, $v_\infty(\frac{1}{l_d}\prod_{k=0}^{d-1}(t - \theta^{q^k})) = q^d - 1$ where v_∞ is the ∞-adic Gauss valuation on $\mathbb{T}_t$.
We set

$$L(t) = \zeta_A(1;1) = \sum_{d\geq 0}\sum_{a\in A_{+,d}} \frac{a(t)}{a} = \prod_{P \text{ monic prime of } A}\left(1 - \frac{P(t)}{P}\right)^{-1} \in \mathbb{T}_t^\times.$$

Then, Proposition 4.3 implies immediately the following log-algebraic result in the sense of Anderson ([1, 2]):

Corollary 4.5. — *We have the following equality in* $\mathbb{T}_t$*:*

$$L(t) = \log_\varphi(1).$$

We refer the interested reader to [6, 10, 12, 22, 23] for the recent developments around Anderson's log-algebraicity Theorem.
We denote by $(-\theta)^{\frac{1}{q-1}}$ a fixed $q-1$-th root of $-\theta$ in $\mathbb{C}_\infty$, and we recall:

$$\tilde{\pi} = (-\theta)^{\frac{1}{q-1}}\theta\prod_{i\geq 1}(1 - \theta^{1-q^i})^{-1} \in \mathbb{C}_\infty^\times,$$

$$\omega(t) = (-\theta)^{\frac{1}{q-1}}\prod_{i\geq 0}(1 - \frac{t}{\theta^{q^i}})^{-1} \in \mathbb{T}_t^\times.$$

The following result is due to F. Pellarin ([24, Theorem 1]):

Theorem 4.6. — *We have the following equality in* $\mathbb{T}_t$*:*

$$\frac{L(t)\omega(t)}{\tilde{\pi}} = \frac{1}{\theta - t}.$$

Proof. — We give a new proof of this result by using Proposition 4.3. Let $d \geq 1$ be an integer. By Carlitz formula (Proposition 4.3):

$$\theta^{-d}t^d\sum_{a\in A_{+,d}}\frac{a(\frac{1}{t})}{a(\frac{1}{\theta})} = \prod_{k=1}^{d}(1 - \theta^{1-q^k})^{-1}\prod_{k=0}^{d-1}\left(1 - \frac{t}{\theta^{q^k}}\right).$$

Now:

$$\theta^{-d}t^d\sum_{a\in A_{+,d}}\frac{a(\frac{1}{t})}{a(\frac{1}{\theta})} = \sum_{a\in A, a(0)=1, \deg_\theta a\leq d}\frac{a(t)}{a}.$$

Furthermore:

$$\sum_{a\in A, a(0)=1, \deg_\theta a\leq d}\frac{a(t)}{a} = -\sum_{a\in A, a(0)\neq 0, \deg_\theta a\leq d}\frac{a(t)}{a}.$$

Letting d tend to $+\infty$, we get:

$$\left(1-\frac{t}{\theta}\right)\sum_{a\in A\setminus\{0\}}\frac{a(t)}{a}=-\frac{\tilde{\pi}}{\theta}\omega(t)^{-1}.$$

Finally observe that:

$$\sum_{a\in A\setminus\{0\}}\frac{a(t)}{a}=-\sum_{a\in A_+}\frac{a(t)}{a}.$$

The Theorem follows.

The function $\omega(t)$ was introduced by G. Anderson and D. Thakur in [3] (see [5] and [22] for generalizations of this special function). The Anderson–Thakur special function is intimately connected to Gauss–Thakur sums as it was highlighted in [8].
Let $C: A\to A\{\tau\}$ be the Carlitz module ([21, Chapter 3]), in other words, C is the homomorphism of $\mathbb{F}_q$-algebras given by $C_\theta=\tau+\theta$. Let us set:

$$\exp_C=\sum_{i\geq 0}\frac{1}{D_i}\tau^i\in\mathbb{T}_t\{\{\tau\}\}.$$

$\exp_C$ is called the Carlitz exponential. We have the following equality in $\mathbb{T}_t\{\{\tau\}\}$:

$$\exp_C\theta=C_\theta\exp_C.$$

Let us observe that $\exp_C$ converges on $\mathbb{T}_t$.

Lemma 4.7. — *We have:*

$$\ker\exp_C|_{\mathbb{C}_\infty}=\tilde{\pi}A.$$

Proof. — Note that the edges of the Newton polygon of $\frac{\exp_C(X)}{X}=\sum_{i\geq 0}\frac{1}{D_i}X^{q^i-1}$ are (q^i-1, iq^i), $i\geq 0$. Since $\ker\exp_C|_{\mathbb{C}_\infty}$ is an A-module, we deduce that there exists $\eta\in\mathbb{C}_\infty$, $v_\infty(\eta)=\frac{-q}{q-1}$ such that:

$$\ker\exp_C|_{\mathbb{C}_\infty}=\eta A.$$

Since $\exp_C$ defines an entire function on $\mathbb{C}_\infty$, we deduce that:

$$\exp_C(X)=\sum_{i\geq 0}\frac{1}{D_i}X^{q^i}=X\prod_{a\in A\setminus\{0\}}\left(1-\frac{X}{\eta a}\right).$$

Recall that, for $n\in\mathbb{N}$, $\sum_{\zeta\in\mathbb{F}_q^\times}\zeta^n=-1$ if $n\geq 1, n\equiv 0 \pmod{q-1}$ and $\sum_{\zeta\in\mathbb{F}_q^\times}\zeta^n=0$ otherwise. We deduce:

$$\frac{X}{\exp_C(X)}=1-\sum_{n\equiv 0 \pmod{q-1}, n\geq 1}\eta^{-n}\left(\sum_{a\in A_+}\frac{1}{a^n}\right)X^n.$$

We therefore get:

$$-\eta^{1-q}\sum_{a\in A_+}\frac{1}{a^{q-1}}=\frac{1}{\theta^q-\theta}.$$

Now, a simple computation shows that $\tau(\omega(t))=(t-\theta)\omega(t)$. Thus, by Theorem 4.6, we get:

$$\frac{(\sum_{d\geq 0}\sum_{a\in A_{+,d}}\frac{a(t)}{a^q})(t-\theta)\omega(t)}{\tilde{\pi}^q}=\frac{1}{\theta^q-t}.$$

We evaluate t at θ to obtain:

$$-\tilde{\pi}^{1-q}\sum_{a\in A_+}\frac{1}{a^{q-1}}=\frac{1}{\theta^q-\theta}.$$

Thus:

$$\frac{\eta}{\tilde{\pi}}\in\mathbb{F}_q^\times.$$

We will need the following crucial result in the sequel:

Proposition 4.8. — *We have the following equality in* $\mathbb{T}_t$:

$$\omega(t)=\exp_C\left(\frac{\tilde{\pi}}{\theta-t}\right).$$

Proof. — This result is a consequence of the formulas established in [24]. We give a detailed proof for the convenience of the reader.
Recall that $\omega(t)\in\mathbb{T}_t^\times$. Let us set

$$F(t)=\exp_C\left(\frac{\tilde{\pi}}{\theta-t}\right).$$

By Lemma 4.7, we observe that:

$$\begin{aligned}C_\theta(F(t))&=\exp_C\left(\frac{\theta\tilde{\pi}}{\theta-t}\right)=\exp_C\left(\frac{(\theta-t+t)\tilde{\pi}}{\theta-t}\right)\\&=\exp_C(\tilde{\pi})+\exp_C\left(\frac{t\tilde{\pi}}{\theta-t}\right)=t\exp_C\left(\frac{\tilde{\pi}}{\theta-t}\right)=tF(t).\end{aligned}$$

Therefore:

$$\tau(F(t))=(t-\theta)F(t).$$

Since $\tau(\omega(t))=(t-\theta)\omega(t)$, we get:

$$\tau\left(\frac{F(t)}{\omega(t)}\right)=\frac{F(t)}{\omega(t)}.$$

We have then:

$$\frac{F(t)}{\omega(t)}\in\mathbb{F}_q[t].$$

Now observe that

$$F(t)=\exp_C\left(\sum_{j\geq 0}\frac{\tilde{\pi}}{\theta^{j+1}}t^j\right)=\sum_{j\geq 0}\lambda_{\theta^{j+1}}t^j,$$

where $\lambda_{\theta^{j+1}}=\exp_C(\frac{\tilde{\pi}}{\theta^{j+1}})$. Note that $\lambda_\theta=(-\theta)^{\frac{1}{q-1}}$. We also observe that for all $j\geq 0$, $v_\infty(\lambda_{\theta^{j+1}})=j+1-\frac{q}{q-1}$. This implies $v_\infty(\frac{F(t)}{\lambda_\theta}-1)>0$. By the definition of $\omega(t)$, we also have $v_\infty(\frac{\omega(t)}{\lambda_\theta}-1)>0$. Thus:

$$v_\infty\left(\frac{F(t)}{\omega(t)}-1\right)>0.$$

Since $\frac{F(t)}{\omega(t)}\in\mathbb{F}_q[t]$, we get $\omega(t)=F(t)$.

Notice that $\omega(t)$ defines a meromorphic function on $\mathbb{C}_\infty$ without zeroes. Its only poles, simple, are located at $t=\theta,\theta^q,\theta^{q^2},\ldots$ As an immediate consequence of Proposition 4.8, we get:

Corollary 4.9. — *For all $j \geq 0$, we have:*

$$(t-\theta^{q^j})\omega(t)|_{t=\theta^{q^j}} = -\frac{\tilde{\pi}^{q^j}}{D_j}.$$

Let $\exp_\varphi \in 1+\tau\mathbb{T}_t\{\{\tau\}\}$ be such that:

$$\exp_\varphi \theta = \varphi_\theta \exp_\varphi .$$

By the same argument as that of the proof of Lemma 4.4, we have:

$$\exp_\varphi = 1 + \sum_{i\geq 1} \frac{\prod_{k=0}^{i-1}(t-\theta^{q^k})}{D_i}\tau^i.$$

Observe that $\exp_\varphi$ converges on $\mathbb{T}_t$.

Lemma 4.10. — *The exponential series $\exp_\varphi$ induces an exact sequence of $\mathbb{F}_q[t]$-modules:*

$$0 \to \frac{\tilde{\pi}}{\omega(t)}A[t] \to \mathbb{T}_t \to \mathbb{T}_t \to 0.$$

Proof. — Let us observe that in $\mathbb{T}_t\{\{\tau\}\}$:

$$\exp_C \omega(t) = \omega(t)\exp_\varphi .$$

Thus $\exp_C$ defines an entire function on $\mathbb{C}_\infty$ and thus $\exp_C(\mathbb{C}_\infty) = \mathbb{C}_\infty$. Therefore:

$$\exp_C(\mathbb{T}_t) = \mathbb{T}_t.$$

Since $\omega(t) \in \mathbb{T}_t^\times$, we get:

$$\exp_\varphi(\mathbb{T}_t) = \mathbb{T}_t,$$

$$\ker \exp_\varphi = \frac{1}{\omega(t)} \ker \exp_C .$$

Now, Lemma 4.7 implies:

$$\ker \exp_C = \tilde{\pi} A[t].$$

Following L. Taelman ([25]), we introduce the module of "units" associated to $\varphi/A[t]$:

$$U(\varphi/A[t]) = \{f \in \mathbb{T}_t \cap K_\infty[[t]] \mid \exp_\varphi(f) \in A[t]\}.$$

Observe that $U(\varphi/A[t])$ is an $A[t]$-module.

Proposition 4.11. — *We have:*

$$U(\varphi/A[t]) = L(t)A[t].$$

Proof. — By Carlitz log-algebraic result (Corollary 4.5):

$$\exp_\varphi(L(t)) = 1.$$

Thus:

$$L(t)A[t] \subset U(\varphi/A[t]).$$

Now, let us set:

$$M = \{f \in \mathbb{T}_t \cap K_\infty[[t]] \mid v_\infty(f) > 0\}.$$

We have:

$$M \cap A[t] = \{0\},$$
$$\exp_\varphi(M) = M,$$
$$\mathbb{T}_t \cap K_\infty[[t]] = A[t] \oplus M.$$

Since $L(t) \in (\mathbb{T}_t \cap K_\infty[[t]])^\times$ and $v_\infty(L(t)) = 0$, we get:

$$\mathbb{T}_t \cap K_\infty[[t]] = L(t)A[t] \oplus M.$$

Thus:

$$\exp_\varphi(\mathbb{T}_t \cap K_\infty[[t]]) \cap A[t] = \exp_\varphi(L(t)A[t]).$$

We deduce that:

$$U(\varphi/A[t]) \subset L(t)A[t] + \ker \exp_\varphi.$$

The Proposition is then a consequence of Lemma 4.10 and Theorem 4.6.

The above Proposition reflects a class formula similar to that obtained in [26]. We refer the interested reader to the references [6, 9, 10, 11, 15, 16, 17, 18, 19].

5. The one digit case

Recall that:

$$L(t) = \zeta_A(1;1) = \sum_{d\geq 0} \sum_{a\in A_{+,d}} \frac{a(t)}{a} \in \mathbb{T}_t.$$

Furthermore, we recall that we have the following equality in $\mathbb{T}_t$ (Theorem 4.6):

$$\frac{L(t)\omega(t)}{\tilde{\pi}} = \frac{1}{\theta - t}.$$

This implies that $L(t)$ extends to an entire function on $\mathbb{C}_\infty$ (see also [7, Proposition 6]). We set:

$$L'(t) = \sum_{d\geq 0} \sum_{a\in A_{+,d}} \frac{a'(t)}{a} \in \mathbb{T}_t,$$

where $a'(t)$ denotes the derivative $\frac{\mathrm{d}}{\mathrm{d}t}a(t)$ of $a(t)$ with respect to t. The derivative $\frac{\mathrm{d}}{\mathrm{d}t}$ induces a continuous endomorphism of the algebra of entire functions on $\mathbb{C}_\infty$, and therefore $L'(t)$ extends to an entire function on $\mathbb{C}_\infty$. Thus, for $j \geq 0$ an integer, $\sum_{d\geq 1} \sum_{a\in A_{+,d}} \frac{a'^{q^j}}{a}$ converges in K_∞ and we have:

$$\delta_{q^j} = \sum_{d\geq 1} \sum_{a\in A_{+,d}} \frac{a'^{q^j}}{a} = L'(t)|_{t=\theta^{q^j}}.$$

Proposition 5.1. — *The following properties hold:*

(1) *We have:*

$$\delta_1 = -\sum_{k\geq 1} \frac{1}{[k]}.$$

(2) *Let $j \geq 1$ be an integer, then:*

$$\delta_{q^j} = \frac{\Pi(q^j)}{[j]}\tilde{\pi}^{1-q^j}.$$

Proof. —

(1) It is well known that, for $n > 0$, $D_n = \prod_{a\in A_{+,n}} a$ [21, Proposition 3.1.6]. Therefore, $\sum_{a\in A_{+,n}} \frac{a'}{a} = -\frac{1}{[n]}$ from which the first formula follows.

(2) By [21], Remark 8.13.10, we have:

$$L(t)|_{t=\theta^{q^j}} = 0.$$

Thus:

$$\delta_{q^j} = L'(t)|_{t=\theta^{q^j}} = \left.\frac{L(t)}{t-\theta^{q^j}}\right|_{t=\theta^{q^j}}.$$

But,

$$\frac{\frac{L(t)}{t-\theta^{q^j}}(t-\theta^{q^j})\omega(t)}{\tilde{\pi}} = \frac{1}{\theta - t}.$$

It remains to apply Corollary 4.9.

Remark 5.2. — The transcendence over K of the "bracket series" $\delta_1 = \sum_{i\geq 1}\frac{1}{[i]}$ was first obtained by Wade [27]. The transcendence of δ_1 directly implies the transcendence of $\tilde{\pi}$.

6. The several digit case

As a consequence of [9, Lemma 7.6] (see also [7, Corollary 21]), the series $L_N(t) = \sum_{d\geq 0}\sum_{a\in A_{+,d}}\frac{a(t)^N}{a}$ has a zero of order at least N at $t = \theta$. Furthermore,

$$\tilde{L}_N(t) = \sum_{d\geq 1}\sum_{a\in A_{+,d}}\frac{a'(t)^N}{a}$$

defines an entire function on $\mathbb{C}_\infty$ such that

$$\delta_N = \tilde{L}_N(\theta).$$

Proof of Theorem 3.1. — Recall that $N = \sum_{i=0}^{k} n_i q^i$ is the q-expansion of N. We set $s = \ell_q(N)$. Recall moreover Equation (3.2):

$$(-1)^{\frac{s-1}{q-1}}B_N(t,\theta) = L_N(t)\left(\prod_{i=0}^{k}\omega(t^{q^i})^{n_i}\right)\tilde{\pi}^{-1} \in A[t].$$

Observe that:

$$L_N(t) = \sum_{a\in A_+}\frac{\prod_{i=0}^{k} a(t^{q^i})^{n_i}}{a}.$$

Let $s = \sum_{i=0}^{k} n_i$ and let $t_1, \dots, t_s$ be s indeterminates over $\mathbb{C}_\infty$. We set:

$$L_s(\underline{t}) = \sum_{a \in A_+} \frac{a(t_1)\cdots a(t_s)}{a}.$$

Since $\delta_N = \tilde{L}_N(\theta)$, it is the evaluation at $t_1 = \dots = t_{n_0} = \theta, t_{n_0+1} = \dots = t_{n_0+n_1} = \theta^q, \dots, t_{n_0+\dots+n_{k-1}+1} = \dots = t_{n_0+\dots+n_{k-1}+n_k} = \theta^{q^k}$ of the function

$$\frac{L_s(\underline{t})}{\prod_{i=0}^{k}\prod_{j=1}^{n_i}(t_{n_0+\dots+n_{i-1}+j} - \theta^{q^i})}.$$

We obtain, by [7, Theorem 1], by Corollary 4.9 and our previous discussions:

$$\beta_N = \frac{\delta_N \prod_{i=0}^{k}\left(\frac{-\tilde{\pi}^{q^i}}{D_i}\right)^{n_i}}{\tilde{\pi}}.$$

Now, by Proposition 5.1, we have, for all $i \geq 1$, $D_i = [i]\delta_{q^i}\tilde{\pi}^{q^i-1}$. We obtain the Theorem by using the fact that:

$$\frac{\Pi(N)}{\Pi([\frac{N}{q}])^q} = \prod_{i \geq 1}[i]^{n_i}.$$

7. Some non vanishing results

Proposition 7.1. — *Let $N \geq 1$ be an integer such that $N \equiv 1 \pmod{q-1}$. Then $\delta_N = 0$.*

Proof. — It follows from Theorem 3.1 and the fact that $B_N(t,\theta)|_{t=\theta} = 0$ ([4]).

The aim of this section is to prove that the series δ_N do not vanish for other values of N:

Theorem 7.2. — *Suppose that $q > 2$. Let $N \geq 1$ be a positive integer such that $2 \leq s := \ell_q(N) \leq q-1$. Then $\delta_N \neq 0$.*

7.1. Decomposition of series in K_∞. — Let i be an integer, $0 \leq i \leq q-2$. We set:

$$K_\infty^{(i)} := \theta^i \mathbb{F}_q((\theta^{1-q})) = \left\{ \sum_{\substack{n \leq n_0 \\ n \equiv i \bmod q-1}} \alpha_n \theta^n ; n_0 \in \mathbb{Z}; \alpha_n \in \mathbb{F}_q \right\} \subset K_\infty.$$

Then, we have the obvious decomposition:

$$K_\infty = \bigoplus_{0 \leq i \leq q-1} K_\infty^{(i)}$$

and the characterization:

$$K_\infty^{(i)} = \left\{ f(\theta) \in K_\infty \,\middle|\, \text{for all } \lambda \in \mathbb{F}_q^*,\ f(\lambda\theta) = \lambda^i f(\theta) \right\}.$$

For simplicity, if $f \in K_\infty$, we will note $f_{|\theta \mapsto \lambda\theta}$ for the image of f under the substitution $\theta \to \lambda\theta$, so that if $f \in K_\infty$, then $f \in K_\infty^{(i)}$ if, and only if $f_{|\theta \mapsto \lambda\theta} = \lambda^i f$ for all $\lambda \in \mathbb{F}_q^*$.
Consider now for an $N \geq 1$, and $d \geq 0$,

$$f = \sum_{a \in A_{+,d}} \frac{a'^N}{a},$$

then if $\lambda \in \mathbb{F}_q^*$, $f_{|\theta\mapsto\lambda\theta} = \lambda^{N(d-1)-d} f$. Thus, $f \in K_\infty^{(d(N-1)-N \mod q-1)}$.

Proposition 7.3. — *Let $N \geq 1$ be an integer, and $m = \frac{q-1}{(N-1,q-1)}$. Then, $\delta_N = 0$ if, and only if, for all $0 \leq j \leq m-1$,*

$$\sum_{d\equiv j \pmod m} \sum_{a\in A_{+,d}} \frac{a'^N}{a} = 0.$$

Proof. — Write for all $d \geq 0$,

$$f_d = \sum_{a\in A_{+,d}} \frac{a'^N}{a}.$$

Then, $f_d \in K_\infty^{(d(N-1)-N \mod m)}$, and if, $d, d' \geq 0$,

$$d(N-1) - N \equiv d'(N-1) - N \mod q-1$$

if and only if $d \equiv d' \mod m$.

Remark 7.4. — The "worst" case in the above proposition occurs when $m = 1$, so that the proposition is empty. But this is equivalent to $N \equiv 1 \mod q-1$ and we already know by Proposition 7.1 that δ_N does not vanish. Otherwise, the vanishing of δ_N is equivalent to the vanishing of at least two series. The worst remaining case is then when $m = 2$, that is, $N \equiv \frac{q+1}{2} \mod q-1$.

7.2. Proof of Theorem 7.2. — For $d \geq 0$, we set:

$$b_d(X) = \prod_{l=0}^{d-1} (X - \theta^{q^l})$$

and recall that:

$$l_d = (\theta - \theta^{q^d})(\theta - \theta^{q^{d-1}}) \ldots (\theta - \theta^q).$$

Observe that:

$$v_\infty(l_d) = -\frac{q^{d+1} - q}{q-1}.$$

Recall that we have expanded N in base q:

$$N = q^{e_1} + \ldots + q^{e_s}$$

with $0 \leq e_1 \leq \ldots \leq e_s$ and $2 \leq s \leq q-1$. Since $2 \leq s \leq q-1$, the log-algebraicity result [10, Proposition 5.6]. (see also [10, Example 5.7]) gives another expression for δ_N:

$$\sum_{a\in A_{+,d}} \frac{a(t_1)\cdots a(t_s)}{a} = \frac{\prod_{i=1}^s b_d(X)}{l_d}$$

so that

$$S_d := \sum_{a\in A_{+,d}} \frac{a'^N}{a} = \frac{\prod_{i=1}^s \frac{\mathrm{d}}{\mathrm{d}X} b_d(X)_{|X=\theta^{q^{e_i}}}}{l_d}$$

and

$$\delta_N = \sum_{d\geq 1} S_d.$$

Let $e \geq 0$ be an integer. We define the function $f_e : \mathbb{N}^* \to \mathbb{N}$ as follows:

$$f_e(d) = \begin{cases} -\left((e-1)q^e + \frac{q^d - q^e}{q-1}\right) & \text{if } d \geq e+1, \\ -(d-1)q^e & \text{if } 1 \leq d \leq e \text{ and } d \equiv 0 \pmod p, \\ -(d-2)q^e - q^{d-1} & \text{if } 1 \leq d \leq e \text{ and } d \equiv 0 \pmod p. \end{cases}$$

This function is strictly decreasing.

Lemma 7.5. — *Let $d \geq 1$ and $e \geq 0$ be integers. Then, we have:*

$$v_\infty\left(\frac{\mathrm{d}}{\mathrm{d}X} b_d(X)\Big|_{X=\theta^{q^e}}\right) = f_e(d).$$

Proof. — Write:

$$\frac{\mathrm{d}}{\mathrm{d}X} b_d(X) = \prod_{l=0}^{d-1}(X - \theta^{q^l}) \sum_{l=0}^{d-1} \frac{1}{X - \theta^{q^l}}.$$

The lemma follows by direct calculations.

Lemma 7.5 implies that for $d \geq 1$,

$$\text{(7.1)} \qquad v_\infty(S_d) = -v_\infty(l_d) + \sum_{i=1}^{s} f_{e_i}(d) = \frac{q^{d+1} - q}{q-1} + \sum_{i=1}^{s} f_{e_i}(d).$$

We will distinguish two cases: $e_s \equiv 0 \pmod p$ and $e_s \equiv 0 \pmod p$.

Proposition 7.6. — *Suppose that $e_s \equiv 0 \pmod p$. Let $d \geq 1$ be an integer such that $d = e_s$. Then:*

$$v_\infty(S_d) > v_\infty(S_{e_s}).$$

In particular, $\delta_N = 0$.

Proof. — Since $e_s \equiv 0 \pmod p$, Equation (7.1) implies:

$$v_\infty(S_{e_s}) = \frac{q^{e_s+1} - q}{q-1} + \sum_{i=1}^{s} f_{e_i}(e_s) = \frac{q^{e_s+1} - q}{q-1} - \sum_{i=1}^{s}\left((e_i - 1)q^{e_i} + \frac{q^{e_s} - q^{e_i}}{q-1}\right).$$

We will distinguish three cases:

Case 1: $d \geq e_s + 1$. — By (7.1), we have:

$$v_\infty(S_d) = \frac{q^{d+1} - q}{q-1} + \sum_{i=1}^{s} f_{e_i}(d) = \frac{q^{d+1} - q}{q-1} - \sum_{i=1}^{s}\left((e_i - 1)q^{e_i} + \frac{q^{d} - q^{e_i}}{q-1}\right).$$

Since $s \leq q-1$, we obtain:

$$v_\infty(S_d) - v_\infty(S_{e_s}) = \frac{q^{d+1} - q^{e_s+1}}{q-1} - \sum_{i=1}^{s} \frac{q^d - q^{e_s}}{q-1} = (q-s)\frac{q^d - q^{e_s}}{q-1} > 0.$$

Thus,

$$v_\infty(S_d) > v_\infty(S_{e_s}) \qquad \text{for } d \geq e_s + 1.$$

Case 2: $d \leq e_s - 2$. — Since the functions f_{e_i} are strictly decreasing, it follows that:

$$\begin{aligned}
v_\infty(S_d) - v_\infty(S_{e_s}) &= \frac{q^{d+1} - q^{e_s+1}}{q-1} + \sum_{i=1}^{s} (f_{e_i}(d) - f_{e_i}(e_s)) \\
&\geq \frac{q^{d+1} - q^{e_s+1}}{q-1} + (f_{e_s}(e_s - 2) - f_{e_s}(e_s)) \\
&\geq \frac{q^{d+1} - q^{e_s+1}}{q-1} + 2q^{e_s} \\
&> \frac{-q^{e_s+1}}{q-1} + 2q^{e_s} \\
&> 0.
\end{aligned}$$

Thus,

$$v_\infty(S_d) > v_\infty(S_{e_s}) \qquad \text{for } d \leq e_s - 2.$$

Case 3: $d = e_s - 1$. — Since the functions f_{e_i} are strictly decreasing, we obtain:

$$\begin{aligned}
v_\infty(S_{e_s-1}) - v_\infty(S_{e_s}) &= \frac{q^{e_s} - q^{e_s+1}}{q-1} + \sum_{i=1}^{s} (f_{e_i}(e_s - 1) - f_{e_i}(e_s)) \\
&\geq \frac{q^{e_s} - q^{e_s+1}}{q-1} + (f_{e_1}(e_s - 1) - f_{e_1}(e_s)) + (f_{e_s}(e_s - 1) - f_{e_s}(e_s)) \\
&\geq \frac{q^{e_s} - q^{e_s+1}}{q-1} + q^{e_s-1} + q^{e_s} = q^{e_s-1} \\
&> 0.
\end{aligned}$$

Thus,

$$v_\infty(S_{e_s-1}) > v_\infty(S_{e_s}).$$

The proof is finished.

Proposition 7.7. — *Suppose that $e_s \equiv 0 \pmod p$. Let $d \geq 1$ be an integer such that $d \notin \{e_s - 1, e_s, e_s + 1\}$. Then:*

$$v_\infty(S_d) > v_\infty(S_{e_s}) > v_\infty(S_{e_s-1}).$$

Proof. — Let t be the integer such that $0 \leq t \leq s-1$ and $e_t < e_{t+1} = \ldots = e_s$. Since $e_s \equiv 0 \pmod p$, Equation (7.1) implies:

$$\begin{aligned}
v_\infty(S_{e_s}) &= \frac{q^{e_s+1} - q}{q-1} + \sum_{i=1}^{s} f_{e_i}(e_s) \\
&= \frac{q^{e_s+1} - q}{q-1} - \sum_{i=1}^{t} \left((e_i - 1)q^{e_i} + \frac{q^{e_s} - q^{e_i}}{q-1} \right) - (s-t)((e_s - 2)q^{e_s} + q^{e_s-1}) \\
&= \frac{q^{e_s+1} - q}{q-1} - \sum_{i=1}^{s} \left((e_i - 1)q^{e_i} + \frac{q^{e_s} - q^{e_i}}{q-1} \right) + (s-t)(q^{e_s} - q^{e_s-1}).
\end{aligned}$$

We will distinguish three cases:

Case 1: $d \geq e_s + 2$. — By (7.1),

$$v_\infty(S_d) = \frac{q^{d+1}-q}{q-1} + \sum_{i=1}^{s} f_{e_i}(d) = \frac{q^{d+1}-q}{q-1} - \sum_{i=1}^{s}\left((e_i-1)q^{e_i} + \frac{q^d - q^{e_i}}{q-1}\right).$$

Since $s \leq q-1$ and $d \geq e_s + 2$, we get:

$$\begin{aligned} v_\infty(S_d) - v_\infty(S_{e_s}) &= \frac{q^{d+1}-q^{e_s+1}}{q-1} - \sum_{i=1}^{s} \frac{q^d - q^{e_s}}{q-1} - (s-t)(q^{e_s} - q^{e_s-1}) \\ &= (q-s)\frac{q^d - q^{e_s}}{q-1} - (s-t)(q^{e_s} - q^{e_s-1}) \\ &\geq \frac{q^{e_s+2}-q^{e_s}}{q-1} - (q-1)(q^{e_s} - q^{e_s-1}) \\ &> 0. \end{aligned}$$

Thus,

$$v_\infty(S_d) > v_\infty(S_{e_s}) \qquad \text{for } d \geq e_s + 2.$$

Case 2: $d \leq e_s - 2$. — We have:

$$\begin{aligned} v_\infty(S_d) - v_\infty(S_{e_s}) &= \frac{q^{d+1}-q^{e_s+1}}{q-1} + \sum_{i=1}^{s}(f_{e_i}(d) - f_{e_i}(e_s)) \\ &\geq \frac{q^{d+1}-q^{e_s+1}}{q-1} + (f_{e_1}(e_s-2) - f_{e_1}(e_s)) + (f_{e_s}(e_s-2) - f_{e_s}(e_s)) \\ &\geq \frac{q^{d+1}-q^{e_s+1}}{q-1} + (q^{e_s-1} + q^{e_s-2}) + (q^{e_s} + q^{e_s-1}) \\ &\geq \frac{-q^{e_s+1}}{q-1} + (q^{e_s} + q^{e_s-1}) + (q^{e_s-1} + q^{e_s-2}) \\ &> 0. \end{aligned}$$

It follows that

$$v_\infty(S_d) > v_\infty(S_{e_s}) \qquad \text{for } d \leq e_s - 2.$$

Case 3: $d = e_s - 1$. — Since $s \geq q-1$, it follows that:

$$\begin{aligned} v_\infty(S_{e_s-1}) - v_\infty(S_{e_s}) &= \frac{q^{e_s}-q^{e_s+1}}{q-1} + \sum_{i=1}^{s}(f_{e_i}(e_s-1) - f_{e_i}(e_s)) \\ &= \frac{q^{e_s}-q^{e_s+1}}{q-1} + sq^{e_s-1} = (-q+s)q^{e_s-1} \\ &< 0 \end{aligned}$$

Thus,

$$v_\infty(S_{e_s}) > v_\infty(S_{e_s-1}).$$

The proof is finished.

Suppose now that $e_s \not\equiv 0 \pmod p$. Set $m = \frac{q-1}{(q-1,N-1)}$ Then $m \geq 2$, and by the above proposition, the series $\sum_{d \equiv e_s \pmod m} S_d$ does not vanish. By Proposition 7.3, we obtain that $\delta_N = 0$. This completes the proof of Theorem 7.2.

8. Taylor expansions in the neighborhood of $(\theta,\ldots,\theta)$

We shall first analyze the function ω in the neighborhood of θ.

8.1. Laurent expansion of ω. — The Tate algebra $\mathbb{T}_t := \widehat{\mathbb{C}_\infty[t]}_{v_\infty}$ is endowed with the family of continuous $\mathbb{C}_\infty$-linear endomorphisms $(D_n)_{n\geq 0}$ where D_n is the n-th higher derivative in t defined by $D_n(t^m) = \binom{m}{n} t^{m-n}$.

Let us suppose that a sequence of elements $(A_i)_{i>0}$ of $\mathbb{T}_t^\times$ is given, so that for all i, A_i is a 1-unit, that is, $v_\infty(A_i-1) > v_\infty(A_i) = 0$, and $\lim_{i\to\infty}(A_i-1) = 0$ in $\mathbb{T}_t$. Then, the product

$$\prod_{i>0} A_i$$

converges to an element $F \in \mathbb{T}_t^\times$. We easily obtain, from Leibniz's rule, and for $n>0$, the formula:

$$D_n(F) = \sum_{\substack{p_1,p_2,\cdots\in\mathbb{N},\\ \sum_{i\geq 1} p_i = n}} \prod_{i\geq 1} D_{p_i}(A_i).$$

Here, by definition, a *composition* of a positive integer n is an s-tuple (with unrestricted s) of positive integers $(n_1,\ldots,n_s)$ such that $\sum_i n_i = n$. We deduce, for $n>0$:

$$\frac{D_n(F)}{F} = \sum_{s\geq 1}\sum_{\substack{P=(p_1,\ldots,p_s)\\ \sum_i p_i = n}}\sum_{0<i_1<\cdots<i_s}\frac{D_{p_1}(A_{i_1})}{A_{i_1}}\cdots\frac{D_{p_s}(A_{i_s})}{A_{i_s}}. \tag{8.1}$$

We now choose

$$F = (t-\theta)\omega,$$

and we recall that the function omega of Anderson and Thakur has the following product expansion:

$$\omega(t) = (-\theta)^{\frac{1}{q-1}}\prod_{i\geq 0}\left(1-\frac{t}{\theta^{q^i}}\right)^{-1}.$$

hence, setting

$$A_i = \left(1-\frac{t}{\theta^{q^i}}\right)^{-1}, \quad i\geq 0$$

we have

$$F = -\theta(-\theta)^{\frac{1}{q-1}}\prod_{i>0} A_i.$$

Observe now that, for all $j>0$ and $i\geq 0$:

$$\frac{D_j(A_i)}{A_i} = (\theta^{q^i}-t)^{-j}.$$

We deduce

$$\frac{D_n(F)}{F} = \sum_{s\geq 1}\sum_{P}\sum_{0<i_1<\cdots<i_s}(\theta^{q^{i_1}}-t)^{-p_1}\cdots(\theta^{q^{i_s}}-t)^{-p_s}$$

where the second sum is over the compositions $P=(p_1,\ldots,p_s)$ of n. Define $\eta(\emptyset) := 1$ and, for $(p_1,\ldots,p_s)$ as above,

$$\eta(p_1,\ldots,p_s) = \sum_{0<i_1<\cdots<i_s}[i_1]^{-p_1}\cdots[i_s]^{-p_s} \in K_\infty,$$

where $[i] = \theta^{q^i} - \theta$. We deduce, taking into account the above computations:

Lemma 8.1. — *The following formula holds:*

$$(t-\theta)\omega(t) = -\tilde{\pi}\sum_{n\geq 0}\left(\sum_{s\geq 1}\sum_{\substack{p=(p_1,\ldots,p_s)\\ \sum_i p_i = n}}\eta(p_1,\ldots,p_s)\right)(t-\theta)^n. \tag{8.2}$$

Note that, by the usual conventions, the coefficient corresponding to $n = 0$ is $-\tilde{\pi}$.

We now compute the higher derivatives of another infinite product. Let us consider:

$$G = \frac{\prod_{i>0}\left(1-\frac{t}{\theta^{q^i}}\right)}{\prod_{i>0}\left(1-\frac{\theta}{\theta^{q^i}}\right)} = \prod_{i>0} B_i,$$

where

$$B_i = \frac{\theta^{q^i} - t}{\theta^{q^i} - \theta}.$$

We know from the formula (4.6) that this coincides with $L(t)$. Since $D_n(B_i)/B_i$ equals 1, $\frac{1}{t-\theta^{q^i}}$ or 0 depending on whether $n = 0, 1$ or $n > 1$, we have the series expansion:

$$\frac{D_n(G)}{G} = \sum_{0<i_1<\cdots<i_n}(t-\theta^{q^{i_1}})^{-1}\cdots(t-\theta^{q^{i_n}})^{-1}.$$

We deduce:

Lemma 8.2. — *The following formula holds:*

$$L(t) = \sum_{n\geq 0}(-1)^n\eta(\underbrace{1,\ldots,1}_{n\ times})(t-\theta)^n.$$

The above formula of course agrees with (1) of Proposition 5.1. Note also that both entire functions $(t-\theta)\omega$ and $L(t)$ define invertible formal series in $K_\infty[[t-\theta]]$.

8.2. The s variable case. — We work in $\mathbb{T}_s$ and we suppose that $n \equiv s \pmod{q-1}$. We recall that

$$\sum_{a\in A^+}\frac{a(t_1)\cdots a(t_s)}{a^n} \in \mathbb{T}_s(K_\infty)^\times$$

represents an entire function in the variables $\underline{t}_s$. We can expand in series

$$\zeta_A(n;s)(\underline{t}_s) = \sum_{\underline{i}} c_{\underline{i}}(t_1-\theta)^{i_1}\cdots(t_s-\theta)^{i_s} \in K_\infty[[t_1-\theta,\ldots,t_s-\theta]].$$

The Newton polyhedron of this series is likely to be interesting. We consider the case $n = 1, s > 1$. Then, in [7], it is proved that

$$L(\underline{t}_s) = \zeta_A(1;s)(\underline{t}_s) = \frac{\tilde{\pi}\lambda_{1,s}}{\omega(t_1)\cdots\omega(t_s)},$$

with $\lambda_{1,s} \in A[\underline{t}_s]$. It is easy to show that, for all $i = 1,\ldots,s$ with $s > 1$, $L(\underline{t}_s)|_{t_i=\theta} = 0$. This means that

$$\mu_{1,s} := \frac{\lambda_{1,s}}{(t_1-\theta)\cdots(t_s-\theta)} \in A[\underline{t}_s].$$

The main conjecture in [7], proved in the paper [4], asserts that if $s>1$, then $\mu_{1,s}$ is a unit of $K[[t_1-\theta,\ldots,t_s-\theta]]$; it does not vanish at $t_i=\theta$. This implies (it is in fact equivalent to) the non-vanishing of δ_s.
In [7] the following result is proved (see Theorem 1):

Theorem 8.3. — *For $n\geq 1$, $n\equiv s \pmod{q-1}$,*

$$V_{n,s}:=\tilde{\pi}^{-n}\zeta_A(n;s)\omega(t_1)\cdots\omega(t_s)\prod_{i=1}^{s}\prod_{j=0}^{\delta-1}\left(1-\frac{t_i}{\theta^{q^j}}\right)\in K[\underline{t}_s],$$

where δ is the smallest non-negative integer such that at once $q^\delta-n\geq 0$ and $s+\ell_q(q^\delta-n)\geq 2$.

8.2.1. *Some examples with $s=n$.* — If $n=s=1$, Theorem 8.3 is sharp by the explicit formula $L(\chi_t,1)=-\frac{\tilde{\pi}}{(t-\theta)\omega}$. We take as another example the case $t=s=q$ for which $\delta=1$. We have

$$\zeta_A(q;s)=\tau(\zeta_A(1;s))=-\frac{\tilde{\pi}^q}{(t_1-\theta)\cdots(t_q-\theta)\omega(t_1)\cdots\omega(t_q)}$$

so, again, the result is sharp.

8.2.2. *More about the polynomials $V_{n,s}$.* — We have observed that $V_{1,s}$ is a unit of $K[[t_1-\theta,\ldots,t_s-\theta]]$ for all $s>1$, $s\equiv 1 \pmod{q-1}$ (recall that $V_{1,s}$ is a polynomial) and the case $s=n=1$ is clear too. What about the more general case of $V_{n,s}$ with $n\equiv s \pmod{q-1}$ and $n>1$? We shall set, for commodity,

$$V_{n,s}^*=\tilde{\pi}^{-n}\zeta_A(n;s)\omega(t_1)\cdots\omega(t_s)\prod_{i=1}^{s}(\theta-t_i).$$

If $n\geq 2$, then $\delta\geq 1$ in Theorem 8.3 and $V_{n,s}\in V_{n,s}^*K[[t_1-\theta,\ldots,t_s-\theta]]^\times$. From now on, we assume that $n\geq 2$. We want to analyze the singularity of $V_{n,s}^*$ at $t_i=\theta$ for all i which is the same as that of $V_{n,s}$.

8.2.3. *Case $s\leq n$.* — This is the simplest case. Indeed, since $\mathrm{ev}_\theta(\zeta_A(n;s))$, the evaluation at $t_1=\cdots=t_s=\theta$ of $\zeta_A(n;s)$, equals the Carlitz zeta value $\zeta_A(n-s)$ which is non-zero, we see that $V_{n,s}$ (or equivalently, $V_{n,s}^*$) are units of $K[[t_1-\theta,\ldots,t_s-\theta]]$.

8.2.4. *Case $s>n$.* — We shall prove:

Theorem 8.4. — *Let us consider integers n,s such that $n\equiv s \pmod{q-1}$, $s>n\geq 2$. For all choices of $(k_1,\ldots,k_s)\in\mathbb{Z}^s$, $(t_1-\theta)^{k_1}\cdots(t_s-\theta)^{k_s}V_{n,s}$, an element of $K((t_1-\theta,\ldots,t_s-\theta))$, is not a unit of $K[[t_1-\theta,\ldots,t_s-\theta]]$.*

Proof. — Observe that the main result of [4] implies that $V_{1,s-n+1}$ is a unit of $K[[t_1-\theta,\ldots,t_{s-n+1}-\theta]]$. In this case, $\mathrm{ev}_\theta(V_{n,s}^*)=0$ (indeed, $n>s$ and $s\equiv n \pmod{q-1}$ implies $\mathrm{ev}_\theta(V_{n,s}^*)=\mathrm{ev}_\theta(\zeta_A(n;s))=0$. Let us choose $I\subset\{1,\ldots,s\}$ a subset with $n-1$ elements and let us denote by J the set $\{1,\ldots,s\}\setminus I$. We have $|J|=s-n+1$. We observe:

$$V_{n,s}^*=\tilde{\pi}^{-n}\prod_{i\in I}[(\theta-t_i)\omega(t_i)]\prod_{j\in J}[(\theta-t_j)\omega(t_j)]\,\zeta_A(n;s).$$

Since

$$\lim_{t_i\to\theta;i\in I}\zeta_A(n;s)=L(\underline{t}_J):=\sum_{d\geq 0}\sum_{a\in A_{+,i}}\frac{\prod_{i\in J}a(t_i)}{a},$$

we deduce that

$$\lim_{t_i\to\theta; i\in I} V^*_{n,s} = V^*_{1,s-n+1}(\underline{t}_J) = V_{1,s-n+1}(\underline{t}_J)\prod_{j\in J}(\theta - t_j).$$

We know that $V_{1,s-n+1}(\underline{t}_J)$ is a unit of $K[[t_j-\theta; j\in J]]$. In particular, expanding

$$V^*_{n,s} = \sum_{i_1,\ldots,i_s\geq 0} c_{i_1,\ldots,i_s}(t_1-\theta)^{i_s}\cdots(t_s-\theta)^{i_s} \in K[[t_1-\theta,\ldots,t_s-\theta]],$$

we have $c_{0,\ldots,0}=0$ and, for any subset $J\subset\{1,\ldots,s\}$ with $s-n+1$ elements, by setting $(i_1,\ldots,i_s)$ with $i_j=1$ for $j\in J$ and $i_j=0$ otherwise, we have $c_{i_1,\ldots,i_s}=0$. This ends the proof of the Proposition.

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David Goss†

Bruno Anglès, Université de Caen Normandie, Laboratoire de Mathématiques Nicolas Oresme, CNRS UMR 6139, Campus II, Boulevard Maréchal Juin, B.P. 5186, 14032 Caen Cedex, France.
E-mail : bruno.angles@unicaen.fr

Tuan Ngo Dac, CNRS and Université de Caen Normandie, Laboratoire de Mathématiques Nicolas Oresme, CNRS UMR 6139, Campus II, Boulevard Maréchal Juin, B.P. 5186, 14032 Caen Cedex, France.
E-mail : tuan.ngodac@unicaen.fr

Federico Pellarin, Institut Camille Jordan, UMR 5208, Site de Saint-Etienne, 23 rue du Dr. P. Michelon,42023 Saint-Etienne, France • *E-mail :* federico.pellarin@univ-st-etienne.fr

Floric Tavares Ribeiro, Université de Caen Normandie, Laboratoire de Mathématiques Nicolas Oresme, CNRS UMR 6139, Campus II, Boulevard Maréchal Juin, B.P. 5186, 14032 Caen Cedex, France.
E-mail : floric.tavares-ribeiro@unicaen.fr