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FAMILIES OF EULERIAN FUNCTIONS INVOLVED IN REGULARIZATION OF DIVERGENT POLYZETAS

by

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Abstract. — Extending the Eulerian functions, we study their relationship with zeta function of several variables. In particular, starting with Weierstrass factorization theorem (and Newton–Girard identity) for the complex Gamma function, we are interested in the ratios of $\zeta(2k)/\pi^{2k}$ and their multiindexed generalization, we obtain an analogue situation and draw some consequences about a structure of the algebra of polyzetas values, by means of some combinatorics of words and noncommutative rational series. The same frameworks also allow to study the independence of a family of eulerian functions.

Résumé. — (*Familles de fonctions eulériennes impliquées dans la régularisation de polyzêtas divergents*) En généralisant les fonctions euleriennes, nous étudions leurs relations avec la fonction zêta en plusieurs variables. En particulier, à partir du théorème de factorisation de Weierstrass (et l'identité de Newton-Girard) pour la fonction Gamma complexe, nous nous intéressons aux rapports $\zeta(2k)/\pi^{2k}$ et leurs généralisations. Nous obtenons une situation analogue et nous tirerons quelques conséquences sur une structure de l'algèbre des valeurs polyzêtas, au moyen de la combinatoire des mots et des séries rationnelles en variables non commutatifs. Le même cadre de travail permet également d'étudier l'indépendance d'une famille de fonctions euleriennes.

1. Introduction

Eulerian functions are most significant for analytic number theory and they are widely applied in Probability theory and in Physical sciences. They are tightly relating to Riemann zeta functions, for instance as follows

$$(1) \qquad \zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t \frac{t^{s-1}}{e^t - 1} \quad \text{and} \quad \Gamma(s) = \int_0^\infty \mathrm{d}u \; u^{s-1} e^{-u}, \text{ for } \Re(s) > 0.$$

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The function Γ is meromorphic, with no zeroes and $-\mathbb{N}^*$ as set of simple poles. Hence Γ^{-1} is entire and admits $-\mathbb{N}^*$ as set of simple zeroes. Moreover, it satisfies[1] $\Gamma(\bar{z}) = \overline{\Gamma(z)}$. From Weierstrass factorization [5] and Newton–Girard identity [14], we have successively

$$\frac{1}{\Gamma(z+1)} = e^{\gamma z} \prod_{n\geq 1} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}} = \exp\left(\gamma z - \sum_{k\geq 2} \zeta(k) \frac{(-z)^k}{k}\right). \tag{2}$$

Using the following functional equation and Euler's complement formula, i.e.

$$\Gamma(1+z) = z\Gamma(z) \quad \text{and} \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(z\pi)},$$

and also introducing the *partial beta function* defined (for any $a, b, z \in \mathbb{C}$ such that $\Re a > 0$, $\Re b > 0$, $|z| < 1$) by

$$\mathrm{B}(z; a, b) := \int_0^z \mathrm{d}t\, t^{a-1}(1-t)^{b-1} \tag{3}$$

and then, classically, $\mathrm{B}(a,b) := \mathrm{B}(1; a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$, one has (for any $u, v \in \mathbb{C}$ such that $|u| < 1, |v| < 1$ and $|u+v| < 1$) the following expression

$$\exp\left(-\sum_{n\geq 2} \zeta(n) \frac{(u+v)^n - (u^n + v^n)}{n}\right) = \frac{\Gamma(1-u)\Gamma(1-v)}{\Gamma(1-u-v)}, \tag{4}$$

$$= \frac{\Gamma(u+v)}{\Gamma(u)\Gamma(v)} \pi \frac{\sin((u+v)\pi)}{\sin(u\pi)\sin(v\pi)} \tag{5}$$

$$= \frac{\pi}{\mathrm{B}(u,v)} (\cot(u\pi) + \cot(v\pi)). \tag{6}$$

In particular, for $v = -u$ ($|u| < 1$), one gets

$$\exp\left(-\sum_{k\geq 1} \zeta(2k) \frac{u^{2k}}{k}\right) = \frac{1}{\Gamma(1-u)\Gamma(1+u)} = \frac{\sin(u\pi)}{u\pi}.$$

Hence, taking the logarithms and considering Taylor expansions, one obtains

$$-\sum_{k\geq 1} \zeta(2k) \frac{u^{2k}}{k} = \log\left(1 + \sum_{n\geq 1} \frac{(u\mathrm{i}\pi)^{2n}}{\Gamma(2n+2)}\right) \tag{7}$$

$$= \sum_{k\geq 1} (u\mathrm{i}\pi)^{2k} \sum_{l\geq 1} \frac{(-1)^{l-1}}{l} \sum_{\substack{n_1,\ldots,n_l \geq 1 \\ n_1+\ldots+n_l = k}} \prod_{i=1}^{l} \frac{1}{\Gamma(2n_i+2)}. \tag{8}$$

One can deduce then the following expression[2] for $\zeta(2k)$:

$$\frac{\zeta(2k)}{\pi^{2k}} = k \sum_{l=1}^{k} \frac{(-1)^{k+l-1}}{l} \sum_{\substack{n_1,\ldots,n_l \geq 1 \\ n_1+\ldots+n_l = k}} \prod_{i=1}^{l} \frac{1}{\Gamma(2n_i+2)} \in \mathbb{Q}. \tag{9}$$

[1] i.e. its coefficients are real, we will see later the combinatorial content of them.
[2] Note that Euler gave another explicit formula using Bernoulli numbers.

Example 1.1. —

$$\frac{\zeta(2)}{\pi^2} = 1.\frac{(-1)^{1+1}}{1}\frac{1}{\Gamma(4)} = \frac{1}{6};$$

$$\frac{\zeta(4)}{\pi^4} = 2\left[\frac{(-1)^{2+1}}{1}\frac{1}{\Gamma(6)} + \frac{(-1)^{2+2}}{2}\frac{1}{\Gamma(4)\Gamma(4)}\right] = \frac{1}{90};$$

$$\frac{\zeta(6)}{\pi^6} = 3\sum_{l=1}^{3}\frac{(-1)^{3+l}}{l}\sum_{\substack{n_1,\ldots,n_l\geq 1\\ n_1+\ldots+n_l=3}}\prod_{i=1}^{l}\frac{1}{\Gamma(2n_i+2)} = \frac{1}{945};$$

$$\frac{\zeta(8)}{\pi^8} = 4\sum_{l=1}^{4}\frac{(-1)^{4+l}}{l}\sum_{\substack{n_1,\ldots,n_l\geq 1\\ n_1+\ldots+n_l=4}}\prod_{i=1}^{l}\frac{1}{\Gamma(2n_i+2)} = \frac{1}{9450};$$

$$\frac{\zeta(10)}{\pi^{10}} = 5\sum\_l = 1^5\frac{(-1)^{5+l}}{l}\sum_{\substack{n_1,\ldots,n_l\geq 1\\ n_1+\ldots+n_l=5}}\prod_{i=1}^{l}\frac{1}{\Gamma(2n_i+2)} = \frac{1}{93555}.$$

Now, more generally, for any $r \in \mathbb{N}_{\geq 1}$ and $(s_1,\ldots,s_r) \in \mathbb{C}^r$, let us consider the following *several variable zeta function*

$$\zeta(s_1,\ldots,s_r) := \sum_{n_1>\ldots>n_r>0} n_1^{-s_1}\ldots n_r^{-s_r} \tag{10}$$

which converges for $(s_1,\ldots,s_r)$ in the open sub-domain of $\mathbb{C}^r, r \geq 1$, [9, 18]

$$\mathcal{H}_r := \{(s_1,\ldots,s_r) \in \mathbb{C}^r \mid \forall\, m = 1,\ldots,r,\ \Re(s_1)+\ldots+\Re(s_m) > m\}.$$

In the convergent cases, from a theorem by Abel, for $n \in \mathbb{N}, z \in \mathbb{C}, |z| < 1$, its values can be obtained as the following limits

$$\zeta(s_1,\ldots,s_r) = \lim_{z\to 1}\mathrm{Li}_{s_1,\ldots,s_r}(z) = \lim_{n\to+\infty}\mathrm{H}_{s_1,\ldots,s_r}(n), \tag{11}$$

where the following *polylogarithms* are well defined

$$\mathrm{Li}_{s_1,\ldots,s_r}(z) := \sum_{n_1>\ldots>n_r>0}\frac{z^{n_1}}{n_1^{s_1}\ldots n_r^{s_r}}, \tag{12}$$

$$\frac{\mathrm{Li}_{s_1,\ldots,s_r}(z)}{1-z} = \sum_{n\geq 0}\mathrm{H}_{s_1,\ldots,s_r}(n)z^n, \tag{13}$$

and so are the Taylor coefficients[3] here simply called *harmonic sums*

$$\mathrm{H}_{s_1,\ldots,s_r} : \mathbb{N} \longrightarrow \mathbb{Q}(\text{i.e. an arithmetic function}), \tag{14}$$

$$n \longmapsto \mathrm{H}_{s_1,\ldots,s_r}(n) = \sum_{n\geq n_1>\ldots>n_r>0} n_1^{-s_1}\ldots n_r^{-s_r}. \tag{15}$$

[3]These quantities are generalizations of the harmonic numbers $H_n = 1 + 2^{-1}\ldots + n^{-1}$ to which they boil down for $r = 1, s_1 = 1$. They are also truncations of the zeta values $\zeta(s_1,\ldots,s_r)$ at order $n+1$.

On $\mathcal{H}_r \cap \mathbb{N}^r$, the polyzetas can be represented by the following integral representation[4] over $]0,1[$ [10] (here, one set $\lambda(z) := z(1-z)^{-1}, t_0 = 1$ and $u_{r+1} = 1$):

$$\begin{aligned}\zeta(s_1,\ldots,s_r) &= \int_0^1 \omega_1(t_1)\frac{\log^{s_1-1}(t_0/t_1)}{\Gamma(s_1)}\ldots\int_0^{t_{r-1}}\omega_1(t_r)\frac{\log^{s_r-1}(t_{r-1}/t_r)}{\Gamma(s_r)}\\ &= \prod_{i=1}^r \frac{1}{\Gamma(s_i)}\int_{[0,1]^r}\prod_{j=1}^r \omega_0(u_j)\lambda(u_1\ldots u_j)\log^{s_j-1}(\frac{1}{u_j})\\ &= \prod_{i=1}^r \frac{1}{\Gamma(s_i)}\int_{\mathbb{R}_+^r}\prod_{j=1}^r \omega_0(u_j)u_j^{s_j}\lambda(e^{-(u_1\ldots u_j)}),\end{aligned} \tag{16}$$

with $\omega_0(z) = \mathrm{d}z/z$ and $\omega_1(z) = \mathrm{d}z/(1-z)$.
As for the Riemann zeta function in (1), we observe that (16) involves again the factors (and products) of eulerian Gamma function and also their quotients (hence, eulerian Beta function). In the sequel, in continuation with [6, 8, 13], we propose to study the ratios $\zeta(s_1,\ldots,s_r)/\pi^{s_1+\ldots+s_r}$ (and others), an analogue of (9), which will be achieved as consequence of regularizations, via the values of *entire* functions, of *divergent* polyzetas and infinite sums of polyzetas (see Theorem 2.22 and Corollaries 2.24, 2.27 in Section 2.4) for which a theorem by Abel (see (11)) could not help any more. This achievement is justified thanks to the extensions of polylogarithms and harmonic sums (see Theorems 2.16 and 2.18 in Section 2.3) and thanks to the study of the independence of a family of eulerian functions which can be viewed as generating series of zeta values (for $r \ge 2$):

$$\frac{1}{\Gamma_{y_r}(z+1)} = \sum_{k\ge0}\zeta(\underbrace{r,\ldots,r}_{k\,\text{times}})z^{kr} = \exp\Big(-\sum_{k\ge1}\zeta(kr)\frac{(-z^r)^k}{k}\Big) \tag{17}$$

(see Propositions 2.7–2.11 and Theorem 2.13 in Section 2.2) via the combinatorial tools introduced in Section 2.1 (see Lemmas 2.1, 2.2 in Section 2.1). Finally, identities among these (convergent or divergent) generating series of zeta values are suitable to obtain relations, at arbitrary weight, among polyzetas (see Examples 2.26 and 2.28 in Section 2.4).

2. Families of eulerian functions

In all the sequel, $\mathbb{C}[\{f_i\}_{i\in I}]$ denotes the algebra generated by $\{f_i\}_{i\in I}$, $\mathbb{C}\{\{(g_i)_{i\in I}\}\}$ denotes the differential $\mathbb{C}$-algebra[5], generated by the family $(g_i)_{i\in I}$ of the $\mathbb{C}$-commutative differential ring $(\mathcal{A},\partial)$ ($1_{\mathcal{A}}$ is its neutral element) and $\mathcal{C}_0$ denotes a differential subring of $\mathcal{A}$ ($\partial\mathcal{C}_0 \subset \mathcal{C}_0$) which is an integral domain containing the field of constants. If the ring $\mathcal{A}$ is without zero divisors then the fields of fractions $\mathrm{Frac}(\mathcal{C}_0)$ and $\mathrm{Frac}(\mathcal{A})$ are naturally differential fields and can be seen as the smallest ones containing $\mathcal{C}_0$ and $\mathcal{A}$, respectively, satisfying $\mathrm{Frac}(\mathcal{C}_0) \subset \mathrm{Frac}(\mathcal{A})$.

2.1. Words and formal power series.

— Let $\mathcal{X}$ denote either the alphabets $X := \{x_0, x_1\}$ or $Y := \{y_k\}_{k\ge1}$, equipped with a total ordering, and let $\mathcal{X}^*$ denote the monoid freely generated by $\mathcal{X}$ (its unit is denoted by $1_{\mathcal{X}^*}$). The set of noncommutative polynomials (resp. series) over $\mathcal{X}$ with coefficients in a commutative ring A, containing $\mathbb{Q}$, is denoted by $A\langle\mathcal{X}\rangle$

[4]On $\mathcal{H}_r$, $\log(a/b)$ is replaced by $\log(a) - \log(b)$.
[5]i.e. the $\mathbb{C}$-algebra generated by g_i and their derivatives [16].

(resp. $A\langle\langle\mathcal{X}\rangle\rangle$) [1]. The algebraic closure of[6] $\widehat{A.\mathcal{X}}$ by the rational operations[7] $\{\mathtt{conc},+,*\}$ within $A\langle\langle\mathcal{X}\rangle\rangle$ is denoted by $A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ [1]. We will also consider the following Hopf algebras and, in the case of $A=\mathbf{k}$ being a field, their Sweedler's dual[8] [7, 13]

$$(A\langle\mathcal{X}\rangle,\mathtt{conc},\Delta_{\sqcup\!\sqcup},1_{\mathcal{X}^*},\epsilon)\quad\text{and}\quad(A\langle Y\rangle,\mathtt{conc},\Delta_{\sqcup\!\sqcup},1_{Y^*},\epsilon), \tag{18}$$

$$(\mathbf{k}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle,\sqcup\!\sqcup,1_{\mathcal{X}^*},\Delta_{\mathtt{conc}},\epsilon)\quad\text{and}\quad(\mathbf{k}^{\mathrm{rat}}\langle\langle Y\rangle\rangle,\sqcup\!\sqcup,1_{Y^*},\Delta_{\mathtt{conc}},\epsilon). \tag{19}$$

In particular, using the set of Lyndon words, denoted by $\mathcal{L}yn\mathcal{X}$, one constructs the basis $\{P_l\}_{l\in\mathcal{L}yn\mathcal{X}}$, for $\mathcal{L}ie_A\langle\mathcal{X}\rangle$, generating the PBW-Lyndon basis $\{P_w\}_{w\in\mathcal{X}^*}$ for $(A\langle\mathcal{X}\rangle,\mathtt{conc},1_{\mathcal{X}^*})$ and then the graded dual basis $\{S_w\}_{w\in\mathcal{X}^*}$ containing the pure transcendence basis $\{S_l\}_{l\in\mathcal{L}yn\mathcal{X}}$ for the shuffle algebra $(A\langle\langle\mathcal{X}\rangle\rangle,\sqcup\!\sqcup,1_{\mathcal{X}^*})$. Similarly, the basis $\{\Pi_l\}_{l\in\mathcal{L}ynY}$ generating the PBW-Lyndon basis $\{\Pi_w\}_{w\in Y^*}$ for $(A\langle Y\rangle,\mathtt{conc},1_{Y^*})$ and then the graded dual basis $\{\Sigma_w\}_{w\in Y^*}$ containing the pure transcendence basis $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$ for the stuffle algebra $(A\langle\langle Y\rangle\rangle,\sqcup\!\sqcup,1_{Y^*})$.

Lemma 2.1. —

1. *The algebras* $(\mathbb{C}[\{x^*\}_{x\in\mathcal{X}}],\sqcup\!\sqcup,1_{\mathcal{X}^*})$ *and* $(\mathbb{C}\langle\mathcal{X}\rangle,\sqcup\!\sqcup,1_{\mathcal{X}^*})$ *are algebraically disjoint over* $\mathbb{C}$ *and*
$$\begin{aligned}(\mathbb{C}[\{x^*\}_{x\in\mathcal{X}}]\langle\mathcal{X}\rangle,\sqcup\!\sqcup,1_{\mathcal{X}^*})&\cong(\mathbb{C}[\{x^*\}_{x\in\mathcal{X}}][\mathcal{L}yn\mathcal{X}],\sqcup\!\sqcup,1_{\mathcal{X}^*})\\&\cong(\mathbb{C}[\{x^*,l\}_{x\in\mathcal{X},l\in\mathcal{L}yn\mathcal{X}}],\sqcup\!\sqcup,1_{\mathcal{X}^*})\end{aligned}$$
which is generated by the transcendent basis $\{x^*,l\}_{x\in\mathcal{X},l\in\mathcal{L}yn\mathcal{X}}$ *over* $\mathbb{C}$.
2. *Let* $K:=\mathbb{C}[\{f(x^*)\}_{x\in\mathcal{X}}]$ *and* $F:=\mathbb{C}[\{f(l)\}_{l\in\mathcal{L}yn\mathcal{X}}]$.

   Let f be the shuffle morphism $(\mathbb{C}[\{x^*\}_{x\in\mathcal{X}}]\langle\mathcal{X}\rangle,\sqcup\!\sqcup,1_{\mathcal{X}^*})\to(\mathcal{A},\times,1_{\mathcal{A}})$.

   Then the following assertions are equivalent
   (a) *The morphism f is injective.*
   (b) *The algebras K and F, satisfying* $K\cap F=\mathbb{C}.1_{\mathcal{A}}$, *are generated by the transcendent bases* $\{f(x^*)\}_{x\in\mathcal{X}}$ *and* $\{f(l)\}_{l\in\mathcal{L}yn\mathcal{X}}$, *respectively, over* $\mathbb{C}$.

   Hence, if (a), *or* (b), *holds then F,K are algebraically disjoint over* $\mathbb{C}$ *and*
$$\mathbb{C}[\{f(x^*)\}_{x\in\mathcal{X}}][\{f(l)\}_{l\in\mathcal{L}yn\mathcal{X}}]\cong\mathbb{C}[\{f(x^*),f(l)\}_{x\in\mathcal{X},l\in\mathcal{L}yn\mathcal{X}}]$$
   which is generated by the transcendent basis $\{f(x^*),f(l)\}_{x\in\mathcal{X},l\in\mathcal{L}yn\mathcal{X}}$ *over* $\mathbb{C}$.

Proof. —

1. — Recall that the algebras $(\mathbb{C}[\{x^*\}_{x\in\mathcal{X}}],\sqcup\!\sqcup,1_{\mathcal{X}^*})$ and $(\mathbb{C}\langle\mathcal{X}\rangle,\sqcup\!\sqcup,1_{\mathcal{X}^*})$ are generated, respectively, by the transcendent bases $\{x^*\}_{x\in\mathcal{X}}$ [7] and $\mathcal{L}yn\mathcal{X}$ [17]. Moreover, $\{x^*\}_{x\in\mathcal{X}}$ is also algebraically independent over $\mathbb{C}\langle\mathcal{X}\rangle$ [7] and then $\mathbb{C}[\{x^*\}_{x\in\mathcal{X}}]\cap\mathbb{C}\langle\mathcal{X}\rangle=\mathbb{C}.1_{\mathcal{X}^*}$. It follows the then expected results.

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[6]In general, $\widehat{A.\mathcal{X}}$ is the module of homogeneous series $S\in A\langle\langle\mathcal{X}\rangle\rangle$ of degree one.
[7]Here $\mathtt{conc}$ stand for the Cauchy product (concatenation) and $\Delta_{\mathtt{conc}}$ is its co-product.
For any $S\in A\langle\langle\mathcal{X}\rangle\rangle$ such that $\langle S|1_{\mathcal{X}^*}\rangle=0$, the Kleene star of S is defined by $S^*:=(1-S)^{-1}=1+S+S^2+\ldots$.
[8]Here, $\sqcup\!\sqcup$ (resp. $\sqcup\!\sqcup$) stand for the shuffle (resp. stuffle) product and $\Delta_{\sqcup\!\sqcup}$ (resp. $\Delta_{\sqcup\!\sqcup}$) is its co-product (see [17] or [6]).
The antipode of the first one is given by $a(w)=(-1)^{|w|}\widetilde{w}$, the antipode of the second one exists because the bialgebra is graded by weight, but is more complicated.

2. — Straightforward. □

Now, for any $r \geq 1$, let us consider the following differential form

(20) $$\omega_r(z) = u_{y_r}(z)\mathrm{d}z \quad \text{with} \quad u_{y_r} \in \mathcal{C}_0 \subset \mathcal{A}.$$

Let us also consider the following noncommutative differential equation (see [7])

(21) $$\mathbf{d}S = MS; \langle S|1_{\mathcal{X}^*}\rangle = 1_{\mathcal{A}}, \quad \text{where } M = \sum_{x\in\mathcal{X}} u_x x \in \widehat{\mathcal{C}_0\mathcal{X}},$$

where $\mathbf{d}$ is the differential operator on $\mathcal{A}\langle\langle\mathcal{X}\rangle\rangle$ extending ∂ as follows:

(22) $$\forall\, S = \sum_{w\in\mathcal{X}^*} \langle S|w\rangle w \in \mathcal{A}\langle\langle\mathcal{X}\rangle\rangle, \quad \mathbf{d}S = \sum_{w\in\mathcal{X}^*} (\partial\langle S|w\rangle)w.$$

In order to prove Proposition 2.8, Theorems 2.13 and 2.16 below, we use the following lemma, a particular case of a general localization result to be proved in a forthcoming paper [7].

Lemma 2.2. — *Suppose that the $\mathbb{C}$-commutative ring $\mathcal{A}$ is without zero divisors and equipped with a differential operator ∂ such that $\mathbb{C} = \ker\partial$.*
Let $S \in \mathcal{A}\langle\langle\mathcal{X}\rangle\rangle$ be a group-like solution of (21), in the following form

$$S = 1_{\mathcal{X}^*} + \sum_{w\in\mathcal{X}^*\mathcal{X}} \langle S|w\rangle w = 1_{\mathcal{X}^*} + \sum_{w\in\mathcal{X}^*\mathcal{X}} \langle S|S_w\rangle P_w = \prod_{l\in\mathcal{L}yn\mathcal{X}}^{\searrow} e^{\langle S|S_l\rangle P_l}.$$

Then

1. *If $H \in \mathcal{A}\langle\langle\mathcal{X}\rangle\rangle$ is another group-like solution of (21) then there exists $C \in \mathcal{L}ie_{\mathcal{A}}\langle\langle\mathcal{X}\rangle\rangle$ such that $S = He^C$ (and conversely).*
2. *The following assertions are equivalent*
   - (a) *$\{\langle S|w\rangle\}_{w\in\mathcal{X}^*}$ is $\mathcal{C}_0$-linearly independent,*
   - (b) *$\{\langle S|l\rangle\}_{l\in\mathcal{L}yn\mathcal{X}}$ is $\mathcal{C}_0$-algebraically independent,*
   - (c) *$\{\langle S|x\rangle\}_{x\in\mathcal{X}}$ is $\mathcal{C}_0$-algebraically independent,*
   - (d) *$\{\langle S|x\rangle\}_{x\in\mathcal{X}\cup\{1_{\mathcal{X}^*}\}}$ is $\mathcal{C}_0$-linearly independent,*
   - (e) *The family $\{u_x\}_{x\in\mathcal{X}}$ is such that, for $f \in \mathrm{Frac}(\mathcal{C}_0)$ and $(c_x)_{x\in\mathcal{X}} \in \mathbb{C}^{(\mathcal{X})}$,*
   $$\sum_{x\in\mathcal{X}} c_x u_x = \partial f \longrightarrow (\forall\, x \in \mathcal{X})(c_x = 0).$$
   - (f) *The family $(u_x)_{x\in\mathcal{X}}$ is free over $\mathbb{C}$ and $\partial\,\mathrm{Frac}(\mathcal{C}_0) \cap \mathrm{span}_{\mathbb{C}}\{u_x\}_{x\in\mathcal{X}} = \{0\}$.*

Sketch of the proof. — The first item has been treated in [11]. The second is a group-like version of the abstract form of Theorem 1 of [4]. It goes as follows

- due to the fact that $\mathcal{A}$ is without zero divisors, we have the following embeddings $\mathcal{C}_0 \subset \mathrm{Frac}(\mathcal{C}_0) \subset \mathrm{Frac}(\mathcal{A})$, $\mathrm{Frac}(\mathcal{A})$ is a differential field, and its derivation can still be denoted by ∂ as it induces the previous one on $\mathcal{A}$,
- the same holds for $\mathcal{A}\langle\langle\mathcal{X}\rangle\rangle \subset \mathrm{Frac}(\mathcal{A})\langle\langle\mathcal{X}\rangle\rangle$ and $\mathbf{d}$
- therefore, equation (21) can be transported in $\mathrm{Frac}(\mathcal{A})\langle\langle\mathcal{X}\rangle\rangle$ and M satisfies the same condition as previously.

– Equivalence between (a)–(d) comes from the fact that $\mathcal{C}_0$ is without zero divisors and then, by denominator chasing, linear independances w.r.t. $\mathcal{C}_0$ and $\mathrm{Frac}(\mathcal{C}_0)$ are equivalent. In particular, supposing condition (d), the family $\{\langle S|x\rangle\}_{x\in\mathcal{X}\cup\{1_{\mathcal{X}^*}\}}$ (basic triangle) is $\mathrm{Frac}(\mathcal{C}_0)$-linearly independent which imply, by the Theorem 1 of [4], condition (e),

– still by Theorem 1 of [4], (e) is equivalent to (f), implying that $\{\langle S|w\rangle\}_{w\in\mathcal{X}^*}$ is $\mathrm{Frac}(\mathcal{C}_0)$-linearly independent which induces $\mathcal{C}_0$-linear independence (i.e. (a)). □

Now, let $\mathcal{A}=\mathcal{H}(\Omega)$, the ring of holomorphic functions on a simply connected domain $\Omega\subset\mathbb{C}$ ($1_{\mathcal{H}(\Omega)}$ is its neutral element). With the notations in (20) and for any path $z_0\rightsquigarrow z$ in Ω, let $\alpha_{z_0}^z:(\mathbb{C}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle,\sqcup\!\sqcup,1_{\mathcal{X}^*})\to(\mathcal{H}(\Omega),\times,1_{\mathcal{A}})$ be the morphism defined, for any $x_{i_1}\ldots x_{i_k})\in\mathcal{X}^*$, by [7]

$$\alpha_{z_0}^z(x_{i_1}\ldots x_{i_k})=\int_{z_0}^{z}\omega_{i_1}(z_1)\ldots\int_{z_0}^{z_{k-1}}\omega_{i_k}(z_k)\quad\text{and}\quad\alpha_{z_0}^z(1_{\mathcal{X}^*})=1_{\mathcal{H}(\Omega)},\tag{23}$$

satisfying $\alpha_{z_0}^z(u\sqcup\!\sqcup v)=\alpha_{z_0}^z(u)\alpha_{z_0}^z(v)$, for $u,v\in\mathcal{X}^*$. By a Ree's theorem [17], the Chen series of $\{\omega_r\}_{r\geq1}$ and along the path $z_0\rightsquigarrow z$ in Ω is group-like:

$$C_{z_0\rightsquigarrow z}=\sum_{w\in\mathcal{X}^*}\alpha_{z_0}^z(w)w=\prod_{l\in\mathcal{L}yn\mathcal{X}}^{\searrow}e^{\alpha_{z_0}^z(S_l)P_l}\in\mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle.\tag{24}$$

Since $\partial\alpha_{z_0}^z(x_{i_1}\ldots x_{i_k})=u_{i_1}(z)\alpha_{z_0}^z(x_{i_2}\ldots x_{i_k})$ then $C_{z_0\rightsquigarrow z}$ is a solution of (21).

Remark 2.3. — For any $w\in\mathcal{X}\mathcal{X}^*$, the value of $\alpha_{z_0}^z(w)$ depends on $\{\omega_i\}_{i\geq1}$, or equivalently on $\{u_x\}_{x\in\mathcal{X}}$ and if $f_x(z)=\alpha_{z_0}^z(x)$ then, for any $n\geq0$, one has [10]

$$\alpha_{z_0}^z(x^n)=\alpha_{z_0}^z(x^{\sqcup\!\sqcup n}/n!)=f_x^n(z)n!\ \text{ and then }\ F_x(z):=\alpha_{z_0}^z(x^*)=e^{f_x(z)}.$$

With data in (20) and shuffle morphism in (23), we will illustrate a bijection, between $(\mathbb{C}\langle\mathcal{X}\rangle\sqcup\!\sqcup\mathbb{C}[\{x^*\}_{x\in\mathcal{X}}],\sqcup\!\sqcup,1_{\mathcal{X}^*})$, the subalgebra of noncommutative rational series and a subalgebra of $\mathcal{H}(\Omega)$ containing the eulerian functions bellow.

2.2. Families of eulerian functions. —

Definition 2.4. — For any $z\in\mathbb{C}$ such that $|z|<1$, we put

$$\ell_1(z):=\gamma z-\sum_{k\geq2}\zeta(k)\frac{(-z)^k}{k}\quad\text{and for}\quad r\geq2,\ell_r(z):=-\sum_{k\geq1}\zeta(kr)\frac{(-z^r)^k}{k}.$$

For any $k\geq1$, let $\Gamma_{y_k}(1+z):=e^{-\ell_k(z)}$ and $\mathrm{B}_{y_k}(a,b):=\dfrac{\Gamma_{y_k}(a)\Gamma_{y_k}(b)}{\Gamma_{y_k}(a+b)}$.

Remark 2.5. —

1. $(\ell_r)_{r\geq1}$ is triangular[9]. So is $(e^{\ell_r}-e^{\ell_r(0)})_{r\geq1}$.

[9] $(g_i)_{i\geq1}$ is said to be *triangular* if the valuation of g_i, $\varpi(g_i)$, equals $i\geq1$. It is easy to check that such a family is $\mathbb{C}$-linearly free and that is also the case of families such that $(g_i-g(0))_{i\geq1}$ is triangular.

2. For any $z \in \Omega = \mathbb{C}, |z| < 1$ and $k \geq 1$, using Remark 2.3, one has

u_{y_k}	$\alpha_0^z(y_k)$	$\alpha_0^z(y_k^*)$
$1_{\mathcal{H}(\Omega)}$	z	e^z
$(1-z)^{-1}$	$-\log(1-z)$	$(1-z)^{-1}$
$\partial\ell_k$	$\ell_k(z)$	$e^{\ell_k(z)} = \Gamma_{y_k}^{-1}(1+z)$
$e^{\ell_k}\partial\ell_k$	$e^{\ell_k(z)} = \Gamma_{y_k}^{-1}(1+z)$	$e^{e^{\ell_k(z)}}$

3. The function ℓ_1 is already considered by Legendre for studying the eulerian Beta and Gamma functions [15], denoted here, repectively, by B_{y_1} and Γ_{y_1}.

4. For any $r \geq 1$, one has $\partial\ell_r = e^{-\ell_r}\partial e^{\ell_r}$.

5. For any $n \geq 0$, one puts classically $\Psi_n := \partial^n \log\Gamma$.

6. Some of these functions cease (unlike Γ) to be hypertranscendental. For example[10] $y(x) = \Gamma_{y_2}^{-1}(1+x)$ is a solution of $(1-\pi^2x^2)y^2 + 2xy\dot{y} + x^2\dot{y}^2 = 1$.

Now, for any $r \geq 1$, let G_r (resp. $\mathcal{G}_r$) denote the set (resp. group) of solutions, $\{\xi_0, \ldots, \xi_{r-1}\}$, of the equation $z^r = (-1)^{r-1}$ (resp. $z^r = 1$). For $r, q \geq 1$, we will need also a system $\mathbb{X}$ of representatives of $\mathcal{G}_{qr}/\mathcal{G}_r$, i.e. $\mathbb{X} \subset \mathcal{G}_{qr}$ such that

$$\mathcal{G}_{qr} = \biguplus_{\tau\in\mathbb{X}} \tau\mathcal{G}_r.$$

It can also be assumed that $1 \in \mathbb{X}$ as with $\mathbb{X} = \{e^{2\mathrm{i}k\pi/qr}\}_{0\leq k\leq q-1}$.

Remark 2.6. — If r is odd then $z^r = (-1)^{r-1} = 1$ and $G_r = \mathcal{G}_r$ as being a group otherwise $G_r = \xi\mathcal{G}_r$ as being an orbit, where ξ satisfies $\xi^r = -1$ (this is equivalent to $\xi \in \mathcal{G}_{2r}$ and $\xi \notin \mathcal{G}_r$).

Proposition 2.7. —

1. *For $r \geq 1, \chi \in \mathcal{G}_r$ and $z \in \mathbb{C}, |z| < 1$, the functions ℓ_r and e^{ℓ_r} have the symmetry, $\ell_r(z) = \ell_r(\chi z)$ and $e^{\ell_r(z)} = e^{\ell_r(\chi z)}$. In particular, for r even, as $-1 \in \mathcal{G}_r$, these functions are even.*

2. *For $|z| < 1$, we have*
$$\ell_r(z) = -\sum_{\chi\in G_r} \log(\Gamma(1+\chi z)) \quad \textit{and} \quad e^{\ell_r(z)} = \prod_{\chi\in G_r} e^{\gamma\chi z} \prod_{n\geq 1}\Big(1+\frac{\chi z}{n}\Big)e^{-\chi z/n}.$$

3. *For any odd $r \geq 2$,*
$$\Gamma_{y_r}^{-1}(1+z) = e^{\ell_r(z)} = \Gamma^{-1}(1+z) \prod_{\chi\in G_r\setminus\{1\}} e^{\ell_1(\chi z)}.$$

4. *In general, for any odd or even $r \geq 2$,*
$$e^{\ell_r(z)} = \prod_{\chi\in G_r} e^{\ell_1(\chi z)} = \prod_{n\geq 1}\Big(1+\frac{z^r}{n^r}\Big).$$

[10]Indeed, we use the fact that $\Gamma_{y_2}^{-1}(1+x) = \sin(\mathrm{i}\pi x)/\mathrm{i}\pi x$ (see Example 2.26 bellow).

Proof. — The results are known for $r = 1$ (i.e. for Γ^{-1}). For $r \geq 2$, we get

1. — By Definition 2.4, with $\chi \in \mathcal{G}_r$, we get

$$\ell_r(\chi z) = -\sum_{k\geq 1} \zeta(kr)\frac{(-\chi^r z^r)^k}{k} = -\sum_{k\geq 1} \zeta(kr)\frac{(-z^r)^k}{k} = \ell_r(z),$$

thanks to the fact that, for any $\chi \in \mathcal{G}_r$, one has $\chi^r = 1$. In particular, if r is even then $\ell_r(z) = \ell_r(-z)$, i.e. ℓ_r is even.

2. — If r is odd, as $G_r = \mathcal{G}_r$ and, applying the symmetrization principle[11], we get

$$\begin{aligned} -\sum_{\chi\in G_r} \ell_1(\chi z) &= -\sum_{\chi\in\mathcal{G}_r} \ell_1(\chi z) \\ &= r\sum_{k\geq 1} \zeta(kr)\frac{(-z)^{kr}}{kr} = \sum_{k\geq 1} \zeta(kr)\frac{(-z^r)^k}{k}. \end{aligned}$$

The last term being due to, precisely, r is odd. If r is even, we have the orbit $G_r = \xi\mathcal{G}_r$ (still with $\xi^r = -1$) and then, by the same principle,

$$\begin{aligned} -\sum_{\chi\in\mathcal{G}_r} \ell_1(\chi\xi z) &= r\sum_{k\geq 1} \zeta(kr)\frac{(-\xi z)^{kr}}{kr} \\ &= \sum_{k\geq 1} \zeta(kr)\frac{\left((-\xi z)^r\right)^k}{k} = \sum_{k\geq 1} \zeta(kr)\frac{\left(-z^r\right)^k}{k}. \end{aligned}$$

3. — Straightforward.

4. — Due to the fact that the external product is finite, we get

$$e^{\ell_r(z)} = \overbrace{\left(\prod_{\chi\in\mathcal{G}_r} e^{\gamma\chi z}\right)}^{=1} \prod_{\substack{n\geq 1\\ \chi\in\mathcal{G}_r}} \Big(1 + \frac{\chi z}{n}\Big) e^{-\chi z/n} = \overbrace{\left(\prod_{\substack{n\geq 1\\ \chi\in\mathcal{G}_r}} e^{-\frac{\chi z}{n}}\right)}^{=1} \prod_{\substack{n\geq 1\\ \chi\in\mathcal{G}_r}} \Big(1 + \frac{\chi z}{n}\Big).$$

Using the elementary symmetric functions of G_r, we get the expected result. □

Proposition 2.8. — *Let $L := \mathrm{span}_{\mathbb{C}}\{\ell_r\}_{r\geq 1}$ and $E := \mathrm{span}_{\mathbb{C}}\{e^{\ell_r}\}_{r\geq 1}$. Let $\mathbb{C}[L]$ and $\mathbb{C}[L]$ be their respective algebra. One has*

1. *The family $(\ell_r)_{r\geq 1}$ is $\mathbb{C}$-linearly free and free from $1_{\mathcal{H}(\Omega)}$.*
2. *The family $(\ell_r)_{r\geq 1}$ and $(e^{\ell_r})_{r\geq 1}$ is $\mathbb{C}$-linearly free and free from $1_{\mathcal{H}(\Omega)}$.*
3. *The families $(\ell_r)_{r\geq 1}$ and $(e^{\ell_r})_{r\geq 1}$ are $\mathbb{C}$-algebraically independent.*
4. *For any $r \geq 1$, one has*
   (a) *The functions ℓ_r and e^{ℓ_r} are $\mathbb{C}$-algebraically independent.*
   (b) *The function ℓ_r is holomorphic on the open unit disc, $D_{<1}$,*

[11]Within the same disk of convergence as f, one has, $f(z) = \sum_{n\geq 1} a_n z^n$ and $\sum_{\chi\in\mathcal{G}_r} f(\chi z) = r\sum_{k\geq 1} a_{rk} z^{rk}$.

(c) *The function* e^{ℓ_r} *(resp.* $e^{-\ell_r}$*) is entire (resp. meromorphic), and admits a countable set of isolated zeroes (resp. poles) on the complex plane which is expressed as* $\biguplus_{\chi\in G_r}\chi\mathbb{Z}_{\leq -1}$.

5. *One has* $E\cap L=\{0\}$ *and, more generally,* $\mathbb{C}[E]\cap\mathbb{C}[L]=\mathbb{C}.1_{\mathcal{H}(\Omega)}$.

Proof. —

1. — Suppose that there is $(a_r)_{r\geq 1}\in\mathbb{C}^{(\mathbb{N})}$ such that

$$\sum_{r\geq 1}a_r\ell_r(z)=a_1\gamma z-\sum_{k\geq 2}a_1\zeta(k)\frac{(-1)^k}{k}z^k-\sum_{r\geq 2}\sum_{k\geq 1}a_r\zeta(kr)\frac{(-1)^k}{k}z^{rk}=0,$$

in which, since $\gamma\neq 0$ then $a_1=0$. It follows that

$$\sum_{r\geq 2}\sum_{k\geq 1}a_r\zeta(kr)\frac{(-1)^k}{k}z^{rk}=0$$

in which $\langle \mathrm{LS}|z^2\rangle=a_2\zeta(2)/2$. Since $\zeta(2)\neq 9$ then $a_2=0$. It also follows that

$$\sum_{r\geq 3}\sum_{k\geq 1}a_r\zeta(kr)\frac{(-1)^k}{k}z^{rk}=0.$$

In similar way, one proves that $a_r=0$, for $r\in\mathbb{N}^+$. Hence, $(\ell_r)_{r\geq 1}$ is $\mathbb{C}$-free.

2. — Suppose that there is $(b_i)_{i\geq 1}\in\mathbb{C}^{(\mathbb{N})}$ such that

$$\sum_{i\geq 1}a_ie^{\ell_i}=0\text{and then}\sum_{i\geq 1}a_i\dot{\ell}_i=0$$

(taking the logarithmic derivative). By integration, one deduces then $(\ell_r)_{r\geq 1}$ is $\mathbb{C}$-linearly dependent contradicting with the item 1. It remains that $(e^{f_i})_{i\in I}$ is $\mathbb{C}$-free.

3. — Using Chen series of $\{\omega_r\}_{r\geq 1}$ defined, as in Remark 2.5, by $u_{x_r}=e^{\ell_r}\partial\ell_r$ (resp. $u_{x_r}=\partial\ell_r$), via items a or b of Lemma 2.2, $\{e^{\ell_r}\}_{r\geq 1}$ (resp. $\{\ell_r\}_{r\geq 1}$) is the $\mathbb{C}$-algebraically independent.

4. —

(a) Since $\ell_r(0)=0,\partial e^{\ell_r}=e^{\ell_r}\partial\ell_r$ then ℓ_r and e^{ℓ_r} are $\mathbb{C}$-algebraically independent.

(b) One has $e^{\ell_1(z)}=\Gamma^{-1}(1+z)$ which proves the claim for $r=1$. For $r\geq 2$, note that $1\leq\zeta(r)\leq\zeta(2)$ which implies that the radius of convergence of the exponent is 1 and means that ℓ_r is holomorphic on the open unit disc. This proves the claim.

(c) $e^{\ell_r(z)}=\Gamma_{y_r}^{-1}(1+z)$ (resp. $e^{-\ell_r(z)}=\Gamma_{y_r}(1+z)$) is entire (resp. meromorphic) as finite product of entire (resp. meromorphic) functions and, by Proposition 2.7, Weierstrass factorization yields zeroes (resp. poles).

5. — Let $f \in E \cap L$ and then there is $\{c_y\}_{y\in Y}$ and $\{d_y\}_{y\in Y} \in \mathbb{C}^{(Y)}$ such that

$$f = \sum_{r\geq 1} c_{y_r}\ell_r = \sum_{r\geq 1} d_{y_r} e^{\ell_r}$$

If $f \neq 0$ then $\ell_{r_0}, e^{\ell_{r_0}}$ could be linearly dependent, for some $r_0 \geq 1$, contradicting with item 1. Hence, $E \cap L = \{0\}$.
$\mathbb{C}[E]$ (resp. $\mathbb{C}[F]$) is generated freely by $(e^{\ell_r})_{r\geq 1}$ (resp. $(\ell_r)_{r\geq 1}$) which are entire (resp. holomorphic on $D_{<1}$) functions. Moreover, any $\mathbb{C}[E] \ni f \neq c1_\Omega$ ($c \in \mathbb{C}$) is entire and then $f \notin \mathbb{C}[L]$ (and conversely). It follows the expected result. □

By Lemma 2.1, Proposition 2.8 and Remark (2.5), one deduces then

Corollary 2.9. — *The map $\alpha_0^z : (\mathbb{C}\langle Y\rangle, ⧢, 1_{Y^*}) \to (\mathrm{span}_{\mathbb{C}}\{\alpha_0^z(w)\}_{w\in Y^*}, \times, 1_{\mathcal{H}(\Omega)})$ is injective, for the inputs $\{\partial\ell_r\}_{r\geq 1}$ or $\{e^{\ell_r}\partial\ell_r\}_{r\geq 1}$, and then $\{\alpha_0^z(w)\}_{w\in Y^*}$ (resp. $\{\alpha_0^z(l)\}_{l\in \mathcal{L}ynY}$ is linearly (resp. algebraically) independent over $\mathbb{C}$.*

From now on the countable set of isolated zeros (resp. poles) of the entire (resp. meromorphic) function e^{ℓ_r} (resp. $e^{-\ell_r}$) is denoted by $\mathcal{O}(e^{\ell_r})$. We have

$$\mathcal{O}(e^{\ell_r}) = \biguplus_{\chi\in G_r} \chi\mathbb{Z}_{\leq -1}. \tag{25}$$

Example 2.10. — One has

$$\begin{aligned}
\mathcal{O}(e^{\ell_1}) &= \mathbb{Z}_{\leq -1},\\
\mathcal{O}(e^{\ell_2}) &= -\mathrm{i}\mathbb{Z}_{\leq -1} \uplus \mathrm{i}\mathbb{Z}_{\leq -1} = \mathrm{i}\mathbb{Z}_{\neq 0},\\
\mathcal{O}(e^{\ell_3}) &= \mathbb{Z}_{\leq -1} \uplus \mathrm{j}\mathbb{Z}_{\leq -1} \uplus \mathrm{j}^2\mathbb{Z}_{\leq -1},\\
\mathcal{O}(e^{\ell_4}) &= (1+\mathrm{i})/\sqrt{2}\mathbb{Z}_{\neq 0} \uplus (1-\mathrm{i})/\sqrt{2}\mathbb{Z}_{\neq 0}.
\end{aligned}$$

Proposition 2.11. — *Let $\mathbb{X}$ denote any system of representatives of $\mathcal{G}_{qr}/\mathcal{G}_r$.*

1. *For any $r \geq 1$ and odd $q \geq 1$, one has, for $|z| < 1$,*
$$e^{\ell_{qr}(z)} = \prod_{\chi\in\mathbb{X}} e^{\ell_r(\chi z)}, \quad \text{or equivalently,} \quad \Gamma^{-1}_{y_{qr}}(1+z) = \prod_{\chi\in\mathbb{X}} \Gamma^{-1}_{y_r}(1+\chi z).$$
2. *e^{ℓ_r} divides $e^{\ell_{qr}}$ if and only if q is odd.*
3. *The full symmetry group of e^{ℓ_r} for the representation $s * f[z] = f(sz)$ is $\mathcal{G}_r$.*

Proof. —

1. — Let ξ be any root of $z^r = (-1)^{r-1}$, one remarks that, in all cases (r be odd or even), we have

$$G_r = \xi\mathcal{G}_r, G_{qr} = \xi\mathcal{G}_{qr}, \mathcal{G}_{qr} = \biguplus_{\chi\in\mathbb{X}} \chi\mathcal{G}_r.$$

Then, by Proposition 2.7, we have

$$\begin{aligned}\ell_{qr}(z) &= \sum_{\chi\in G_{qr}} \ell_1(\chi z)\\ &= \sum_{\rho_1\in\mathcal{G}_{qr}} \ell_1(\xi\rho_1 z) = \sum_{\chi\in\mathbb{X},\,\rho_2\in\mathcal{G}_r} \ell_1(\xi\rho_2\chi z)\\ &= \sum_{\chi\in\mathbb{X},\,\rho_2\in\mathcal{G}_r} \ell_1(\xi\rho_2(\chi z)) = \sum_{\chi\in\mathbb{X}} \ell_r(\chi z)\\ &= \ell_r(z) + \sum_{\chi\in\mathbb{X}\setminus\{1\}} \ell_r(\chi z).\end{aligned}$$

Last equality assumes that $1\in\mathbb{X}$. Taking exponentials, we get

$$e^{\ell_{qr}(z)} = \prod_{\chi\in\mathbb{X}} e^{\ell_r(\chi z)} = e^{\ell_r(z)} \prod_{\chi\in\mathbb{X}\setminus\{1\}} e^{\ell_r(\chi z)}. \tag{26}$$

Again, first equality is general and the last assumes that $1\in\mathbb{X}$.

2. — The fact that e^{ℓ_r} divides $e^{\ell_{qr}}$ if q is odd comes from the factorization (26). Now, when q even, it suffices to remark, from (25), that the opposite of any solution of $z^r=-1$ is a zero of[12] e^{ℓ_r} and $\mathcal{O}(e^{\ell_{qr}})\cap\mathbb{U}=-G_{qr}$. But when q is even one has $-G_r\cap -G_{qr}=\emptyset$. Hence, in this case, e^{ℓ_r} cannot divide $e^{\ell_{qr}}$.

3. — Let $\mathcal{G}$ denote the symmetry group of e^{ℓ_r} and remark that the distance of $\mathcal{O}(e^{\ell_r})$ to zero is 1. Hence, as $\mathcal{O}(e^{\ell_r}(s.z)) = s^{-1}\mathcal{O}(e^{\ell_r})$, we must have $\mathcal{G}\subset\mathbb{U}$. Then, by Remark 2.6, as $\mathcal{O}(e^{\ell_r})\cap\mathbb{U}=G_r$, we must have $\mathcal{G}\subset\mathcal{G}_r$, the reverse inclusion is exactly the first point of Proposition 2.7. □

Example 2.12. —

1. For $r=1, q=2, \mathbb{X}=\{1,-1\}$, one has the Euler's complement like formula, i.e. $\Gamma_{y_2}(1+\mathrm{i}z) = \Gamma_{y_1}(1+z)\Gamma_{y_1}(1-z) = z\pi/\sin(z\pi)$. Changing $z\mapsto -\mathrm{i}z$, one also has $\Gamma_{y_2}(1+z) = \Gamma_{y_1}(1+\mathrm{i}z)\Gamma_{y_1}(1-\mathrm{i}z)$.
2. For $r=2, q=3, \mathbb{X}=\{1,\mathrm{j},\mathrm{j}^2\}$, one has $\Gamma_{y_6}(1+z) = \Gamma_{y_2}(1+z)\Gamma_{y_2}(1+\mathrm{j}z)\Gamma_{y_2}(1+\mathrm{j}^2z)$.

With the notations of Proposition 2.8, the algebra $\mathbb{C}[L]$ (resp. $\mathbb{C}[E]$) is generated freely by $(\ell_r)_{r\geq 1}$ (resp. $(e^{\ell_r})_{r\geq 1}$) which are holomorphic on $D_{<1}$ (resp. entire) functions. Moreover,

$$E\cap L=\{0\}, \text{ and more generally, } \mathbb{C}[E]\cap\mathbb{C}[L]=\mathbb{C}.1_{\mathcal{H}(\Omega)}. \tag{27}$$

We are in a position to consider the following differential subalgebras of $(\mathcal{H}(\Omega),\partial)$:

$$\mathcal{L} := \mathbb{C}\{\{(\ell_r^{\pm 1})_{r\geq 1}\}\} \text{ and } \mathcal{E} := \mathbb{C}\{\{(e^{\pm\ell_r})_{r\geq 1}\}\}. \tag{28}$$

Since $\partial\ell_r^{-1} = -\ell_r^{-2}\partial\ell_r$ then $\mathcal{L}=\mathbb{C}[\{\ell_r^{\pm1},\partial^i\ell_r\}_{r,i\geq 1}]$. Let

$$\mathcal{L}^+ := \mathbb{C}[\{\partial^i\ell_r\}_{r,i\geq 1}]. \tag{29}$$

[12]More precisely, denoting $\mathbb{U}$ the unit circle, one has $\mathcal{O}(e^{\ell_r})\cap\mathbb{U}=-G_r\neq\emptyset$.

This $\mathbb{C}$-differential subalgebra $\mathcal{L}^+$ is an integral domain generated by holomorphic functions and $\mathrm{Frac}(\mathcal{L}^+)$ is generated by meromorphic functions. Since there is $0 \neq q_{i,l,k} \in \mathcal{L}^+$ such that $(\partial^i e^{\pm\ell_k})^l = q_{i,l,k} e^{\pm l\ell_k}$ $(i,l,k \geq 1)$ then let

$$\mathcal{E}^+ := \mathrm{span}_{\mathbb{C}}\{(\partial^{i_1} e^{\pm\ell_{r_1}})^{l_1} \ldots (\partial^{i_k} e^{\pm\ell_{r_k}})^{l_k}\}_{(i_1,l_1,r_1),\ldots,(i_k,l_k,r_k)\in(\mathbb{N}^*)^3,k\geq 1} \tag{30}$$

$$= \mathrm{span}_{\mathbb{C}}\{q_{i_1,l_1,r_1} \ldots q_{i_k,l_k,r_k} e^{l_1\ell_{r_1}+\ldots+l_k\ell_{r_k}}\}_{(i_1,l_1,r_1),\ldots,(i_k,l_k,r_k)\in\mathbb{N}^*\times\mathbb{Z}^*\times\mathbb{N}^*,k\geq 1} \tag{31}$$

$$\subset \mathrm{span}_{\mathcal{L}^+}\{e^{l_1\ell_{r_1}+\ldots+l_k\ell_{r_k}}\}_{(l_1,r_1),\ldots,(l_k,r_k)\in\mathbb{Z}^*\times\mathbb{N}^*,k\geq 1} \tag{32}$$

$$=: \mathcal{C}. \tag{33}$$

Note that in (33), $\mathcal{C}$ is a differential subring of $\mathcal{A} = \mathcal{H}(\Omega)$ (hence, $\mathrm{Frac}(\mathcal{C})$ is a differential subfield of $\mathrm{Frac}(\mathcal{A})$) and

$$\mathcal{E}^+ \cap E = \{0\}. \tag{34}$$

Theorem 2.13. —

1. *The family $(e^{\ell_r})_{r\geq 1}$ (resp. $(\ell_r)_{r\geq 1}$) is algebraically free over $\mathcal{E}^+$ (resp. $\mathcal{L}^+$).*

2. *$\mathbb{C}[E]$ and $\mathbb{C}[L]$ are algebraically disjoint, within $\mathcal{A}$.*

Proof. —

1. — Considering the Chen series of the differential forms $\{\omega_r\}_{r\geq 1}$ defined, for any $r \geq 1$, by $u_{y_r} = e^{\ell_r}\partial\ell_r$. Let $Q \in \mathrm{Frac}(\mathcal{L}) \cap E$ (resp. $\mathrm{Frac}(\mathcal{C}) \cap E$):

i. since $Q \in E$ then there is $\{c_y\}_{y\in Y} \in \mathbb{C}^{(Y)}$ such that

$$Q = \sum_{r\geq 1} c_{y_r} e^{\ell_r} \quad \text{and then} \quad \partial Q = \sum_{r\geq 1} c_{y_r} e^{\ell_r}\partial\ell_r, \tag{35}$$

ii. since $Q \in \mathrm{Frac}(\mathcal{L}) \supset \mathcal{L} \supset \mathbb{C}[L]$ (resp. $\mathrm{Frac}(\mathcal{C}) \supset \mathcal{C} \supset \mathcal{E}^+$) then, by (27) (resp. (34)) it remains that $Q = 0$.

Hence, by Proposition 2.8, since $\{e^{\ell_k}\}_{k\geq 1}$ is $\mathbb{C}$-free and $Q = 0$ then

i. on the one hand, for any $r \geq 1$, one has $c_{y_r} = 0$,

ii. on the other hand, $\{\alpha_0^z(S_l)\}_{l\in\mathcal{L}ynY}$ (including $\{\alpha_0^z(S_y)\}_{y\in Y}$) is algebraically free over $\mathcal{L}$ (resp. $\mathcal{C}$).

It follows that $\{e^{\ell_r}\}_{r\geq 1}$ is algebraically free over $\mathbb{C}[L]$ (resp. $\mathcal{E}^+$).
Now, suppose there is an algebraic relation among $(\ell_k)_{k\geq 1}$ over $\mathcal{L}^+$ in which, by differentiating and substituting $\partial\ell_k$ by $e^{-\ell_k}\partial e^{\ell_k}$, we get an algebraic relation among $\{e^{\ell_r}\}_{r\geq 1}$ over $\mathbb{C}[L]$ and $\mathcal{E}^+$ contradicting with previous results. It follows then $(\ell_k)_{k\geq 1}$ is $\mathcal{L}^+$-algebraically independent.

2. — $\{e^{\ell_k}\}_{k\geq 1}$ (resp. $\{\ell_k\}_{k\geq 1}$) is algebraically independent over $\mathbb{C}[L]$ (resp. $\mathbb{C}[E]$). Hence, $\{e^{\ell_k}, \ell_k\}_{k\geq 1}$ generates freely $\mathbb{C}[E+L]$ and $\mathbb{C}[E] \cap \mathbb{C}[L] = \mathbb{C}.1_{\mathcal{H}(\Omega)}$.
It follows that $\mathbb{C}[E]$ and $\mathbb{C}[L]$ are algebraically disjoint, within $\mathcal{A}$. □

Corollary 2.14. —

1. *Using the inputs* $\{\partial\ell_r\}_{r\geq 1}$ *(resp.* $\{e^{\ell_r}\partial\ell_r\}_{r\geq 1}$*), the following morphism is injective (see also Remark* (2.5)*)*

$$\alpha_0^z : (\mathcal{L}^+\langle Y\rangle, \sqcup, 1_{Y^*}) \longrightarrow (\mathrm{span}_{\mathcal{L}^+}\{\alpha_0^z(w)\}_{w\in Y^*}, \times, 1_{\mathcal{H}(\Omega)}),$$
$$(\textit{resp. } \alpha_0^z : (\mathcal{E}^+\langle Y\rangle, \sqcup, 1_{Y^*}) \longrightarrow (\mathrm{span}_{\mathcal{E}^+}\{\alpha_0^z(w)\}_{w\in Y^*}, \times, 1_{\mathcal{H}(\Omega)})).$$

   Hence, $\{\alpha_0^z(w)\}_{w\in Y^*}$ *(resp.* $\{\alpha_0^z(l)\}_{l\in\mathcal{L}ynY}$*) is linearly (resp. algebraically) independent over* $\mathcal{L}^+$ *(resp.* $\mathcal{E}^+$*).*

2. *Using the inputs* $\{\partial\ell_r\}_{r\geq 1}$ *and denoting the set of exchangeable polynomials (over* Y *and with coefficients in* $\mathbb{C}$*) by* $\mathbb{C}_{\mathrm{exc}}\langle Y\rangle$ *(see* [7] *for example), the family* $\{\alpha_0^z(\lambda)\}_{\lambda\in\mathcal{L}ynY\cup\{y_r^*\}_{r\geq 1}}$ *is* $\mathbb{C}$*-algebraically independent and the restricted morphism* α_0^z : $(\mathbb{C}_{\mathrm{exc}}\langle Y\rangle \sqcup \mathbb{C}[\{y_r^*\}_{r\geq 1}], \sqcup, 1_{Y^*}) \to \mathbb{C}[L+E]$ *is bijective.*

   Hence, $\{(e^{\ell_r})_{r\geq 1}, (\ell_r)_{r\geq 1}\}$ *is* $\mathbb{C}$*-algebraically independent.*

3. *Let* $\mathcal{C}_k := \mathrm{span}_{\mathcal{L}^+}\{e^{l_1\ell_{r_1}+\ldots+l_k\ell_{r_k}}\}_{(l_1,r_1),\ldots,(l_k,r_k)\in\mathbb{Z}^*\times\mathbb{N}^*}$*. Then*

$$\mathcal{C} = \bigoplus_{k\geq 1}\mathcal{C}_k$$

Proof. —

1. — It is a consequence of Theorem 2.13.

2. — The free algebras $(\mathbb{C}_{\mathrm{exc}}\langle Y\rangle, \sqcup, 1_{Y^*})$ and $(\mathbb{C}[\{y_r\}_{r\geq 1}], \sqcup, 1_{Y^*})$ are algebraically disjoint and their images by α_0^z, by Proposition 2.8, are, respectively, the free algebras $\mathbb{C}[L]$ and $\mathbb{C}[E]$ which are, by Theorem 2.13, algebraically disjoint. Moreover, since $\mathbb{C}_{\mathrm{exc}}\langle Y\rangle = \mathbb{C}[\{y\}_{y\in Y}]$ and $Y \subset \mathcal{L}ynY$ then we deduce the respected results.

3. — For any $k \geq 1$, let $\Phi_k := \mathrm{span}_{\mathbb{C}}\{e^{l_1\ell_{r_1}+\ldots+l_k\ell_{r_k}}\}_{\text{distinct } r_1,\ldots,r_k\in\mathbb{N}^*, l_1,\ldots,l_k\in\mathbb{Z}^*}$. Let $\mathbb{C}[\Phi]$ be the algebra of $\Phi := \mathrm{span}_{\mathbb{C}}\{e^{\pm\ell_r}\}_{r\geq 1}$. Since $(\ell_r)_{r\geq 1}$ is $\mathbb{C}$-free then $\Phi_1 \subsetneq \Phi_2 \subsetneq \ldots$ and then $\mathbb{C}[\Phi] = \bigoplus_{k\geq 1}\Phi_k$. Moreover, the disjunction of $\mathbb{C}[E]$ and $\mathbb{C}[L]$ leads to $\mathcal{C}_k \cong \mathcal{L}^+ \otimes_{\mathbb{C}} \Phi_k$ and then yields the expected result. □

Remark 2.15. — Let us back the second point of Proposition 2.11 and then the formula (26), for any $q \in \mathbb{N}_{\geq 1}$ such that $q \equiv 1 \pmod 2$,

$$e^{\ell_{qr}(z)} = e^{\ell_r(z)} \prod_{\chi\in\mathbb{X}\setminus\{1\}} e^{\ell_r(\chi z)}.$$

Since $(e^{\ell_r})_{r\geq 1}$ is algebraically free over $\mathcal{E}^+$ then

$$\prod_{\chi\in\mathbb{X}\setminus\{1\}} e^{\ell_r(\chi z)} \notin \mathcal{E}^+[(e^{\ell_k})_{k\geq 1}].$$

2.3. Polylogarithms and harmonic sums indexed by rational series. — Using the projector $\pi_X : (\mathbb{C}\langle Y\rangle, \cdot, 1_{Y^*}) \to (\mathbb{C}\langle X\rangle, \cdot, 1_{X^*})$ defined as the concatenation morphism, mapping y_s to $x_0^{s-1}x_1$ and admitting π_Y as adjoint, one has the following one-to-one correspondences

$$(s_1, \ldots, s_r) \in (\mathbb{N}^*)^r \leftrightarrow y_{s_1} \ldots y_{s_r} \in Y^* \underset{\pi_Y}{\overset{\pi_X}{\rightleftharpoons}} x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1 \in X^*x_1.$$

In all the sequel, $\Omega := \widetilde{\mathbb{C} \setminus \{0, 1\}}$ and

$$\omega_0(z) := z^{-1}\mathrm{d}z \quad\text{and}\quad \omega_1(z) := (1-z)^{-1}dz. \tag{36}$$

By (23), $\mathrm{Li}_{s_1,\ldots,s_r}(z) = \alpha_0^z(x_0^{s_1-1}x_1 \ldots x_0^{s_k-1}x_1)$, for $s_1, \ldots, s_r \in \mathbb{N}^*$. Thus, putting $\mathrm{Li}_{x_0}(z) := \log(z)$, the following morphisms are injective

$$\mathrm{Li}_\bullet : (\mathbb{Q}\langle X\rangle, ш, 1_{X^*}) \longrightarrow (\mathbb{Q}\{\mathrm{Li}_w\}_{w\in X^*}, \cdot, 1), \tag{37}$$

$$x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1 \longmapsto \mathrm{Li}_{x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1} = \mathrm{Li}_{s_1,\ldots,s_r}, \tag{38}$$

$$\mathrm{H}_\bullet : (\mathbb{Q}\langle Y\rangle, ш, 1_{Y^*}) \longrightarrow (\mathbb{Q}\{\mathrm{H}_w\}_{w\in Y^*}, \cdot, 1), \tag{39}$$

$$y_{s_1} \ldots y_{s_r} \longmapsto \mathrm{H}_{y_{s_1}\ldots y_{s_r}} = \mathrm{H}_{s_1,\ldots,s_r}. \tag{40}$$

In order to extend $\mathrm{Li}_\bullet, \mathrm{H}_\bullet$ (in (38), (40)) over some subdomain of $\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ (resp. $\mathbb{C}^{\mathrm{rat}}\langle\langle Y\rangle\rangle$), let us call $\mathrm{Dom}_R(\mathrm{Li}_\bullet)$ the set of series

$$S = \sum_{n\geq 0} S_n \quad\text{with}\quad S_n := \sum_{|w|=n} \langle S|w\rangle w \tag{41}$$

such that $\sum_{n\geq 0} \mathrm{Li}_{S_n}$ converge uniformly in any compact of Ω.
For any $0 < R \leq 1$, such that $\sum_{n\geq 0} \mathrm{Li}_{S_n}$ converge uniformly in the open disc $D_{|z|<R}$, one has $(1-z)^{-1}\mathrm{Li}_S = \sum_{N\geq 0} a_N z^N$ converge in the same disc and then $\mathrm{H}_{\pi_Y(S_n)}(N) = a_N$, for $N \geq 0$. Hence, let us define

$$\mathrm{Dom}_R(\mathrm{Li}_\bullet) := \{S \in \mathbb{C}1_{X_*} \oplus \mathbb{C}\langle\langle X\rangle\rangle x_1 | \textstyle\sum_{n\geq 1} \mathrm{Li}_{S_n} \text{ converge in } D_{|z|<R}\},$$

$$\mathrm{Dom}(\mathrm{H}_\bullet) := \pi_Y \, \mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li}_\bullet), \text{ where } \mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li}_\bullet) := \textstyle\bigcup_{0<R\leq 1} \mathrm{Dom}_R(\mathrm{Li}_\bullet).$$

Under suitable convergence condition this extension can be realized and [2, 6, 13]

1. $\mathrm{Dom}(\mathrm{Li}_\bullet)$ (resp. $\mathrm{Dom}(\mathrm{H}_\bullet)$) is closed by shuffle (resp. quasi-shuffle) products.

2. $\mathrm{Li}_{S ш T} = \mathrm{Li}_S \mathrm{Li}_T$ and $\mathrm{H}_{S ш T} = \mathrm{H}_S \mathrm{H}_T$, for $S, T \in \mathrm{Dom}(\mathrm{Li}_\bullet)$ (resp. $\mathrm{Dom}(\mathrm{H}_\bullet)$).

Any series $S \in \mathbb{C}\langle\langle \mathcal{X}\rangle\rangle$ is syntactically exchangeable iff it is of the form

$$S = \sum_{\alpha\in\mathbb{N}^{(\mathcal{X})}, \mathrm{supp}(\alpha)=\{x_1,\ldots,x_k\}} s_\alpha x_1^{\alpha(x_1)} ш \ldots ш x_k^{\alpha(x_k)}. \tag{42}$$

The set of these series, a ш-subalgebra of $A\langle\langle \mathcal{X}\rangle\rangle$, will be denoted by $\mathbb{C}_{\mathrm{exc}}^{\mathrm{synt}}\langle\langle \mathcal{X}\rangle\rangle$.

***Theorem 2.16* (extension of $\mathrm{Li}_\bullet$).** — *Let $\mathcal{C}_\mathbb{C} := \mathbb{C}[\{z^a, (1-z)^b\}_{a,b\in\mathbb{C}}]$. Then*

1. *The algebra $\mathcal{C}_\mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*}$ is closed under the differential operators $\theta_0 := z\partial_z$ and $\theta_1 := (1-z)\partial_z$ and under their sections ι_0, ι_1 ($\theta_0\iota_0 = \theta_1\iota_1 = \mathrm{Id}$).*

2. *The bi-integro differential algebra* $(\mathcal{C}_{\mathbb{C}}\{\mathrm{Li}_w\}_{w\in X^*},\theta_0,\theta_1,\iota_0,\iota_1)$ *is closed under the action of the group of transformations,* $\mathcal{G}$, *generated by* $\{z\mapsto 1-z, z\mapsto 1/z\}$, *permuting* $\{0,1,+\infty\}$*:*
$$\forall\ h\in\mathcal{C}_{\mathbb{C}}\{\mathrm{Li}_w\}_{w\in X^*},\ \ \forall\ g\in\mathcal{G}, h(g)\in\mathcal{C}_{\mathbb{C}}\{\mathrm{Li}_w\}_{w\in X^*}.$$

3. *If* $R\in\mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle\sqcup\mathbb{C}\langle X\rangle$ *(resp.* $\mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle$*) then* $\mathrm{Li}_R\in\mathcal{C}_{\mathbb{C}}\{\mathrm{Li}_w\}_{w\in X^*}$ *(resp.* $\mathcal{C}_{\mathbb{C}}[\log(z),\log(1-z)]$*).*

4. *The family* $\{\mathrm{Li}_w\}_{w\in X^*}$ *(resp.* $\{\mathrm{Li}_l\}_{l\in\mathcal{L}ynX}$*) is linearly (resp. algebraically) independent over* $\mathcal{C}_{\mathbb{C}}$*.*

Proof. — The three first items are immediate. Only the last one needs a proof:
Let then $B=\mathbb{C}\setminus\{0,1\}$, $\Omega=\mathbb{C}\setminus(]-\infty,0]\cup[1,+\infty[)$ and choose a basepoint $b\in\Omega$, one has the following diagram

$$\begin{array}{ccc} & & (\widetilde{B},\widetilde{b}) \\ & \overset{s}{\nearrow} & \downarrow p \\ (\Omega,b) & \overset{j}{\hookrightarrow} & (B,b) \end{array}$$

Any holomorphic function $f\in\mathcal{H}(\Omega)$ such that $f'=df/dz$ admits an analytic continuation to B can be lifted to $\widetilde{B}$ by $\widetilde{f}(z):=f(b)+\int_b^z f'(s)ds$.
Let L be the noncommutative series of the polylogarithms $\{\mathrm{Li}_w\}_{w\in X^*}$, which is group-like, and $C_{z_0\rightsquigarrow z}$ be the Chen series, of $\{\omega_0,\omega_1\}$ along $z_0\rightsquigarrow z\in\widetilde{B}$, $C_{z_0\rightsquigarrow z}=\mathrm{L}(z)\mathrm{L}^{-1}(z_0)$ (see [13]). Now, in view of Lemma 2.2, as the algebra $\mathcal{C}$ is without zero divisors and contains the field of constants $\mathbb{C}$, it suffices to prove that $\mathrm{Li}_{x_0},\mathrm{Li}_{x_1}$ and 1_Ω are $\mathcal{C}$-linearly independent. It is an easy exercise to check that $s_*(\widetilde{f}):=\widetilde{f}\circ s$ coincides with the given f. This is the case, in particular of the functions $\log(z)$ and $\log((1-z)^{-1})$ whose liftings will be denoted $\log_0$ and $\log_1$, respectively. So, we lift the functions z^a and $(1-z)^b$ as, respectively,

$$e_0^a(\widetilde{z}):=e^{a\log_0(\widetilde{z})}\ \text{ and }\ e_1^b:=e^{b\log_1(\widetilde{z})} \tag{43}$$

and, of course, by construction,

$$e_0^a\circ s=(z\mapsto z^a)\ \text{ and }\ e_1^b\circ s=(z\mapsto(1-z)^b) \tag{44}$$

We suppose a dependence relation, in $\mathcal{H}(\Omega)$

$$P_0(z^a,(1-z)^b)\mathrm{Li}_{x_0}+P_1(z^a,(1-z)^b)\mathrm{Li}_{x_1}+P_2(z^a,(1-z)^b).1_\Omega=0 \tag{45}$$

where $P_i\in\mathbb{C}[X,Y]$ are two-variable polynomials. From (44) and the fact that $\Omega\neq\emptyset$, we get

$$P_0(e_0^a,e_1^b)\log_0+P_1(e_0^a,e_1^b)\log_1+P_2(e_0^a,e_1^b).1_{\tilde{B}}=0. \tag{46}$$

Now, we consider D_0 (resp. D_1), the deck transformation corresponding to the path $\sigma_0(t)=e^{2\mathrm{i}\pi t}/2$ (resp. $\sigma_1(t)=(1-e^{-2\mathrm{i}\pi t})/2$, one gets

$$\log_0\circ(D_0^r)(\widetilde{z})=\log_0(\widetilde{z})+2\mathrm{i}r\pi\ \text{ and }\ \log_1\circ(D_1^s)(\widetilde{z})=\log_1(\widetilde{z})+2\mathrm{i}s\pi \tag{47}$$

Now we remark that

$$e_0^{[a]}\circ D_0(\widetilde{z})=e_0^{[a]}(\widetilde{z})e^{2a\mathrm{i}\pi}\ \text{ and }\ e_1^{[b]}\circ D_0=e_1^{[b]} \tag{48}$$

and, similarly

$$e_1^{[b]}\circ D_1(\widetilde{z})=e_1^{[b]}(\widetilde{z})e^{2b\mathrm{i}\pi}\ \text{ and }\ e_0^{[a]}\circ D_1=e_0^{[a]} \tag{49}$$

so that $P_i(e_0^a, e_1^b)$ remain bounded through the actions of D_0^r and D_1^s, from (47), we get that $P_i = 0$, $i = 0..2$ which proves the claim. □

Example 2.17 **([6]).** — Let us use the noncommutative multivariate exponential transforms i.e., for any syntactically exchangeable series, we get

$$\sum_{i_0,i_1\geq 0} s_{i_0,i_1} x_0^{i_0} \sqcup\!\sqcup\, x_1^{i_1} \longmapsto \sum_{i_0,i_1\geq 0} \frac{s_{i_0,i_1}}{i_0!i_1!} \mathrm{Li}_{x_0}^{i_0} \mathrm{Li}_{x_1}^{i_1}.$$

Hence, $x_0^n \longmapsto \mathrm{Li}_{x_0}^n/n!$ and $x_1^n \longmapsto \mathrm{Li}_{x_1}^n/n!$, for $n \in \mathbb{N}$, yielding some polylogarithms indexed by series,

$$\mathrm{Li}_{x_0^*}(z) = z, \quad \mathrm{Li}_{x_1^*}(z) = (1-z)^{-1}, \quad \mathrm{Li}_{(ax_0+bx_1)^*}(z) = z^a(1-z)^{-b}.$$

Moreover, for any $(s_1, \ldots, s_r) \in \mathbb{N}_+^r$, there exists an unique series $R_{y_{s_1}\cdots y_{s_r}}$ belonging to $(\mathbb{Z}[x_1^*], \sqcup\!\sqcup, 1_{X^*})$ such that $\mathrm{Li}_{-s_1,\ldots,-s_r} = \mathrm{Li}_{R_{y_{s_1}\cdots y_{s_r}}}$. More precisely (by convention $\rho_0 = x_1^* - 1_{X^*}$),

$$R_{y_{s_1}\ldots y_{s_r}} = \sum_{k_1=0}^{s_1} \cdots \sum_{k_r=0}^{(s_1+\ldots+s_r)-(k_1+\ldots+k_{r-1})} \binom{s_1}{k_1} \cdots \binom{\sum_{i=1}^r s_i - \sum_{i=1}^{r-1} k_i}{k_r} \rho_{k_1} \sqcup\!\sqcup \ldots \sqcup\!\sqcup\, \rho_{k_r},$$

and using the Stirling numbers of second kind, $S_2(k_i, j)$, one has

$$\rho_{k_i} = x_1^* \sqcup\!\sqcup \sum_{j=1}^{k_i} S_2(k_i, j) j! (x_1^* - 1_{X^*})^{\sqcup\!\sqcup j}, (k_i \neq 0).$$

Theorem 2.18 **(extension of $\mathrm{H}_\bullet$).** — *For any $r \geq 1$, one has, for any $t \in \mathbb{C}, |t| < 1$,*

$$\mathrm{H}_{(t^r y_r)^*} = \sum_{k\geq 0} \mathrm{H}_{y_r^k} t^{kr} = \exp\left(\sum_{k\geq 1} \mathrm{H}_{y_{kr}} \frac{(-t^r)^{k-1}}{k}\right).$$

Moreover, for $|a_s| < 1, |b_s| < 1$ and $|a_s + b_s| < 1$,

$$\mathrm{H}_{(\sum_{s\geq 1}(a_s+b_s)y_s + \sum_{r,s\geq 1} a_s b_r y_{s+r})^*} = \mathrm{H}_{(\sum_{s\geq 1} a_s y_s)^*} \mathrm{H}_{(\sum_{s\geq 1} b_s y_s)^*}.$$

Hence,

$$\mathrm{H}_{(a_s y_s + a_r y_r + a_s a_r y_{s+r})^*} = \mathrm{H}_{(a_s y_s)^*} \mathrm{H}_{(a_r y_r)^*}, \mathrm{H}_{(-a_s^2 y_{2s})^*} = \mathrm{H}_{(a_s y_s)^*} \mathrm{H}_{(-a_s y_s)^*}.$$

Proof. — For $t \in \mathbb{C}, |t| < 1$, since $\mathrm{Li}_{(tx_1)^*}$ is well defined then so is the arithmetic function, expressed via Newton–Girard formula (see [3]), for $n \geq 0$, by

$$\mathrm{H}_{(ty_1)^*}(n) = \sum_{k\geq 0} \mathrm{H}_{y_1^k}(n) t^k = \exp\left(-\sum_{k\geq 1} \mathrm{H}_{y_k}(n) \frac{(-t)^k}{k}\right) = \prod_{l=1}^n \left(1 + \frac{t}{l}\right).$$

Similarly, for any $r \geq 2$, the transcendent function $\mathrm{H}_{(t^r y_r)^*}$ can be expressed via Newton–Girard formula (see [3]) once again and via Adam's transform, by

$$\mathrm{H}_{(t^r y_r)^*}(n) = \sum_{k\geq 0} \mathrm{H}_{y_r^k}(n) t^{kr} = \exp\left(-\sum_{k\geq 1} \mathrm{H}_{y_{kr}}(n) \frac{(-t^r)^k}{k}\right) = \prod_{l=1}^N \left(1 - \frac{(-t^r)}{l^r}\right).$$

Since $\|\mathrm{H}_{y_r}\|_\infty \le \zeta(r)$ then $-\sum_{k\ge1}\mathrm{H}_{kr}(-t^r)^k/k$ is termwise dominated by $\|\ell_r\|_\infty$ and then $\mathrm{H}_{(t^r y_r)^*}$ by e^{ℓ_r} (see also Theorem 2.22 bellow). It follows then the last results by using the following identity [7]

$$\Big(\sum_{s\ge1} a_s y_s\Big)^* \sqcup\!\sqcup \Big(\sum_{s\ge1} b_s y_s\Big)^* = \Big(\sum_{s\ge1}(a_s+b_s)y_s + \sum_{r,s\ge1} a_s b_r y_{s+r}\Big)^*. \qquad \square$$

From the estimations from above of the previous proof, it follows then

Corollary 2.19. — *For any $r\ge2$, one has*

$$\frac{1}{\Gamma_{y_r}(1+t)} = \sum_{k\ge0}\zeta(\underbrace{r,\ldots,r}_{k\ times})t^{kr} = \exp\Big(-\sum_{k\ge1}\zeta(kr)\frac{(-t^r)^k}{k}\Big) = \prod_{n\ge1}\Big(1-\frac{(-t^r)}{n^r}\Big).$$

Corollary 2.20. — *For any $r\ge1$*

$$y_r^* = \exp_{\sqcup\!\sqcup}\Big(\sum_{k\ge1} y_{kr}\frac{(-1)^{k-1}}{k}\Big).$$

Hence, for any $k\ge0$, one has

$$y_r^n = \frac{(-1)^n}{n!}\sum_{\substack{s_1,\ldots,s_n>0\\ s_1+\ldots+ns_n=n}} \frac{(-y_r)^{\sqcup\!\sqcup s_1}}{1^{s_1}} \sqcup\!\sqcup \ldots \sqcup\!\sqcup \frac{(-y_{nr})^{\sqcup\!\sqcup s_n}}{n^{s_n}}$$

and, for any $r,s\ge1$, one also has

$$y_r^* \sqcup\!\sqcup y_r^* = \sum_{k=0}^{r}\binom{r+s-k}{s}\binom{s}{k} y_{r+s-k}.$$

2.4. Extended double regularization by Newton–Girard formula. — By (38)–(40), the following polymorphism is, by definition, surjective (see [13])

$$\zeta : \frac{(\mathbb{Q}1_{X^*}\oplus x_0\mathbb{Q}\langle X\rangle x_1, \sqcup\!\sqcup, 1_{X^*})}{(\mathbb{Q}1_{Y^*}\oplus (Y-\{y_1\})\mathbb{Q}\langle Y\rangle, \sqcup\!\sqcup, 1_{Y^*})} \twoheadrightarrow (\mathcal{Z},\cdot,1), \tag{50}$$

mapping both $x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1$ and $y_{s_1}\ldots y_{s_r}$ to $\zeta(s_1,\ldots,s_r)$, where $\mathcal{Z}$ denotes the $\mathbb{Q}$-algebra (algebraically) generated by $\{\zeta(l)\}_{l\in\mathcal{L}ynX-X}$, or equivalently, $\{\zeta(l)\}_{l\in\mathcal{L}ynY-\{y_1\}}$. It can be extended as characters

$$\zeta_{\sqcup\!\sqcup} : (\mathbb{R}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*}) \longrightarrow (\mathbb{R},\cdot,1), \tag{51}$$

$$\zeta_{\sqcup\!\sqcup}, \gamma_\bullet : (\mathbb{R}\langle Y\rangle, \sqcup\!\sqcup, 1_{Y^*}) \longrightarrow (\mathbb{R},\cdot,1) \tag{52}$$

such that, for any $l\in\mathcal{L}ynX$, one has (see [13])

$$\zeta_{\sqcup\!\sqcup}(l) = \mathrm{f.p.}_{z\to1}\mathrm{Li}_l(z), \{(1-z)^a\log^b(1-z)\}_{a\in\mathbb{Z},b\in\mathbb{N}}, \tag{53}$$

$$\zeta_{\sqcup\!\sqcup}(\pi_Y l) = \mathrm{f.p.}_{n\to+\infty}\mathrm{H}_{\pi_Y l}(n), \{n^a\mathrm{H}_1^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}, \tag{54}$$

$$\gamma_{\pi_Y l} = \mathrm{f.p.}_{n\to+\infty}\mathrm{H}_{\pi_Y l}(n), \{n^a\log^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}. \tag{55}$$

It follows that, for any $l \in \mathcal{L}ynX - X$, $\zeta_{\sqcup\!\sqcup}(l) = \zeta_{\sqcup\!\sqcup}(\pi_Y l) = \gamma_{\pi_Y l} = \zeta(l)$, and, for the algebraic generator x_0, $\zeta_{\sqcup\!\sqcup}(x_0) = 0 = \log(1)$ and, for the algebraic generators x_1 and y_1 (divergent cases),

$$\zeta_{\sqcup\!\sqcup}(x_1) = 0 = \mathrm{f.p.}_{z\to 1} \log(1-z), \{(1-z)^a \log^b(1-z)\}_{a\in\mathbb{Z},b\in\mathbb{N}}, \tag{56}$$

$$\zeta_{\sqcup\!\sqcup}(y_1) = 0 = \mathrm{f.p.}_{n\to+\infty} \mathrm{H}_1(n), \{n^a \mathrm{H}_1^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}, \tag{57}$$

$$\gamma_{y_1} = \gamma = \mathrm{f.p.}_{n\to+\infty} \mathrm{H}_1(n), \{n^a \log^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}. \tag{58}$$

As in [6, 13], considering a character $\chi_\bullet$ on $(\mathbb{C}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*})$ and considering $\mathrm{Dom}(\chi_\bullet) \subset \mathbb{C}\langle\langle X\rangle\rangle$ as in (41), we can also check easily that [2]:

- $\mathbb{C}\langle X\rangle \sqcup\!\sqcup \mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle \subset \mathrm{Dom}(\chi_\bullet)$ which is closed by shuffle product,
- for any $S, T \in \mathrm{Dom}(\chi_\bullet)$, one has $\chi_{S\sqcup\!\sqcup T} = \chi_S \chi_T$,
- if $S \in \mathrm{Dom}(\chi_\bullet)$ then $\exp_{\sqcup\!\sqcup}(S) \in \mathrm{Dom}(\chi_\bullet)$ and $\chi_{\exp_{\sqcup\!\sqcup}(S)} = e^{\chi_S}$.

Similarly, considering a character $\chi_\bullet$ on $(\mathbb{C}\langle Y\rangle, \sqcup\!\sqcup, 1_{Y^*})$ and considering $\mathrm{Dom}(\chi_\bullet) \subset \mathbb{C}\langle\langle Y\rangle\rangle$ as in (41), we can also check easily that [2]:

- $\mathbb{C}\langle Y\rangle \sqcup\!\sqcup \mathbb{C}^{\mathrm{rat}}\langle\langle Y\rangle\rangle \subset \mathrm{Dom}(\chi_\bullet)$ which is closed by quasi-shuffle product,
- for any $S, T \in \mathrm{Dom}(\chi_\bullet)$, one has $\chi_{S\sqcup\!\sqcup T} = \chi_S \chi_T$,
- if $S \in \mathrm{Dom}(\chi_\bullet)$ then $\exp_{\sqcup\!\sqcup}(S) \in \mathrm{Dom}(\chi_\bullet)$ and $\chi_{\exp_{\sqcup\!\sqcup}(S)} = e^{\chi_S}$.

Example 2.21. — For any $z \in \mathbb{C}, |z| < 1, x \in X = \{x_0, x_1\}, y_r \in Y = \{y_k\}_{k\geq 1}$, since

$$(zx)^* = \exp_{\sqcup\!\sqcup}(z) \quad \text{and} \quad (zy_r)^* = \exp_{\sqcup\!\sqcup}\left(\sum_{k\geq 1} y_{kr} \frac{(-z)^{k-1}}{k}\right)$$

then

$$\zeta_{\sqcup\!\sqcup}((zx)^*) = e^{z\zeta_{\sqcup\!\sqcup}(x)} \quad \text{and} \quad \gamma_{(zy_r)^*} = \exp\left(\sum_{k\geq 1} \zeta_{\sqcup\!\sqcup}(y_{kr}) \frac{(-z)^{k-1}}{k}\right).$$

We now in situation to state that

***Theorem 2.22* (Regularization by Newton–Girard formula).** — *The characters $\zeta_{\sqcup\!\sqcup}$ and $\gamma_\bullet$ are extended algebraically as follows:*

$$\zeta_{\sqcup\!\sqcup} : (\mathbb{C}\langle X\rangle \sqcup\!\sqcup \mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle, \sqcup\!\sqcup, 1_{X^*}) \longrightarrow (\mathbb{C}, \cdot, 1),$$
$$\forall\, t \in \mathbb{C}, |t| < 1, (tx_0)^*, (tx_1)^* \longmapsto 1_{\mathbb{C}}.$$

and

$$\gamma_\bullet : (\mathbb{C}\langle Y\rangle \sqcup\!\sqcup \mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle Y\rangle\rangle, \sqcup\!\sqcup, 1_{Y^*}) \longrightarrow (\mathbb{C}, \cdot, 1),$$
$$\forall\, t \in \mathbb{C}, |t| < 1, \forall\, r \geq 1, (t^r y_r)^* \longmapsto \Gamma^{-1}_{y_r}(1+t).$$

Moreover, the morphism $(\mathbb{C}[\{y_r^*\}_{r\geq 1}], \sqcup\!\sqcup, 1_{Y^*}) \to \mathbb{C}[E]$ mapping y_r^* to $\Gamma^{-1}_{y_r}$, is injective and $\Gamma_{y_{2r}}(1 + \sqrt[2r]{-1}t) = \Gamma_{y_r}(1+t)\Gamma_{y_r}(1+\sqrt[r]{-1}t)$, for $r \geq 1$.

Proof. — By Definition 2.4, Propositions 2.8, 2.11 and Theorems 2.16, 2.18, we get the expected results (see also Proposition 2.7). □

Example 2.23 **([6]).** —

$$\begin{aligned}
\mathrm{Li}_{-1,-1} &= \mathrm{Li}_{-x_1^*+5(2x_1)^*-7(3x_1)^*+3(4x_1)^*},\\
\mathrm{Li}_{-2,-1} &= \mathrm{Li}_{x_1^*-11(2x_1)^*+31(3x_1)^*-33(4x_1)^*+12(5x_1)^*},\\
\mathrm{Li}_{-1,-2} &= \mathrm{Li}_{x_1^*-9(2x_1)^*+23(3x_1)^*-23(4x_1)^*+8(5x_1)^*},\\
\mathrm{H}_{-1,-1} &= \mathrm{H}_{-y_1^*+5(2y_1)^*-7(3y_1)^*+3(4y_1)^*},\\
\mathrm{H}_{-2,-1} &= \mathrm{H}_{y_1^*-11(2y_1)^*+31(3y_1)^*-33(4y_1)^*+12(5y_1)^*},\\
\mathrm{H}_{-1,-2} &= \mathrm{H}_{y_1^*-9(2y_1)^*+23(3y_1)^*-23(4y_1)^*+8(5y_1)^*}.
\end{aligned}$$

Hence, $\zeta_{\sqcup\!\sqcup}(-1,-1)=0$, $\zeta_{\sqcup\!\sqcup}(-2,-1)=-1$, $\zeta_{\sqcup\!\sqcup}(-1,-2)=0$, and

$$\begin{aligned}
\gamma_{-1,-1} &= -\Gamma^{-1}(2)+5\Gamma^{-1}(3)-7\Gamma^{-1}(4)+3\Gamma^{-1}(5)=11/24,\\
\gamma_{-2,-1} &= \Gamma^{-1}(2)-11\Gamma^{-1}(3)+31\Gamma^{-1}(4)-33\Gamma^{-1}(5)+12\Gamma^{-1}(6)=-73/120,\\
\gamma_{-1,-2} &= \Gamma^{-1}(2)-9\Gamma^{-1}(3)+23\Gamma^{-1}(4)-23\Gamma^{-1}(5)+8\Gamma^{-1}(6)=-67/120.
\end{aligned}$$

From Theorems 2.18 and 2.22, one deduces

Corollary 2.24. —

1. *With the notations of* (23)*, Definition 2.4 and with the differential forms* $\{(\partial\ell_r)dz\}_{r\ge1}$*, for any* $z\in\mathbb{C},|z|<1$*, one has*

$$\gamma_{\sqcup\!\sqcup_{r\ge1}(z^ry_r)^*}=\prod_{r\ge1}\gamma_{(z^ry_r)^*}=\prod_{r\ge1}e^{\ell_r(z)}=\prod_{r\ge1}\frac{1}{\Gamma_{y_r}(1+z)}=\alpha_0^z(\sqcup\!\sqcup_{r\ge1}y_r^*).$$

2. *One has, for* $|a_s|<1,|b_s|<1$ *and* $|a_s+b_s|<1$*,*

$$\gamma_{(\sum_{s\ge1}(a_s+b_s)y_s+\sum_{r,s\ge1}a_sb_ry_{s+r})^*}=\gamma_{(\sum_{s\ge1}a_sy_s)^*}\gamma_{(\sum_{s\ge1}b_sy_s)^*}.$$

Hence, $\gamma_{(a_sy_s+a_ry_r+a_sa_ry_{s+r})^*}=\gamma_{(a_sy_s)^*}\gamma_{(a_ry_r)^*},\gamma_{(-a_s^2y_{2s})^*}=\gamma_{(a_sy_s)^*}\gamma_{(-a_sy_s)^*}.$

Remark 2.25. — The restriction $\alpha_0^z:(\mathbb{C}[\{y_r,y_r^*\}_{r\ge1}],\sqcup\!\sqcup,1_{Y^*})\to\mathbb{C}[L+E]$ is injective (see Corollary 2.14) while $\ker(\gamma_\bullet)\neq\{0\}$, over $\mathbb{C}\langle Y\rangle\sqcup\!\sqcup\,\mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle Y\rangle\rangle$ [13].

Example 2.26 **([11, 12]).** — By Theorem 2.22,

$$\gamma_{(-t^2y_2)^*}=\Gamma_{y_2}^{-1}(1+\mathrm{i}t),\gamma_{(ty_1)^*}=\Gamma_{y_1}^{-1}(1+t),\gamma_{(-ty_1)^*}=\Gamma_{y_1}^{-1}(1+t).$$

Then, by Corollary 2.24, $\gamma_{(-t^2y_2)^*}=\gamma_{(ty_1)^*}\gamma_{(-ty_1)^*}$ meaning that

$$\Gamma_{y_2}^{-1}(1+\mathrm{i}t)=\Gamma_{y_1}^{-1}(1+t)\Gamma_{y_1}^{-1}(1-t).$$

Or equivalently,

$$\exp\left(-\sum_{k\ge2}\zeta(2k)\frac{t^{2k}}{k}\right)=\sum_{k\ge2}\zeta(\overbrace{2,\dots,2}^{k\text{ times}})(-1)^kt^{2k}=\frac{\sin(t\pi)}{t\pi}=\sum_{k\ge1}(-1)^k\frac{(t\pi)^{2k}}{(2k+1)!}.$$

Since $\gamma_{(-t^2y_2)^*}=\zeta((-t^2x_0x_1)^*)$ then, identifying the coefficients of t^{2k}, we get

$$\frac{\zeta(\overbrace{2,\dots,2}^{k\text{ times}})}{\pi^{2k}}=\frac{1}{(2k+1)!}\in\mathbb{Q}.$$

Similarly, by Theorem 2.22,

$$\gamma_{(-t^4y_4)^*} = \Gamma_{y_4}^{-1}(1+\sqrt[4]{-1}t), \gamma_{(t^2y_2)^*} = \Gamma_{y_2}^{-1}(1+t), \gamma_{(-t^ty_2)^*} = \Gamma_{y_2}^{-1}(1+\mathrm{i}t).$$

Then, by Corollary 2.24, $\gamma_{(-t^4y_4)^*} = \gamma_{(t^2y_2)^*}\gamma_{(-t^2y_2)^*}$ meaning that

$$\Gamma_{y_4}^{-1}(1+\sqrt[4]{-1}t) = \Gamma_{y_2}^{-1}(1+t)\Gamma_{y_2}^{-1}(1+\mathrm{i}t).$$

Or equivalently,

$$\begin{aligned}\exp\left(-\sum_{k\geq 1}\zeta(4k)\frac{t^{4k}}{k}\right) &= \sum_{k\geq 2}\zeta(\underbrace{4,\ldots,4}_{k \text{ times}})(-1)^k\frac{(t\pi)^{4k}}{(4k+2)!}\\ &= \frac{\sin(\mathrm{i}t\pi)}{\mathrm{i}t\pi}\frac{\sin(t\pi)}{t\pi}\\ &= \sum_{k\geq 1}\frac{2(-4t\pi)^{4k}}{(4k+2)!}.\end{aligned}$$

Since $\gamma_{(-t^4y_4)^*} = \zeta((-t^4y_4)^*)$, $\gamma_{(-t^2y_2)^*} = \zeta((-t^2y_2)^*)$, $\gamma_{(t^2y_2)^*} = \zeta((t^2y_2)^*)$ then, using the poly-morphism ζ and identities on rational series, we get

$$\begin{aligned}\zeta((-t^4y_4)^*) &= \zeta((-t^2y_2)^*)\zeta((t^2y_2)^*)\\ &= \zeta((-t^2x_0x_1)^*)\zeta((t^2x_0x_1)^*))\\ &= \zeta((-4t^4x_0^2x_1^2)^*).\end{aligned}$$

Thus, by identification the coefficients of t^{4k}, we obtain

$$\frac{\zeta(\overbrace{4,\ldots,4}^{k \text{ times}})}{4^k\pi^{4k}} = \frac{\zeta(\overbrace{3,1,\ldots,3,1}^{k \text{ times}})}{\pi^{4k}} = \frac{2}{(4k+2)!} \in \mathbb{Q}.$$

***Corollary 2.27* (comparison formula,** [8, 13]**).** — *For any $z,a,b \in \mathbb{C}$ such that $|z|<1$ and $\Re(a)>0, \Re(b)>0$, we have*

$$\mathrm{B}(z;a,b) = \mathrm{Li}_{x_0[(ax_0)^*\sqcup\!\sqcup((1-b)x_1)^*]}(z) = \mathrm{Li}_{x_1[((a-1)x_0)^*\sqcup\!\sqcup(-bx_1)^*]}(z).$$

Hence, on the one hand

$$\mathrm{B}(a,b) = \zeta_{\sqcup\!\sqcup}(x_0[(ax_0)^* \sqcup\!\sqcup ((1-b)x_1 = \zeta_{\sqcup\!\sqcup}(x_1[((a-1)x_0)^* \sqcup\!\sqcup (-bx_1)^*])$$

and, on the other hand

$$\mathrm{B}(a,b) = \frac{\gamma_{((a+b-1)y_1)^*}}{\gamma_{((a-1)y_1)^*\sqcup\!\sqcup((b-1)y_1)^*}} = \frac{\gamma_{((a+b-1)y_1)^*}}{\gamma_{((a+b-2)y_1+(a-1)(b-1)y_2)^*}}.$$

Proof. — The results, of $\mathrm{B}(z;a,b)$, are the computations of iterated integrals associated to different rational series, using the differential forms in (36). Those of $\mathrm{B}(a,b)$, are then immediate consequences, by evaluating these iterated integrals at $z=1$ and by using Definition 2.4 and Corollary 2.24. □

***Example 2.28* (**[11, 12]**).** — Let us consider, for $t_0,t_1 \in \mathbb{C}, |t_0|<1, |t_1|<1$,

$$R := t_0x_0(t_0x_0+t_0t_1x_1)^*(t_0t_1x_1) = t_0^2t_1x_0[(t_0x_0)^* \sqcup\!\sqcup (t_0t_1x_1)^*]x_1.$$

Then, with the differential forms in (36), we get successively

$$\begin{aligned}\mathrm{Li}_R(z) &= t_0^2 t_1 \int_0^z \frac{\mathrm{d}s}{s} \int_0^s \left(\frac{s}{r}\right)^{t_0} \left(\frac{1-r}{1-s}\right)^{t_0 t_1} \frac{\mathrm{d}r}{1-r} \\ &= t_0^2 t_1 \int_0^z (1-s)^{-t_0 t_1} s^{t_0-1} \int_0^s (1-r)^{t_0 t_1 - 1} r^{-t_0} \mathrm{d}s\mathrm{d}r\end{aligned}$$

By change of variable, $r = st$, we obtain then

$$\mathrm{Li}_R(z) = t_0^2 t_1 \int_0^z (1-s)^{-t_0 t_1} \int_0^1 (1-st)^{t_0 t_1 - 1} t^{-t_0} \mathrm{d}t\mathrm{d}s.$$

It follows then

$$\zeta(R) = t_0^2 t_1 \int_0^1 \int_0^1 (1-s)^{-t_0 t_1} (1-st)^{t_0 t_1 - 1} t^{-t_0} \mathrm{d}t\mathrm{d}s.$$

By change of variable, $y = \dfrac{1-s}{1-st}$, we obtain also

$$\zeta(R) = t_0^2 t_1 \int_0^1 \int_0^1 (1-ty)^{-1} t^{-t_0} y^{-t_0 t_1} \mathrm{d}y\mathrm{d}t.$$

By expending $(1-ty)^{-1}$ and then integrating, we get on the one hand

$$\zeta(R) = \sum_{n \geq 1} \frac{t_0}{n - t_0} \frac{t_0 t_1}{n - t_0 t_1} = \sum_{k > l > 0} \zeta(k) t_0^k t_1^l.$$

On the other hand, using the expansion of R, we get also

$$\zeta(R) = \sum_{k>0} \sum_{l>0} \sum_{\substack{s_1 + \ldots + s_l = k \\ s_1 \ldots, s_l \geq 1, s1 \geq 2}} \zeta(s_1, \ldots, s_l) t_0^k t_1^l.$$

Finally, by identification the coefficients of $\langle \zeta(R) | t_0^k t_1^l \rangle$, we deduce the *sum formula*

$$\zeta(k) = \sum_{\substack{s_1 + \ldots + s_l = k \\ s_1 \ldots, s_l \geq 1, s_1 \geq 2}} \zeta(s_1, \ldots, s_l).$$

3. Conclusion

In this work, we illustrated a bijection, between a sub shuffle algebra of noncommutative rational series (recalled in 2.1) and a subalgebra of holomorphic functions, $\mathcal{H}(\Omega)$, on a simply connected domain $\Omega \subset \mathbb{C}$ containing the family of extended eulerian functions $\{\Gamma_y^{-1}(1+z)\}_{y \in Y}$ and the family of their logarithms, $\{\log \Gamma_y^{-1}(1+z)\}_{y \in Y}$ (introduced in 2.2), involved in summations of polylogarithms and harmonics sums (studied in 2.3) and in regularizations of divergent polyzetas (achieved, for this stage, in 2.4).

These two families are algebraically independent over a differential subring of $\mathcal{H}(\Omega)$ and generate freely two disjoint functional algebras. For any $y_r \in Y$, the special functions $\Gamma_{y_r}^{-1}(1+z)$ and $\log \Gamma_{y_r}^{-1}(1+z)$ are entire and holomorphic on the unit open disc, respectively. In particular, $\Gamma_{y_r}^{-1}(1+z)$ admits a countable set of isolated zeroes on the complex plane, i.e. $\biguplus_{\chi \in G_r} \chi \mathbb{Z}_{\leq -1}$, where G_r is the set of solutions of the equation $z^r = (-1)^{r-1}$.

These functions allow to obtain identities, at arbitrary weight, among polyzetas and an analogue situation, as the ratios $\zeta(2k)/\pi^{2k}$, drawing out consequences about a structure of polyzetas. This work will be completed, in the forth comming works, by a study a family of functions obtained as image of rational series for which their linear representation (ν, μ, η) are such that the Lie algebra generated by the matrices $\{\mu(y)\}_{y\in Y}$ is *solvable*.

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