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p-ADIC DIRECTIONS OF PRIMITIVE VECTORS

by

Antonin Guilloux and Tal Horesh

Abstract. — Linnik type problems concern the distribution of projections of integral points on the unit sphere as their norm increases, and different generalizations of this phenomenon. Our work addresses a question of this type: we prove the uniform distribution of the projections of primitive $\mathbb{Z}^2$ points in the p-adic unit sphere, as their (real) norm tends to infinity. The proof is via counting lattice points in semi-simple S-arithmetic groups.

Résumé. — (*Directions p-adique de vecteurs primitifs*) Les problèmes de type Linnik concernent la distribution des projections des points entiers sur la sphère unitaire lorsque leur norme augmente et différentes généralisations de ce phénomène. Notre travail s'intéresse à une question de ce type : nous prouvons la distribution uniforme des projections des points primitifs de $\mathbb{Z}^2$ sur la sphère unitaire p-adique lorsque leur norme (réelle) tend vers l'infini. La preuve se fait en comptant les points d'un réseau dans des S-groupes arithmétiques semi-simples.

A primitive vector is an n-tuple $(a_1, \ldots, a_n)$ of co-prime integers, and we let $\mathbb{Z}^n_{\text{prim}}$ denote the set of primitive vectors in $\mathbb{Z}^n$. Since every integral vector is an integer multiple of a unique primitive vector, it is very natural to restrict questions about equidistribution of integer vectors to the set of primitive vectors. For example, one question about an equidistribution property for integer vectors that has been studied in the past (e.g. in [28]) is whether the directions of integral vectors, i.e. their projections to the unit sphere in $\mathbb{R}^n$, distribute uniformly in the unit sphere as their norm tends to ∞. This question belongs to the well known family of *Linnik type problems* (e.g. [7, 8, 23]), and the answer is positive: for every "reasonable" subset Θ of the sphere, it holds that

$$(1) \qquad \frac{\#\left\{\frac{v}{\|v\|} \in \Theta : v \in \mathbb{Z}^n, \|v\| \leq R\right\}}{\#\left\{\frac{v}{\|v\|} \in \mathbb{S}^{n-1} : v \in \mathbb{Z}^n, \|v\| \leq R\right\}} \underset{R\to\infty}{\longrightarrow} \frac{\mathrm{Leb}(\Theta)}{\mathrm{Leb}(\mathbb{S}^1)},$$

where Leb is the Lebesgue measure on the sphere. While in the above quotient every "integral direction" on the unit sphere is hit several times (the first time for a primitive vector, and then another time for each one of its integer multiples), restricting to $v \in \mathbb{Z}^n_{\text{prim}}$ allows every integral direction to be considered exactly once.

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Questions about equidistribution of directions, as well as of other parameters of primitive vectors, have been studied recently using dynamical methods in [1, 2, 10, 11, 24]. In the present paper we restrict to dimension $n = 2$, and study the equidistribution of *p-adic directions* of primitive vectors. Indeed, since primitive vectors have integer coordinates, they can be seen as vectors over any field that contains the rationals, and in particular over the field of p-adic numbers $\mathbb{Q}_p$ for a positive prime number p. There, just like the direction of a real vector is its projection to the (real) unit sphere through multiplication by inverse of the norm, the p-adic direction of a vector is its projection to the p-adic unit sphere. However, the primitive vectors have p-adic norm one, so in fact they are already contained in the p-adic unit sphere $\mathbb{S}^1_p$ (we will observe this below, where we recall some basic definitions in the p-adic setting). So, $\mathbb{Z}^2_{\text{prim}}$ is a countable subset of $\mathbb{S}^1_p$ which is equipped with a natural height function: the real norm. One is then led to ask whether the set $\mathbb{Z}^2_{\text{prim}}$ equidistributes in $\mathbb{S}^1_p$, i.e., if an analog to (1) holds when $\mathbb{Z}^n$ is replaced by $\mathbb{Z}^2_{\text{prim}}$, and $\mathbb{S}^{n-1}$ is replaced by $\mathbb{S}^1_p$. To formulate such an analog, we need to declare what are the analogous objects in $\mathbb{S}^1_p$ for an arc on the (real) unit circle, and for the Lebesgue measure on it. Below, we will define the concept of a p-adic arc, $\Theta_p \subset \mathbb{S}^1_p$ (Definition 1.4), and recall a Haar measure μ_p on $\mathbb{Q}_p$. Since $\mathbb{S}^1_p$ is an open and compact subset of the p-adic plane, then the restriction of the Haar measure μ_p^2 on the plane $\mathbb{Q}_p^2$ to $\mathbb{S}^1_p$ is a finite non-zero measure on $\mathbb{S}^1_p$. As we shall see below, it corresponds to the Lebesgue measure on the real unit circle.
The theorem below establishes the uniform distribution of the primitive vectors in $\mathbb{S}^1_p$ as their real norm tends to infinity; even more, it establishes *joint equidistribution* of their real and p-adic directions in the product of unit circles $\mathbb{S}^1 \times \mathbb{S}^1_p$.

Theorem A. — *For $v \in \mathbb{Z}^2_{\text{prim}}$, the pairs of real and p-adic directions*

$$\left(\frac{v}{\|v\|}, v\right) \in \mathbb{S}^1 \times \mathbb{S}^1_p$$

become uniformly distributed in $\mathbb{S}^1 \times \mathbb{S}^1_p$ w.r.t. $\mathrm{Leb} \times \mu_p^2|_{\mathbb{S}^1_p}$ as $\|v\| \to \infty$, meaning that for every product of arcs $\Theta \times \Theta_p \subset \mathbb{S}^1 \times \mathbb{S}^1_p$ it holds that

$$\frac{\#\left\{v \in \mathbb{Z}^2_{\text{prim}} : \left(\frac{v}{\|v\|}, v\right) \in \Theta \times \Theta_p, \quad \|v\| \le R\right\}}{\#\left\{v \in \mathbb{Z}^2_{\text{prim}} : \quad \|v\| \le R\right\}} \xrightarrow[R\to\infty]{} \frac{\mathrm{Leb}(\Theta) \cdot \mu_p^2(\Theta_p)}{\mathrm{Leb}(\mathbb{S}^1) \cdot \mu_p^2(\mathbb{S}^1_p)}.$$

The convergence is at rate at most $O\left(R^{-2\tau_p+\delta}\right)$ for every $\delta > 0$, where $\tau_p = \frac{1}{28}$.

Remark. — The parameter τ_p in the error term exponent depends on the rate of decay of the matrix coefficients of automorphic representations of $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{Q}_p)$, and can be improved to $\frac{1}{14}$ when assuming the Ramanujan conjecture (cf. Remark 3.5).

A similar theorem holds when considering a finite set of primes instead of a single p. Theorem 4.1 in Section 4 states a related equidistribution result in this S-arithmetic setting, that translates into a counting statement as above, see Remark 4.3. However, we have chosen to deal with only one prime here in order to ease the exposition.

Organization of the paper. — The first section of the paper is a collection of general facts on p-adic numbers and arithmetic lattices. Sections 2 and 3 are devoted to the proof of Theorem A, along the lines of the proof given in [19, 20] for the uniform distribution of the real directions of primitive vectors in the unit sphere. It consists of two stages: the first is a translation of the theorem to a statement about counting lattice points in the group $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{Q}_p)$ (Section 2), and the second is proving the counting statement via a method developed in [17] (Section 3). In Section 4, we show that the same method gives a stronger statement than the one in Theorem A, namely the joint equidistribution of real and p-adic directions for any finite number of primes p — cf. Theorem 4.1 (and Remark 4.3). Section 5 is devoted to proving a technical result on well-roundedness in the p-adic setting that is used in Section 4.

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1. p-adic numbers and arithmetic lattices

Let us now recall some basic facts on the p-adic numbers and arithmetic lattices.

1.1. p-adic numbers and vector spaces. —

Definition 1.1 **(p-adic valuation and absolute value).** — For a non-zero p-adic number a, the *p-adic valuation* of a is defined to be the biggest integer $\nu(a)$ such that

$$a = \sum_{i=\nu(a)}^{\infty} \alpha_i p^i, \quad \alpha_i \in \{0, 1, \dots, p-1\},$$

where for $a = 0$ one defines $\nu(0) = \infty$. The *p-adic absolute value* is given by

$$|a|_p = p^{-\nu(a)}.$$

The ring of *p-adic integers* $\mathbb{Z}_p < \mathbb{Q}_p$ is the p-adic unit ball, namely the set of p-adic numbers with absolute value at most 1 (equivalently, of non-negative valuation). Inside $\mathbb{Z}_p$, the set of *invertible* p-adic integers $\mathbb{Z}_p^\times \subset \mathbb{Z}_p$ is the set of p-adic numbers with absolute value 1 (equivalently, of valuation 0).

Note 1.2. — It is easy to see that every $a \in \mathbb{Q}_p$ can be written (uniquely) as

$$a = u_a p^{\nu(a)} = u_a |a|_p^{-1}$$

where $u_a \in \mathbb{Z}_p^\times$.

A norm on $\mathbb{Q}_p^2$ (and therefore a unit circle) is then defined as follows.

Definition 1.3 **(norm and unit circle in $\mathbb{Q}_p^2$).** — The p-adic norm (or just "norm") of a vector $(a, b) \in \mathbb{Q}_p^2$ is defined to be

$$\|(a,b)\|_p = \max\left\{|a|_p, |b|_p\right\} = p^{-\min\{\nu(a),\nu(b)\}}.$$

Accordingly, the *p-adic unit circle* in $\mathbb{Q}_p^2$ is the set of vectors of norm one:

$$\mathbb{S}_p^1 := \left\{ v \in \mathbb{Q}_p^2 : \|v\|_p = 1 \right\}.$$

Note that (a,b) is in $\mathbb{S}_p^1$ if and only if both a,b are in $\mathbb{Z}_p$, and at least one of them is in $\mathbb{Z}_p^\times$. In particular, since the real integers are also p-adic integers, and since an integer lies in $\mathbb{Z}_p^\times$ if and only if it is not divisible by p, we have that $\mathbb{Z}^2_{\text{prim}} \subset \mathbb{S}_p^1$.
Since $\mathbb{Z}_p$ is the unit ball in $\mathbb{Q}_p$, then a ball of radius p^{-N} around $\alpha \in \mathbb{Q}_p$ is $\alpha + p^N\mathbb{Z}_p$. Similarly, a ball of radius p^{-N} around $(\alpha,\beta) \in \mathbb{Q}_p^2$ is $(\alpha,\beta) + p^N\mathbb{Z}_p^2$. The analog in $\mathbb{Q}_p^2$ for an arc Θ in the unit circle $\mathbb{S}^1$ is a ball that is contained in the p-adic circle:

Definition 1.4 **(p-adic arc).** — Let $N > 0$ be an integer. A *p-adic arc* of radius p^{-N} is a ball $\theta + p^N\mathbb{Z}_p^2$, where $\theta \in \mathbb{S}_p^1$. It will be denoted by $\Theta_p = \Theta_p(\theta, p^{-N})$.

Note that it is sufficient that $\theta \in \mathbb{S}_p^1$ in order to have $\Theta_p \subset \mathbb{S}_p^1$!
A Haar measure μ_p is defined on $\mathbb{Q}_p$ (see, e.g. [5]) by assigning to a ball of radius p^{-N} the volume p^{-N}. We let $\mu_p^2 := \mu_p \times \mu_p$ denote the resulting Haar measure on $\mathbb{Q}_p^2$, which then assigns to a ball of radius p^{-N} the volume p^{-2N}. Being a Haar measure on $\mathbb{Q}_p^2$, μ_p^2 is invariant under the group $\mathrm{SL}_2(\mathbb{Q}_p)$ of p-adic 2 by 2 matrices with determinant one. We note that unlike the real case, in the p-adic plane the unit circle has positive Haar measure. Indeed, it contains the subset $\mathbb{Z}_p^\times \times \mathbb{Z}_p$, which has measure

$$\mu_p(\mathbb{Z}_p^\times) \cdot \mu_p(\mathbb{Z}_p) = \mu_p(\mathbb{Z}_p - p\mathbb{Z}_p) \cdot \mu_p(\mathbb{Z}_p) = \left(1 - \frac{1}{p}\right) \cdot 1 = 1 - \frac{1}{p}.$$

Hence, it is possible to restrict μ_p^2 to $\mathbb{S}_p^1$. This measure is the analog of the Lebesgue measure on the real unit circle, in the sense that it is invariant under the group of norm preserving linear transformations of $\mathbb{Q}_p^2$. Indeed, it is known (e.g. [15, Corollary 3.3]) that the group $\mathrm{SL}_2(\mathbb{Z}_p)$ is the stabilizer of the norm $\|\cdot\|_p$ on $\mathbb{Q}_p^2$, so in particular it preserves and acts transitively on $\mathbb{S}_p^1$.

1.2. *S*-arithmetic lattices. — The field $\mathbb{Q}_p$ does not contain a lattice; but, inside the ring

$$\mathbb{F} := \mathbb{R} \times \mathbb{Q}_p,$$

the subring $\mathbb{Z}[\frac{1}{p}]$ of polynomials in $\frac{1}{p}$ with integer coefficients embeds diagonally as a co-compact lattice. It is an integral domain, and as such it plays the role of the integral lattice inside $\mathbb{F}$. According to the Borel Harish–Chandra Theorem [4], the diagonal embedding of the subgroup $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$, which by abuse of notation we denote by $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$, is a lattice inside

$$\mathrm{SL}_2(\mathbb{F}) := \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{Q}_p).$$

Both these lattices, $\mathbb{Z}[\frac{1}{p}] < \mathbb{F}$ and $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}]) < \mathrm{SL}_2(\mathbb{F})$, are a special case of *S-arithmetic lattices* [26]. To familiarize the reader with the S-arithmetic framework, we include a proof that these two discrete subgroups are indeed lattices, by establishing the existence of finite-volume fundamental domains. Here a fundamental domain means a full set of representatives.

Fact 1.5. — *Let $\mathcal{D}_\infty$ denote a fundamental domain for* $\mathrm{SL}_2(\mathbb{Z})$ *in* $\mathrm{SL}_2(\mathbb{R})$*. Then*

1. $\left[\frac{1}{2},\frac{1}{2}\right) \times \mathbb{Z}_p$ *is a fundamental domain for* $\mathbb{Z}[\frac{1}{p}]$ *in* $\mathbb{F}$.

2. $\mathcal{D}_\infty \times \mathrm{SL}_2(\mathbb{Z}_p)$ *is a fundamental domain for* $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$ *in* $\mathrm{SL}_2(\mathbb{F})$.

Proof. —

1. — Given $(x,\alpha) \in \mathbb{R} \times \mathbb{Q}_p$, with $\alpha = \sum_{i=\nu(\alpha)}^{\infty} \alpha_i p^i$, write

$$\{\alpha\} = \sum_{i=\nu(\alpha)}^{-1} \alpha_i p^i \in \mathbb{Z}\left[\frac{1}{p}\right].$$

Then clearly $\alpha - \{\alpha\} = \sum_{i=0}^{\infty} \alpha_i p^i \in \mathbb{Z}_p$, and so

$$(x,\alpha) - (\{\alpha\}, \{\alpha\}) \in \mathbb{R} \times \mathbb{Z}_p.$$

Now let $m \in \mathbb{Z}$ be an integer such that $x - \{\alpha\} - m \in \left[\frac{1}{2},\frac{1}{2}\right)$. Since $\mathbb{Z} \subset \mathbb{Z}_p$, then

$$(x,\alpha) - (\{\alpha\} + m, \{\alpha\} + m) \in \left[\frac{1}{2},\frac{1}{2}\right) \times \mathbb{Z}_p.$$

For uniqueness, assume (x,α) and (y,β) are both in $\left[\frac{1}{2},\frac{1}{2}\right) \times \mathbb{Z}_p$, with $f \in \mathbb{Z}[\frac{1}{p}]$ such that

$$(x,\alpha) + (f,f) = (y,\beta).$$

Then $f = \beta - \alpha \in \mathbb{Z}_p$ must be an integer since $\mathbb{Z}[\frac{1}{p}] \cap \mathbb{Z}_p = \mathbb{Z}$, and the real coordinate forces that $f = 0$.

2. — For the second part, let $g = (g_\infty, g_p) \in \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{Q}_p)$; we show that there exists a unique $(\gamma,\gamma) \in \mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$ such that

$$(g_\infty, g_p) \cdot (\gamma,\gamma) \in \mathcal{D}_\infty \times \mathrm{SL}_2(\mathbb{Z}_p).$$

It is a consequence of row reduction that any $g_p \in \mathrm{SL}_2(\mathbb{Q}_p)$ can be written (non-uniquely) as

$$g_p = k_p \gamma_p$$

with $k_p \in \mathrm{SL}_2(\mathbb{Z}_p)$ and $\gamma_p \in \mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$. Then $(g_\infty, g_p) = (g_\infty, k_p\gamma_p)$ meaning that

$$(g_\infty, g_p)\left(\gamma_p^{-1}, \gamma_p^{-1}\right) = \left(g_\infty \gamma_p^{-1}, k_p\right).$$

Write

$$\mathrm{SL}_2(\mathbb{R}) \ni g_\infty \gamma_p^{-1} = x_\infty \gamma_\infty$$

where $x_\infty \in \mathcal{D}_\infty$ and $\gamma_\infty \in \mathrm{SL}_2(\mathbb{Z})$. Then

$$(g_\infty, g_p)\left(\gamma_p^{-1}\gamma_\infty^{-1}, \gamma_p^{-1}\gamma_\infty^{-1}\right) = (x_\infty, k_p\gamma_\infty^{-1}) \in \mathcal{D}_\infty \times \mathrm{SL}_2(\mathbb{Z}_p),$$

which establishes existence. For uniqueness of $(\gamma,\gamma) = (\gamma_p^{-1}\gamma_\infty^{-1}, \gamma_p^{-1}\gamma_\infty^{-1})$, assume that $(g_\infty, g_p)\cdot(\gamma,\gamma) = (x_p, x_\infty)$ and $(g_\infty, g_p)\cdot(\gamma',\gamma') = \left(x'_\infty, x'_p\right)$, where both (x_∞, x_p) and $\left(x'_\infty, x'_p\right)$ lie in $\mathcal{D}_\infty \times \mathrm{SL}_2(\mathbb{Z}_p)$. Then

$$(x_\infty, x_p) \cdot \left(\gamma^{-1}\gamma', \gamma^{-1}\gamma'\right) = \left(x'_\infty, x'_p\right) \in \mathcal{D}_\infty \times \mathrm{SL}_2(\mathbb{Z}_p)$$

where $\gamma^{-1}\gamma' \in \mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$. From the p-adic component of the equation we have that $\gamma^{-1}\gamma' \in \mathrm{SL}_2(\mathbb{Z}_p)$. Then $\gamma^{-1}\gamma' \in \mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}]) \cap \mathrm{SL}_2(\mathbb{Z}_p) = \mathrm{SL}_2(\mathbb{Z})$. On the other hand, from the real component of the equation, we have that $\gamma^{-1}\gamma'$ cannot lie in $\mathrm{SL}_2(\mathbb{Z})$, unless $\gamma^{-1}\gamma' = \mathrm{id}$. We conclude that $\gamma = \gamma'$. $\square$

2. From primitive vectors to lattice points in the group $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{Q}_p)$

It is well known that there exists a connection between primitive vectors in $\mathbb{Z}^2$, and integral matrices inside $\mathrm{SL}_2(\mathbb{R})$; the goal of this section is to establish an analogous connection in the setting of $\mathbb{R} \times \mathbb{Q}_p$, hence reducing the proof of Theorem A to counting lattice points in the group $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{Q}_p)$. To exhibit such a connection, we first extend the notion of primitive vectors (subsection 2.1), then find suitable coordinates on $\mathrm{SL}_2(\mathbb{F})$ through a Bruhat-Iwasawa decomposition (Definition 2.6). Proposition 2.7 states the precise connection.

2.1. Primitive vectors over $\mathbb{Z}[\frac{1}{p}]$. — We aim to formulate a connection between primitive vectors in $\mathbb{Z}[\frac{1}{p}]^2$ and matrices inside $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$. First, let us extend the definition of a primitive vector from $\mathbb{Z}$ to a general integral domain:

Definition 2.1. — Let $\mathcal{O}$ be an integral domain. A vector $v = (a, b) \in \mathcal{O}^2$ is called *primitive* if the prime ideals $\langle a \rangle$ and $\langle b \rangle$ satisfy that $\langle a \rangle + \langle b \rangle = \mathcal{O}$. In other words, if there exists a solution $(x, y) \in \mathcal{O}^2$ to the $\mathcal{O}$–diophantine equation

$$ax + by = 1.$$

We refer to this equation as the *gcd equation* of v, and denote the set of primitive elements in $\mathcal{O}^2$ by $\mathcal{O}^2_{\mathrm{prim}}$.

Clearly, the set of primitive $\mathbb{Z}[\frac{1}{p}]^2$ (vectors contains the set of primitive $\mathbb{Z}^2$) vectors, but is not equal to it; e.g. the vector $(p, 0)$ is primitive in $\mathbb{Z}[\frac{1}{p}]^2$ but not in $\mathbb{Z}^2$. However, these sets are very much related to each other: every element in $\mathbb{Z}[\frac{1}{p}]^2_{\mathrm{prim}}$ is a multiplication by a power of p of an element in $\mathbb{Z}^2_{\mathrm{prim}}$.

Lemma 2.2. — *The primitive vectors in $\mathbb{Z}[\frac{1}{p}]^2$ are $\{p^\alpha v : \alpha \in \mathbb{Z},\, v \in \mathbb{Z}^2_{\mathrm{prim}}\}$.*

For the proof, we observe that every element f of $\mathbb{Z}[\frac{1}{p}]$ can be written as $f = \frac{m}{p^n}$ where $m, n \in \mathbb{Z}$ and m is coprime to p. If $\deg(f) = 0$ (as a polynomial in $\frac{1}{p}$), then f is an integer and this fact is clear. Otherwise, let $n > 0$ and write

$$f = \frac{a_n}{p^n} + \frac{a_{n-1}}{p^{n-1}} + \cdots + \frac{a_1}{p} + a_0$$

where $a_0, \ldots, a_n$ are integers that are (except maybe a_0) coprime to p, and $a_n \neq 0$. Then

$$f = \frac{a_n + a_{n-1}p + \cdots + a_1 p^{n-1} + a_0 p^n}{p^n},$$

and the denominator is coprime to p when $n > 0$.

Proof of Lemma 2.2. — Let $(a,b) \in \mathbb{Z}\left[\frac{1}{p}\right]^2_{\text{prim}}$ and write $a = a'p^\alpha$ and $b = b'p^\beta$ where $a', b' \in \mathbb{Z}$ are co-prime to p and $\alpha, \beta \in \mathbb{Z}$. We claim that $\gcd(a', b') = 1$. To see this, substitute a and b into the gcd equation $ax + by = 1$ (x, y variables in $\mathbb{Z}\left[\frac{1}{p}\right]$) to obtain $a'p^\alpha x + b'p^\beta y = 1$. Multiply both sides by a non-negative power of p to obtain an equation in integers: $a'x' + b'y' = p^m$ where $x', y' \in \mathbb{Z}$. It then follows that $\gcd(a', b')$ is a power of p, but since both a', b' are co-prime to p, then $\gcd(a', b') = 1$. Without loss of generality, assume $\beta \geq \alpha$. Then $(a,b) = p^\alpha \left(a', b'p^{\beta-\alpha}\right) = p^\alpha v$ where $v = \left(a', b'p^{\beta-\alpha}\right) \in \mathbb{Z}^2_{\text{prim}}$. $\square$

To formulate the connection between primitive $\mathbb{Z}\left[\frac{1}{p}\right]^2$–vectors to matrices in $\mathrm{SL}_2(\mathbb{Z}\left[\frac{1}{p}\right])$, we introduce the Bruhat-Iwasawa decomposition of $\mathrm{SL}_2(\mathbb{F})$.

2.2. Iwasawa decomposition of $\mathrm{SL}_2(\mathbb{R})$**.** — Let us first recall the KAN decomposition of $\mathrm{SL}_2(\mathbb{R})$. Here $\mathrm{SL}_2(\mathbb{R}) = K_\infty A_\infty N_\infty$ where $K_\infty = \mathrm{SO}_2(\mathbb{R})$ is maximal compact, $A_\infty = \left\{\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right)\right\}$ is the diagonal subgroup and $N_\infty = \left\{\left(\begin{smallmatrix} 1 & \mathbb{R} \\ 0 & 1 \end{smallmatrix}\right)\right\}$ is the subgroup of upper unipotent matrices. Then, for $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$,

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \underbrace{\frac{1}{\sqrt{a^2+b^2}} \begin{pmatrix} a & -b \\ b & a \end{pmatrix}}_{\in K_\infty} \underbrace{\begin{pmatrix} \sqrt{a^2+b^2} & 0 \\ 0 & \frac{1}{\sqrt{a^2+b^2}} \end{pmatrix}}_{\in A_\infty} \underbrace{\begin{pmatrix} 1 & \frac{ac+bd}{a^2+b^2} \\ 0 & 1 \end{pmatrix}}_{\in N_\infty}.$$

By letting $v := (a,b)^{\mathrm{t}}$ and $w := (c,d)^{\mathrm{t}}$, we obtain

$$(v \quad w) = \underbrace{\left(\widehat{v} \quad \widehat{v}^\perp\right)}_{\in K_\infty} \underbrace{\begin{pmatrix} \|v\| & 0 \\ 0 & \|v\|^{-1} \end{pmatrix}}_{\in A_\infty} \underbrace{\begin{pmatrix} 1 & \frac{\langle w, \hat{v}\rangle}{\|v\|} \\ 0 & 1 \end{pmatrix}}_{\in N_\infty}, \tag{2}$$

where

$$\widehat{v} := v/\|v\|$$

is the unit vector pointing in the direction of v,

$$v^\perp := (-b, a)$$

is the vector pointing in the orthogonal direction to v, and $\langle \cdot, \cdot \rangle$ is the standard dot product in $\mathbb{R}^2$.

We note that any pair of the three subgroups K_∞, A_∞ and N_∞ intersect trivially, and therefore the decomposition $\mathrm{SL}_2(\mathbb{R}) = K_\infty A_\infty N_\infty$ induces coordinates on $\mathrm{SL}_2(\mathbb{R})$: every element $g \in \mathrm{SL}_2(\mathbb{R})$ has a unique presentation as $g = kan$. In addition, a Haar measure on $\mathrm{SL}_2(\mathbb{R})$ can be decomposed in the Iwasawa coordinates as

$$\mathrm{d}\mu_{\mathrm{SL}_2(\mathbb{R})}(g) = \mathrm{d}\mu_{\mathrm{SL}_2(\mathbb{R})}(kan) = \frac{\mathrm{d}\mu_{K_\infty}(k)\mathrm{d}\mu_{A_\infty}(a)\mathrm{d}\mu_{N_\infty}(n)}{\alpha}$$

with $\alpha \in \mathbb{R}$ being the first diagonal coefficient of a, and where: μ_{N_∞} is the Haar measure on N_∞ corresponding to the Lebesgue measure on $\mathbb{R}$ under the isomorphism $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \leftrightarrow x$; μ_{K_∞} is the Haar measure on K_∞ corresponding to the Lebesgue measure on $\mathbb{S}^1$ under the isomorphism $\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right) \leftrightarrow \theta$; and μ_{A_∞} is a Haar measure on A_∞ corresponding to the Lebesgue measure on the multiplicative group of $\mathbb{R}_{>0}$ under the isomorphism $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right) \leftrightarrow \alpha$.

With these isomorphisms in mind, and recalling that the Haar measure on $(\mathbb{R}_{>0}, \cdot\,)$ is $\frac{\mathrm{d}\alpha}{\alpha}$, we have

$$\mathrm{d}\mu_{\mathrm{SL}_2(\mathbb{R})}(g) = \mathrm{d}\mu_{\mathrm{SL}_2(\mathbb{R})}\left(\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right) = \frac{\mathrm{d}\theta\mathrm{d}\alpha\mathrm{d}x}{\alpha^2}, \tag{3}$$

where $\mathrm{d}y$ stands for integration w.r.t. the Lebesgue measure on $\mathbb{R}$.

2.3. Bruhat decomposition of $\mathrm{SL}_2(\mathbb{Q}_p)$**.** — Proceeding to the p-adic case, the group $\mathrm{SL}_2(\mathbb{Q}_p)$ can also be written as $K_pA_pN_p$, but these are not coordinates. Indeed, set

$$\begin{aligned} K_p &:= \left\{\begin{pmatrix}a & c\\ b & d\end{pmatrix} : a,b,c,d \in \mathbb{Z}_p, ad-bc=1\right\} = \mathrm{SL}_2(\mathbb{Z}_p)\\ A_p &:= \left\{\begin{pmatrix}p^{-t} & 0\\ 0 & p^t\end{pmatrix} : t \in \mathbb{Z}\right\} \\ N_p &:= \left\{\begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix} : \alpha \in \mathbb{Q}_p\right\}. \end{aligned} \tag{4}$$

Since K_p and N_p intersect non-trivially, the Iwasawa decomposition on $\mathrm{SL}_2(\mathbb{Q}_p)$ is not unique. To remedy this, we use the Bruhat decomposition instead. Let

$$M_p := \left\{\begin{pmatrix}u & 0\\ 0 & u^{-1}\end{pmatrix} : u \in \mathbb{Z}_p^*\right\}$$

be the centralizer of A_p in K_p, $D_p := \left\{\left(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{smallmatrix}\right) : \alpha \in \mathbb{Q}_p\right\}$ be the diagonal subgroup (satisfying $D_p = M_pA_p$), and

$$N_p^- := \left\{\begin{pmatrix}1 & 0\\ \alpha & 1\end{pmatrix} : \alpha \in \mathbb{Q}_p\right\}$$

be the lower unipotent subgroup. Each two of the subgroups N_p^-, M_p, A_p and N_p intersect trivially, which means that they induce coordinates — the Bruhat coordinates — on the subset

$$N_p^- M_p A_p N_p \subset \mathrm{SL}_2(\mathbb{Q}_p).$$

This set is not the whole of $\mathrm{SL}_2(\mathbb{Q}_p)$, but its complement in $\mathrm{SL}_2(\mathbb{Q}_p)$ is of Haar measure zero. We choose the following Haar measures on the above subgroups. The natural isomorphisms between N_p and N_p^- to $\mathbb{Q}_p$ equip N_p and N_p^- with Haar measures that correspond to the Haar measure μ_p on $\mathbb{Q}_p$:

$$\mathrm{d}\mu_{N_p}\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix} = \mathrm{d}\mu_p(\beta), \qquad \mathrm{d}\mu_{N_p^-}\begin{pmatrix}1 & 0\\ \alpha & 1\end{pmatrix} = \mathrm{d}\mu_p(\alpha).$$

Similarly, the natural isomorphism between A_p to $\mathbb{Z}$ induces A_p with the Haar measure on $\mathbb{Z}$ that is the counting measure:

$$\mu_{A_p} = \sum_{a \in A_p} \Delta_a,$$

where Δ_a is the Dirac measure supported on $a \in A_p$. Finally, D_p is naturally isomorphic with the multiplicative group $\mathbb{Q}_p^\times$ of the field $\mathbb{Q}_p$, from which it inherits the Haar measure:

$$\mathrm{d}\mu_{D_p}\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix} = \mathrm{d}\mu_{\mathbb{Q}_p^\times}(\alpha) = \frac{\mathrm{d}\mu_p(\alpha)}{|\alpha|_p}.$$

Since the measure on $\mathbb{Q}_p^\times$ restricts to the measure on $\mathbb{Z}_p^\times$, and the latter is isomorphic to M_p in the same way that $\mathbb{Q}_p^\times$ is isomorphic to D_p, we obtain a Haar measure on M_p:

$$\mathrm{d}\mu_{M_p}\begin{pmatrix} u & 0 \\ 0 & u^{-1}\end{pmatrix} = \mathrm{d}\mu_{\mathbb{Z}_p^\times}(u) = \frac{\mathrm{d}\mu_p(u)}{|u|_p} = \mathrm{d}\mu_p(u).$$

A Haar measure on $\mathrm{SL}_2(\mathbb{Q}_p)$ can be expressed in the Bruhat coordinates as follows.

Lemma 2.3. — *The Haar measure on $G_p = \mathrm{SL}_2(\mathbb{Q}_p)$ w.r.t. the Bruhat coordinates is*

$$\mathrm{d}\mu_{G_p}(g) = \mathrm{d}\mu_{G_p}\left(\begin{pmatrix} 1 & 0 \\ \alpha & 1\end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & u^{-1}\end{pmatrix}\begin{pmatrix} p^{-t} & 0 \\ 0 & p^t\end{pmatrix}\begin{pmatrix} 1 & \beta \\ 0 & 1\end{pmatrix}\right) = p^{2t}\mathrm{d}\mu_p(\alpha)\mathrm{d}\mu_p(u)\Delta_t\mathrm{d}\mu_p(\beta)$$

Proof. — Let P_p^- denote the subgroup of lower triangular matrices in $\mathrm{SL}_2(\mathbb{Q}_p)$, meaning that $P_p^- = N_p^- D_p = N_p^- M_p A_p$. By [12, Lemma 11.31], since G_p is unimodular, a Haar measure $\mu_{G_p} = \mu_{\mathrm{SL}_2(\mathbb{Q}_p)}$ is given by $\mu_{G_p} = \mu^L_{P_p^-} \times \mu^R_{N_p}$, where $\mu^L_{P_p^-}$ is a left Haar measure on P_p^- and $\mu^R_{N_p}$ is a right Haar measure on N_p, which is simply μ_{N_p}. It is left to compute a left Haar measure on P_p^-. Note that P_p^- can be introduced in two ways as $N_p^- D_p = D_p N_p^-$, but the expression for the Haar measure will correspond to the choice of coordinates. It is easy to show that $\mu_{D_p} \times \mu_{N_p^-}$ is a left Haar measure on $P_p^- = D_p N_p^-$; to obtain this left Haar measure in the coordinates $N_p^- D_p$ we will perform a change of variables:

$$\begin{aligned}\int\!\!\int f\left(\begin{bmatrix} \alpha^{-1} & 0 \\ o & \alpha\end{bmatrix}\begin{bmatrix} 1 & 0 \\ y & 1\end{bmatrix}\right)\mathrm{d}\mu_{D_p}\left(\begin{bmatrix} \alpha^{-1} & 0 \\ o & \alpha\end{bmatrix}\right)\mathrm{d}\mu_{N_p^-}\left(\begin{bmatrix} 1 & 0 \\ y & 1\end{bmatrix}\right)& \\ = \int\!\!\int f\left(\begin{bmatrix} 1 & 0 \\ y\alpha^{-2} & 1\end{bmatrix}\begin{bmatrix} \alpha & 0 \\ o & \alpha^{-1}\end{bmatrix}\right)\mathrm{d}\mu_{D_p}\left(\begin{bmatrix} \alpha^{-1} & 0 \\ o & \alpha\end{bmatrix}\right)\mathrm{d}\mu_{N_p^-}\left(\begin{bmatrix} 1 & 0 \\ y & 1\end{bmatrix}\right)& \\ = \int\!\!\int f\left(\begin{bmatrix} 1 & 0 \\ x & 1\end{bmatrix}\begin{bmatrix} \alpha & 0 \\ o & \alpha^{-1}\end{bmatrix}\right)|\alpha|_p^2\,\mathrm{d}\mu_{D_p}\left(\begin{bmatrix} \alpha^{-1} & 0 \\ o & \alpha\end{bmatrix}\right)\mathrm{d}\mu_{N_p^-}\left(\begin{bmatrix} 1 & 0 \\ x & 1\end{bmatrix}\right)&\end{aligned}$$

Since the integral on the left-hand side is invariant under replacing $f(g)$ by $f(hg)$ for any $h \in P_p^-$, then so is the integral on the right-hand side. Hence $\mu^L_{P_p^-} = |\alpha|_p^2\left(\mu_{D_p} \times \mu_{N_p^-}\right)$. Now, since $D_p = M_p A_p$ where M_p and A_p commute and are abelian, we have that $\mu_{D_p} = \mu_{M_p} \times \mu_{A_p}$. We conclude that a left Haar measure on G_p is given in the Bruhat coordinates as

$$\mu_{G_p} = |\alpha|_p^2\left(\mu_{N_p^-} \times \mu_{M_p} \times \mu_{A_p} \times \mu_{N_p}\right). \qquad \square$$

Remark 2.4. — Under this choice of Haar measure, the (compact and open) subgroup $\mathrm{SL}_2(\mathbb{Z}_p)$ has mass $1 - \frac{1}{p}$. Indeed, we have

$$\mathrm{SL}_2(\mathbb{Z}_p) \simeq N_p^-(\mathbb{Z}_p) \times M_p \times N_p(\mathbb{Z}_p) \simeq \mathbb{Z}_p \times \mathbb{Z}_p^\times \times \mathbb{Z}_p$$

and therefore

$$\mu_{G_p}\left(\mathrm{SL}_2(\mathbb{Z}_p)\right) = \mu_p(\mathbb{Z}_p)\mu_p\left(\mathbb{Z}_p^\times\right)\mu_p(\mathbb{Z}_p) = 1 \cdot \left(1 - \frac{1}{p}\right) \cdot 1 = 1 - \frac{1}{p}.$$

2.4. The Bruhat-Iwasawa coordinates. — While the Bruhat decomposition of $\mathrm{SL}_2(\mathbb{Q}_p)$ provides uniqueness, the Iwasawa decomposition provides an arithmetic interpretation of the different components, as suggested in (2) for the $\mathrm{SL}_2(\mathbb{R})$ case. Luckily, these two decompositions coincide on a "large" subset of $\mathrm{SL}_2(\mathbb{Q}_p)$, on which we are going to focus from now on. Consider $g = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Q}_p)$; assuming that $a \neq 0$ we have that

$$\begin{aligned}\begin{pmatrix} a & c \\ b & d \end{pmatrix} &\overset{a\neq 0}{=} \underbrace{\begin{pmatrix} 1 & 0 \\ \frac{b}{a} & 1 \end{pmatrix}}_{\in N_p^-}\underbrace{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}}_{\in D_p}\underbrace{\begin{pmatrix} 1 & \frac{c}{a} \\ 0 & 1 \end{pmatrix}}_{\in N_p} \\ &\overset{a=u_a|a|_p^{-1}}{=} \underbrace{\begin{pmatrix} 1 & 0 \\ \frac{b}{a} & 1 \end{pmatrix}}_{\in N_p^-}\underbrace{\begin{pmatrix} u_a & 0 \\ 0 & u_a^{-1} \end{pmatrix}}_{\in M_p}\underbrace{\begin{pmatrix} |a|_p^{-1} & 0 \\ 0 & |a|_p \end{pmatrix}}_{\in A_p}\underbrace{\begin{pmatrix} 1 & \frac{c}{a} \\ 0 & 1 \end{pmatrix}}_{\in N_p}.\end{aligned}$$

Indeed, the set of $g \in \mathrm{SL}_2(\mathbb{Q}_p)$ with $a \neq 0$ is exactly $N_p^- M_p A_p N_p$. If we assume further that $|a|_p \geq |b|_p$, then $\frac{b}{a} \in \mathbb{Z}_p$ and therefore the N_p^- component lies also in K_p; letting

$$G_p^+ = \mathrm{SL}_2(\mathbb{Q}_p)^+ := \left\{\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q}_p) : |a|_p \geq |b|_p\right\},$$

in which necessarily $a \neq 0$, we conclude that the Bruhat decomposition of G_p^+, which is unique, coincides with the Iwasawa decomposition. This is due to the fact that the $N_p^- M_p$ component in the Bruhat decomposition coincides with the K_p component in the Iwasawa decomposition. Denoting this component by

$$Q_p := \left\{\begin{smallmatrix}\text{lower triangular} \\ \text{matrices in } K_p\end{smallmatrix}\right\} = \left\{\begin{pmatrix} u & 0 \\ m & u^{-1} \end{pmatrix} : m \in \mathbb{Z}_p, u \in \mathbb{Z}_p^\times\right\} < K_p$$

(note that it indeed lies in G_p^+), we have that $G_p^+ = Q_p A_p N_p$, and these are coordinates on G_p^+.

Moving forward to the arithmetic interpretation of these coordinates, it is clear that the first columns of the elements in G_p^+ lie in the p-adic "right half plane":

$$\mathbb{Q}_p^{2\,+} := \left\{(a,b) \in \mathbb{Q}_p^2 : |a|_p \geq |b|_p\right\}.$$

Accordingly, the right half of the p-adic unit sphere is denoted

$$\mathbb{S}_p^{1,+} := \mathbb{S}_p^1 \cap \mathbb{Q}_p^{2\,+} = \left\{(u,a) : u \in \mathbb{Z}_p^\times, a \in \mathbb{Z}_p\right\},$$

and the $\mathbb{Z}[\frac{1}{p}]$– vectors in $\mathbb{Q}_p^{2\,+}$ are denoted by $\mathbb{Z}[\frac{1}{p}]^{2,+}$.

Fact 2.5. — *The half-sphere $\mathbb{S}_p^{1,+}$ is homeomorphic to Q_p and to $\mathbb{Z}_p^\times \times \mathbb{Z}_p$. Its Haar measure $\mu_p^2\left(\mathbb{S}_p^{1,+}\right)$ equals $\mu_p(\mathbb{Z}_p^\times)\mu_p(\mathbb{Z}_p) = 1 - \frac{1}{p} = \mu_{G_p}(\mathrm{SL}_2(\mathbb{Z}_p))$.*

To conclude, we have that for $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \in G_p^+$ (i.e. $v = (a,b)^{\mathrm{t}} \in \mathbb{Q}_p^{2\,+}$),

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \overset{a=u_a|a|_p^{-1}}{=} \underbrace{\begin{pmatrix} u_a & 0 \\ \frac{b}{a}u_a & u_a^{-1} \end{pmatrix}}_{\in Q_p}\underbrace{\begin{pmatrix} |a|_p^{-1} & 0 \\ 0 & |a|_p \end{pmatrix}}_{\in A_p}\underbrace{\begin{pmatrix} 1 & \frac{c}{a} \\ 0 & 1 \end{pmatrix}}_{\in N_p},$$

which means that the p-adic analog to (2) is the following. For $g = \begin{pmatrix} v & w \end{pmatrix} \in G_p^+$, it holds that

$$(5) \qquad \begin{pmatrix} v & w \end{pmatrix} = \underbrace{\begin{pmatrix} \check{v} & * \end{pmatrix}}_{\in Q_p} \underbrace{\begin{pmatrix} \|v\|_p^{-1} & 0 \\ 0 & \|v\|_p \end{pmatrix}}_{\in A_p} \underbrace{\begin{pmatrix} 1 & \frac{y(w)}{y(v)} \\ 0 & 1 \end{pmatrix}}_{\in N_p},$$

where $v = (x(v), y(v))^{\mathrm{t}}$, $w = (x(w), y(w))^{\mathrm{t}}$, and

$$\check{v} := \|v\|_p \, v$$

is the unit vector pointing in the p-adic direction of v (the projection of v to the p-adic unit sphere). (Note that the fact that $\check{v}$ is a p-adic unit vector is completely analogous to the fact that $u_a = |a|_p \, a$ is a unit in $\mathbb{Z}_p$).

Definition 2.6. — The Iwasawa–Bruhat decomposition of

$$\mathrm{SL}_2(\mathbb{F})^+ := \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{Q}_p)^+$$

is

$$\mathrm{SL}_2(\mathbb{F})^+ = K_\infty A_\infty N_\infty \times Q_p A_p N_p = \underbrace{(K_\infty \times Q_p)}_{:=Q} \underbrace{(A_\infty \times A_p)}_{:=A} \underbrace{(N_\infty \times N_p)}_{:=N}.$$

2.5. Correspondence between primitive vectors in $\mathbb{Z}[\frac{1}{p}]^2$ and matrices in $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$. We now define natural subsets in the Bruhat-Iwasawa components.

– For $\mathcal{D} \subset \mathbb{F}$, we consider

$$N_{\mathcal{D}} = \left\{ \left(\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \right) \in N_\infty \times N_p : (\alpha, a) \in \mathcal{D} \right\};$$

– for $R > 1$, $t_1 \leq t_2 \in \mathbb{Z}$, let

$$A_{R,t_1,t_2} = \left\{ \left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \begin{pmatrix} p^{-t} & 0 \\ 0 & p^t \end{pmatrix} \right) \in A_\infty \times A_p : \begin{matrix} 1 < \alpha \leq R \\ t_1 \leq t \leq t_2 \end{matrix} \right\};$$

– for a real arc $\Theta \subset \mathbb{S}^1$ and a p-adic arc $\Theta_p \subset \mathbb{S}_p^1$, let

$$Q_{\Theta,\Theta_p} = \left\{ \left(\begin{pmatrix} \widehat{u} & \widehat{u}^\perp \end{pmatrix}, \begin{pmatrix} \check{v} & * \end{pmatrix} \right) \in K_\infty \times Q : \begin{matrix} \widehat{u} \in \Theta \subset \mathbb{S}^1 \\ \check{v} \in \Theta_p \subset \mathbb{S}_p^1 \end{matrix} \right\}.$$

The following proposition establishes a 1-to-1 correspondence between vectors in $\mathbb{Z}[\frac{1}{p}]^2_{\mathrm{prim}}$ and certain matrices in $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$.

Proposition 2.7. — *Let $\mathcal{D} \subset \mathbb{F}$ be a fundamental domain for the lattice $\mathbb{Z}[\frac{1}{p}]$ in $\mathbb{F}$.*

1. *There is a bijection $v \leftrightarrow \gamma_{v,D}$ between primitive $\mathbb{Z}[\frac{1}{p}]^{2,+}$–vectors and $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])^+$ matrices in $QAN_{\mathcal{D}}$.*

2. *For $\mathcal{D} \subset \mathbb{F}$, $R > 1$, $t_1 \leq t_2 \in \mathbb{Z}$ and arcs $\Theta \subset \mathbb{S}^1$, $\Theta_p \subset \mathbb{S}_p^1$, the bijection $v \leftrightarrow \gamma_{v,D}$ restricts to being between*

(a) *primitive* $\mathbb{Z}[\frac{1}{p}]^{2,+}$*–vectors of real norm* $\|v\| \leq R$*, p-adic norm* $p^{t_1} \leq \|v\|_p \leq p^{t_2}$ *and directions* $(\widehat{v}, \check{v}) \in \Theta \times \Theta_p \subseteq \mathbb{S}^1 \times \mathbb{S}^1_p$

(b) $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])^+$ *matrices in* $Q_{\Theta\times\Theta_p} A_{R,t_1,t_2} N_{\mathcal{D}}$.

Proof of Proposition 2.7. — If $a, b \in \mathbb{Z}[\frac{1}{p}]$ are such that $v = (a, b)$ is primitive, then there exist (infinitely many) solutions (x, y) to the gcd equation of v over $\mathbb{Z}[\frac{1}{p}]$, $ax + by = 1$. If (x_0, y_0) is such a solution, then the set of all solutions is

$$\left\{(x_0, y_0) + m\,(-b, a) : m \in \mathbb{Z}[\frac{1}{p}]\right\}.$$

A choice of $m \in \mathbb{Z}[\frac{1}{p}]$ sets a unique solution to this equation. The gcd equation of v can also be written in the form

$$\det\left(\begin{bmatrix} b & x \\ -a & y \end{bmatrix}\right) = 1,$$

which is equivalent to setting

$$v^{\perp} = (b, -a)^{\mathrm{t}}$$
$$w = (x, y)^{\mathrm{t}}$$

and requiring that

$$\begin{bmatrix} v^{\perp} & w \end{bmatrix} \in \mathrm{SL}_2\left(\mathbb{Z}\left[\frac{1}{p}\right]\right).$$

The possibilities for the second column w are all the solutions to the gcd equation of v; the matrices obtained from the different possibilities are

$$\left\{\begin{bmatrix} v^{\perp} & w_0 + m v^{\perp} \end{bmatrix} : m \in \mathbb{Z}\left[\frac{1}{p}\right]\right\}$$

where $w_0 = (x_0, y_0)^{\mathrm{t}}$ is some solution. This set of matrices is an orbit for the group $\left\{\begin{bmatrix} 1 & \mathbb{Z}[\frac{1}{p}] \\ 0 & 1 \end{bmatrix}\right\} = N_p \cap \mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$ acting by right multiplication:

$$\left\{\begin{bmatrix} v^{\perp} & w_0 \end{bmatrix}\begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix} : m \in \mathbb{Z}\left[\frac{1}{p}\right]\right\}.$$

According to (2) and (5), when viewing this set of matrices (via the diagonal embedding) in $\mathrm{SL}_2(\mathbb{F})$, it equals

$$\left\{\left(k_\infty a_\infty \begin{bmatrix} 1 & \frac{\langle w, \hat{v}\rangle}{\|v\|} + m \\ 0 & 1 \end{bmatrix}, k_p a_p \begin{bmatrix} 1 & \frac{y(w)}{y(v)} + m \\ 0 & 1 \end{bmatrix}\right) : m \in \mathbb{Z}\left[\frac{1}{p}\right]\right\}.$$

Since $\mathcal{D}$ is a fundamental domain for $\mathbb{Z}[\frac{1}{p}]^{\mathrm{diag}}$ in $\mathbb{F}$, there exists a unique m for which $\left(\frac{\langle w, \hat{v}\rangle}{\|v\|} + m, \frac{y(w)}{y(v)} + m\right)$ lies in $\mathcal{D}$. This $m = m\,(\mathcal{D})$ determines a unique solution $w_{v,\mathcal{D}}$ which in turn determines a unique matrix

$$\gamma_{v,D} := \begin{bmatrix} v^{\perp} & w_{v,\mathcal{D}} \end{bmatrix} \in \mathrm{SL}_2\left(\mathbb{Z}\left[\frac{1}{p}\right]\right).$$

This establishes a one to one correspondence $v \leftrightarrow \gamma_{v,\mathcal{D}}$ between $\mathbb{Z}[\frac{1}{p}]^2$ primitive vectors and matrices in $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}]) \cap QAN_{\mathcal{D}}$, and proves part 1 of the proposition. As for part 2: the fact that $(\widehat{v}, \check{v}) \in \Theta \times \Theta_p$ if and only if the Q–component of γ_v lies in Q_{Θ,Θ_p} follows from (2) and the definition of Q_{Θ,Θ_p}, and the fact that $\|v\| \leq R$ and $p^{t_1} \leq \|v\|_p \leq p^{t_2}$ if and only if the A–component of γ_v lies in A_{R,t_1,t_2} follows from (5) and the definition of A_{R,t_1,t_2}. □

3. Counting lattice points inside well-rounded sets

Proposition 2.7 gives a reformulation of Theorem A in the form of counting matrices of the lattice $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$ inside the S-arithmetic group $\mathrm{SL}_2(\mathbb{F})$, $S = \{\infty, p\}$. The next step on the way to prove Theorem A is to solve this lattice point counting problem. This is the topic of this section. The method we will apply was established in [17], and it relies on ergodic theory. The corpus of work on equidistribution and counting lattice points by dynamical methods is rather vast; for a short survey on the applied techniques in the case of lattices in real algebraic Lie groups, we refer to [17, p. 7]. As for the techniques in (as well as an introduction to) the S-arithmetic setting, we refer to the survey [18].

3.1. Well-roundedness and counting lattice points. — Proposition 2.7 allows us to translate the question on the number of primitive $\mathbb{Z}[\frac{1}{p}]^{2,+}$–vectors of real norm $\|v\| \leq R$, p-adic norm $p^{t_1} \leq \|v\|_p \leq p^{t_2}$ and directions $(\widehat{v}, \check{v}) \in \Theta \times \Theta_p \subseteq \mathbb{S}^1 \times \mathbb{S}^1_p$, into the problem of counting $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])^+$ matrices in $Q_{\Theta\times\Theta_p} A_{R,t_1,t_2} N_{\mathcal{D}}$, where $\mathcal{D} \subset \mathbb{F}$ is a fundamental domain for $\mathbb{Z}[\frac{1}{p}]$. From now on we fix the fundamental domain from Fact 1.5,

$$\mathcal{D} := \left(-\frac{1}{2}, \frac{1}{2}\right] \times \mathbb{Z}_p.$$

We now describe a method to approach this counting problem.

Notation 3.1. — In what follows, $\mathbf{G}$ will denote the product $\prod_{v\in S} \mathbf{G}_v(\mathbb{K}_v)$ where $\mathbb{K}$ is a number field, $\mathbb{K}_v$ is the localization of $\mathbb{K}$ over a place v, $\mathbf{G}_v$ is a simple algebraic group defined over $\mathbb{K}$ and S is a finite set of places that contains ∞.

Definition 3.2. — Let $\mathbf{G}$ be as in Notation 3.1, μ a Borel measure on $\mathbf{G}$, and $\{\mathcal{O}_\epsilon\}_{\epsilon>0}$ a family of identity neighborhoods in G.

1. For a measurable subset $\mathcal{B} \subset \mathbf{G}$, we define
$$\mathcal{B}^+(\epsilon) := \mathcal{O}_\epsilon \mathcal{B} \mathcal{O}_\epsilon = \bigcup_{u,v\in\mathcal{O}_\epsilon} u\,\mathcal{B}\,v,$$
$$\mathcal{B}^-(\epsilon) := \bigcap_{u,v\in\mathcal{O}_\epsilon} u\,\mathcal{B}\,v.$$
2. The set $\mathcal{B}$ is *Lipschitz well-rounded (LWR)* with (positive) parameters $(\mathcal{C}, \epsilon_0)$ if for every $0 < \epsilon < \epsilon_0$
$$\mu\left(\mathcal{B}^+(\epsilon)\right) \leq (1 + \mathcal{C}\epsilon)\ \mu\left(\mathcal{B}^-(\epsilon)\right). \tag{6}$$
3. A family $\{\mathcal{B}_R\}_{R>0} \subset \mathbf{G}$ of measurable domains is Lipschitz well-rounded with positive parameters $(\mathcal{C}, R_0, \epsilon_0)$ if for every $0 < \epsilon < \epsilon_0$ and $R > R_0$, the set $\mathcal{B}_R$ is LWR with $(\mathcal{C}, \epsilon_0)$.

Remark 3.3. — Indeed the definition of LWR depends on the choice of a family $\{\mathcal{O}_\epsilon\}_{\epsilon>0}$ of identity neighborhoods; however, we disregard this fact since we will only work with a specific family, cf. Definition 3.6.

The notion of well-roundedness of a family is by now standard (see e.g. [13]), but it is less common to define a well-rounded *set*; however, this notion will be useful for us, as the sets under our consideration will project to a well-rounded family in the real component, but to a well-rounded set in the p-adic one.

Theorem 3.4 **([17, Theorems 1.9, 4.5, and Remark 1.10]).** — *Let* $\mathbf{G}$ *as in Notation 3.1 and with Haar measure* μ*, and let* $\Gamma < \mathbf{G}$ *be a lattice. Assume that* $\{\mathcal{B}_R\} \subset \mathbf{G}$ *is a family of finite-measure domains which satisfy* $\mu(\mathcal{B}_R) \to \infty$ *as* $T \to \infty$*. If the family* $\{\mathcal{B}_R\}$ *is Lipschitz well-rounded, then there exists a parameter* $\tau(\Gamma) \in \left(0, \frac{1}{2(1+\dim \mathbf{G})}\right)$ *such that for* R *large enough and every* $\delta > 0$*:*

$$\left| \#\left(\mathcal{B}_R \cap \Gamma\right) - \frac{\mu\left(\mathcal{B}_R\right)}{\mu\left(\mathbf{G}/\Gamma\right)} \right| \ll_{\mathbf{G},\Gamma,\delta} \mathrm{const} \cdot \mu\left(\mathcal{B}_R\right)^{1-\tau(\Gamma)+\delta}$$

as $T \to \infty$*, where* $\mu(\mathbf{G}/\Gamma)$ *is the measure of a fundamental domain of* Γ *in* $\mathbf{G}$*.*

Remark 3.5 **(The error exponent).** — The parameter $\tau(\Gamma)$ depends on estimates on the rate of decay of matrix coefficients of automorphic representations of $\mathbf{G}$. For $\mathbf{G} = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{Q}_p)$, any bound toward the generalized Ramanujan conjecture, from Gelbart–Jacquet [14] to Kim–Sarnak [22], implies that these coefficients are L^{q+} for some $2 < q \leq 4$ (see e.g. [6]). These bounds give $\tau_p = \tau(\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])) = \frac{1}{4(1+\dim(\mathbf{G}))} = \frac{1}{28}$, and only the full Ramanujan conjecture would give a better exponent, namely $\tau_p = \frac{1}{2(1+\dim(\mathbf{G}))} = \frac{1}{14}$. This exponent is obtained by a combination of Theorems 1.9, 4.5 and Definition 3.1 in [17].

According to Theorem 3.4, the goal of counting lattice points inside $Q_{\Theta\times\Theta_p} A_{R,t_1,t_2} N_D$ will be achieved by establishing that these sets are Lipschitz well-rounded w.r.t. a certain choice of identity neighborhoods in $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{Q}_p)$. The LWR of these sets reduces to the LWR of their projections to both the real and to the p-adic components; the LWR of the real component is known (see more details in the proof of Theorem A below), and so it remains to verify the LWR of the projection to the p-adic part. To this end, we will consider the following identity neighborhoods inside $\mathrm{SL}_2(\mathbb{Q}_p)$:

Definition 3.6. — For any subgroup H_p of $G_p = \mathrm{SL}_2(\mathbb{Q}_p)$ and a positive integer N we set

$$\begin{aligned}\mathcal{O}_{p^{-N}} &= \ker\left\{G_p(\mathbb{Z}_p) \to G_p\left(\mathbb{Z}_p/p^N\mathbb{Z}_p\right)\right\} \\ &= \left(\mathrm{I}_2 + p^N \operatorname{Mat}_2(\mathbb{Z}_p)\right) \cap G_p(\mathbb{Z}_p) \subset K_p\end{aligned}$$

and

$$\mathcal{O}^{H_p}_{p^{-N}} = \mathcal{O}_{p^{-N}} \cap H_p \subset K_p \cap H_p.$$

Let us now state explicitly the LWR property for the p-adic factor. We note that the LWR property is rather strong here, since the Lipschitz constant equals zero. The following proposition concerns a family of sets inside $\mathrm{SL}_2(\mathbb{Q}_p)$, and it should be understood that the notations for the Bruhat-Iwasawa subsets are the p-adic parts of the Bruhat-Iwasawa subsets of

$\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{Q}_p)$ that were defined in Section 2.5; e.g., $(A_p)_{t_1,t_2}$ is the p-adic part of A_{R,t_1,t_2}, $(Q_p)_{\Theta_p}$ is the p-adic part of Q_{Θ,Θ_p}, etc.

Proposition 3.7. — *Consider the family $\mathcal{B}_{t_1,t_2} = (Q_p)_{\Theta_p}(A_p)_{t_1,t_2}(N_p)_{\alpha+p^{\psi}\mathbb{Z}_p}$ of subsets in $\mathrm{SL}_2(\mathbb{Q}_p)$ with:*

1. *$\Theta_p \subseteq \mathbb{S}^1_p$ a fixed p-adic arc;*
2. *$\psi \in \mathbb{Z}$;*
3. *t_1 and t_2 two real parameters satisfying $t_1 \leq t_2$.*

Then the family $\{\mathcal{B}_{t_1,t_2}\}_{t_0<t_1\leq t_2}$ for an arbitrary $t_0 \in \mathbb{R}$ is Lipschitz well-rounded, with Lipschitz constant zero.

The proof of Proposition 3.7 is quite technical, we postpone it to the end of the paper; see Section 5. Note that in the following proof of Theorem A, we will only need the very special case where $t_1 = t_2 = 0$. In this case the family is indeed reduced to a single set!
We now have all the tools to prove Theorem A.

Proof of Theorem A. — We first note that according to Lemma 2.2,

$$\mathbb{Z}^2_{\text{prim}} = \mathbb{Z}[\frac{1}{p}]^2_{\text{prim}} \cap \mathbb{S}^1_p.$$

It then follows from Proposition 2.7 that there is a bijection between the sets

$$\left\{v \in \mathbb{Z}^2_{\text{prim}} : \begin{matrix}(\widehat{v}, v) \in \Theta \times \Theta_p, \\ \|v\| \leq R\end{matrix}\right\} \longleftrightarrow \left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}]) \cap \left(Q_{\Theta\times\Theta_p} A_{R,0,0} N_{[-\frac{1}{2},\frac{1}{2}]\times\mathbb{Z}_p}\right)\right\}.$$

The sets $Q_{\Theta\times\Theta_p} A_{R,0,0} N_{[-\frac{1}{2},\frac{1}{2}]\times\mathbb{Z}_p}$ are a product of a real factor $(K_\infty)_\Theta (A_\infty)_R (N)_{[-\frac{1}{2},\frac{1}{2}]}$ and a p-adic factor $(Q_p)_{\Theta_p}(A_p)_{0,0}(N_p)_{\mathbb{Z}_p}$; the family of projections to real component is LWR by [19, Theorem 1.1], and the projection to the p-adic part is LWR according to Proposition 3.7. Since a product of LWR sets is LWR [21, Corollary 4.3 and Remark 4.4], it follows that the family $\left\{Q_{\Theta\times\Theta_p} A_{R,0,0} N_{[-\frac{1}{2},\frac{1}{2}]\times\mathbb{Z}_p}\right\}_{R>0}$ is LWR. In the notations of Theorem 3.4, let $\tau_p = \tau\left(\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])\right) = \frac{1}{4(1+\dim(G))}$, see Remark 3.5. By Theorem 3.4,

$$\begin{aligned}
&\#\left\{\gamma \in \mathrm{SL}_2\left(\mathbb{Z}\left[\frac{1}{p}\right]\right) \cap \left(Q_{\Theta\times\Theta_p} A_{R,0,0} N_{[-\frac{1}{2},\frac{1}{2}]\times\mathbb{Z}_p}\right)\right\} \\
&\quad= \frac{\mu_{\mathrm{SL}_2(\mathbb{F})}\left(Q_{\Theta\times\Theta_p} A_{R,0,0} N_{[-\frac{1}{2},\frac{1}{2}]\times\mathbb{Z}_p}\right)}{\mu_{\mathrm{SL}_2(\mathbb{F})}\left(\mathrm{SL}_2(\mathbb{F})/\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])\right)} + O\left(\mu\left(Q_{\Theta\times\Theta_p} A_{R,0,0} N_{[-\frac{1}{2},\frac{1}{2}]\times\mathbb{Z}_p}\right)^{1-\tau_p+\delta}\right) \\
&\quad= \frac{\mu_{\mathrm{SL}_2(\mathbb{R})}\left((K_\infty)_\Theta (A_\infty)_R (N_\infty)_{[-\frac{1}{2},\frac{1}{2}]}\right)}{\mu_{\mathrm{SL}_2(\mathbb{R})}\left(\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})\right)} \cdot \frac{\mu_{\mathrm{SL}_2(\mathbb{Q}_p)}\left((Q_p)_{\Theta_p}(A_p)_{0,0}(N_p)_{\mathbb{Z}_p}\right)}{\mu_{\mathrm{SL}_2(\mathbb{Q}_p)}\left(\mathrm{SL}_2(\mathbb{Z}_p)\right)} \\
&\qquad+ O\left(\mu\left(Q_{\Theta\times\Theta_p} A_{R,0,0} N_{[-\frac{1}{2},\frac{1}{2}]\times\mathbb{Z}_p}\right)^{1-\tau_p+\delta}\right),
\end{aligned}$$

where the last equality was deduced using Fact 1.5. The main term is a product of two factors, one real and one p-adic. The real factor equals

$$\frac{\mathrm{Leb}(\Theta)R^2}{\pi^2/3},$$

by (3). We turn to compute the p-adic factor. By Lemma 2.3 and Fact 2.5,

$$\mu_{\mathrm{SL}_2(\mathbb{Q}_p)}\left(\left(K_p\right)_{\Theta_p}\left(A_p\right)_{0,0}\left(N_p\right)_{\mathbb{Z}_p}\right)=\mu_p^2\left(\Theta_p\right)\cdot 1\cdot\mu_p(\mathbb{Z}_p)=\mu_p^2\left(\Theta_p\right).$$

Furthermore, as noted in Fact 2.5, we have $\mu_{\mathrm{SL}_2(\mathbb{Q}_p)}\left(\mathrm{SL}_2(\mathbb{Z}_p)\right)=1-\frac{1}{p}$. Then we may conclude:

$$\text{(7)}\quad \#\left\{v\in\mathbb{Z}^2_{\mathrm{prim}}:\begin{array}{c}(\widehat{v},v)\in\Theta\times\Theta_p,\\ \|v\|\le R\end{array}\right\}=\frac{3}{\pi^2}\cdot\frac{p}{p-1}\cdot\mathrm{Leb}(\Theta)\mu_p^2\left(\Theta_p\right)R^2+O\left(R^{2(1-\tau_p+\delta)}\right).$$

Applying (7) to $\Theta\times\Theta_p=\mathbb{S}^1\times\mathbb{S}_p^{1,+}$, we have that the total number of primitive vectors up no norm R is asymptotic to

$$\frac{3}{\pi^2}\cdot\frac{p}{p-1}\cdot\mathrm{Leb}(\mathbb{S}^1)\mu_p^2\left(\mathbb{S}_p^{1,+}\right)R^2,$$

and upon dividing (7) by the above and applying symmetry considerations to pass from $\mathbb{S}_p^{1,+}$ to $\mathbb{S}_p^1$, we obtain the desired limit. □

As an evidence for the asymptotics in (7), notice that when taking $\Theta=\mathbb{S}^1$ and $\Theta_p=\mathbb{S}_p^{1,+}$, and recalling that $\mathrm{Leb}(\mathbb{S}^1)=2\pi$ and $\mu_p^2\left(\mathbb{S}_p^{1,+}\right)=1-\frac{1}{p}$ (Fact 2.5), we have that

$$\#\left\{v\in\mathbb{Z}^2_{\mathrm{prim}}:\begin{array}{c}(\widehat{v},v)\in\mathbb{S}^1\times\mathbb{S}_p^{1,+},\\ \|v\|\le R\end{array}\right\}=\#\left\{v\in\mathbb{Z}^2_{\mathrm{prim}}:\|v\|\le R\right\}=\frac{6}{\pi}R^2+o\left(R\right),$$

which coincides with the well known asymptotics for the primitive circle problem.

4. Equidistribution of Iwasawa components in the S-arithmetic case

We have used a special case of Proposition 3.7 to prove Theorem A. The full proposition (together with Theorem 3.4) implies a stronger equidistribution result: the equidistribution of both Q and N components of the Bruhat-Iwasawa decomposition for an S-arithmetic lattice, in product of SL_2's. Let us begin by introducing this set up.

So far we have only dealt with the group $\mathrm{SL}_2\left(\mathbb{R}\right)\times\mathrm{SL}_2(\mathbb{Q}_p)$, meaning the case of one infinite place, and one finite place. But in fact, we can allow any finite number of finite or infinite places, and consider the group

$$G=\left(\mathrm{SL}_2\left(\mathbb{R}\right)\right)^{n_\infty}\times\prod_{p\in S_f}\left(\mathrm{SL}_2(\mathbb{Q}_p)\right)^{n_p},$$

where S_f is a finite set of primes. The notion of arithmetic lattice, which generalizes $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])<\mathrm{SL}_2\left(\mathbb{R}\right)\times\mathrm{SL}_2(\mathbb{Q}_p)$, is described e.g. in [26, Section 5.4]. As noted in [17, Remark 4.6], the ergodic method of Gorodnik and Nevo applies to these lattices. The analogous sets to $Q_{\Theta\times\Theta_p}A_{R,t_1,t_2}N_{\Psi_\infty\times\Psi_p}\subset\mathrm{SL}_2(\mathbb{F})$ inside G are the following. For $1\le i\le n_\infty$, let $\Theta_i\subset\mathbb{S}^1$ be arcs on the unit circle, $\Psi_i\subset\mathbb{R}$ intervals, and $R_i\ge 1$ positive real numbers. Set

$$\underline{R}=(R_1,\dots,R_{n_\infty}),$$

and consider

$$B_{\underline{R}}^{\mathrm{Iw}} := \prod_{i=1}^{n_\infty} (K_\infty)_{\Theta_i} (A_\infty)_{R_i} (N_\infty)_{\Psi_i} \times \prod_{p\in S_f} (Q_p)^{n_p}_{\Theta_p} (A_p)^{n_p}_{t_1^p, t_2^p} (N_p)^{n_p}_{\alpha_p + p^{\psi_p}\mathbb{Z}_p}$$

where for every $p \in S_f$:

1. $\Theta_p \subseteq \mathbb{S}_p^1$ is a fixed p-adic arc,
2. $\psi_p \in \mathbb{Z}$,
3. t_1^p and t_2^p are two real parameters that satisfy $t_1^p \leq t_2^p$ and are bounded from below, namely there exist $t_0^p \in \mathbb{R}$ such that $t_0^p < t_1^p \leq t_2^p$.

Let μ denote a Haar measure on G.

Theorem 4.1. — *Let $\Gamma < G$ be an S-arithmetic lattice, and $\|\cdot\|$ any norm on $\mathbb{R}^{n_\infty}$. Then, for $\tau = \frac{1}{4\left(1+3\left(n_\infty + \sum n_p\right)\right)}$, the following asymptotic formula holds for every $\delta > 0$ as $\|\underline{R}\| \to \infty$:*

$$\begin{aligned}
\#\left\{\left(B_{\underline{R}}^{\mathrm{Iw}}\right) \cap \Gamma^+\right\} &= \frac{\mu\left(B_{\underline{R}}^{\mathrm{Iw}}\right)}{\mu(G/\Gamma)} + O\left(\mu\left(B_{\underline{R}}^{\mathrm{Iw}}\right)^{1-\tau+\delta}\right)\\
&= \frac{6}{\pi^2}\cdot \prod_{i=1}^{n_\infty}\left(\mathrm{Leb}_{\mathbb{S}^1}(\Theta_i)\,\mathrm{Leb}_{\mathbb{R}}(\Psi_i)\cdot R_i^2\right)\cdot \prod_{p\in S_f}\left(\frac{\mu_p^2(\Theta_p)\left(\sum_{t=t_1^p}^{t_2^p} p^{-2t}\right)\left(1-p^{-\psi_p}\right)}{1-\frac{1}{p}}\right)^{n_p}\\
&\qquad + O\left(\left(\prod_{i=1}^{n_\infty} R_i^2\right)^{1-\tau+\delta}\right).
\end{aligned}$$

The implied constant depends on Θ, Ψ, and ψ_p, Θ_p, t_1^p, t_2^p for every p.

The exponent τ, as discussed in Remark 3.5, would be improved to $\tau = \frac{1}{2\left(1+3\left(n_\infty+\sum n_p\right)\right)}$ with the full Ramanujan conjecture.

Remark 4.2. — If t_2^p grows to infinity, then the O-constant does not depend on it.

Proof. — We first note that the family $\left\{B_{\underline{R}}^{\mathrm{Iw}}\right\}$ is LWR, since it is a product of the projections $(K_\infty)_{\Theta_i} (A_\infty)_{R_i} (N)_{I_i}$ to the real components, which are LWR according to [19, Theorem 1.1], and the projections $(Q_p)_{\Theta_p} (A_p)_{t_1^p, t_2^p} (N_p)_{\alpha_p + p^{\psi_p}\mathbb{Z}_p}$ to the finite components, which are LWR according to Proposition 3.7. Now the result follows from Theorem 3.4, combined with the fact that the $\mu_{\mathrm{SL}_2(\mathbb{R})}^{\times n_\infty} \times \prod_p \mu_{\mathrm{SL}_2(\mathbb{Q}_p)}$-volume of $B_{\underline{R}}^{\mathrm{Iw}}$ is the expression appearing in the main term, according to 3 and to Lemma 2.3. ☐

Remark 4.3. — We note that when considering the more general S-arithmetic setting as in Theorem 4.1, we obtain a generalization of Theorem A to joint equidistribution of several p-adic directions of primitive vectors. For this, one should apply Theorem 4.1 to the lattice $\Gamma = \mathrm{SL}_2\left(\mathbb{Z}[\{\frac{1}{p}\}_{p\in S_f}]\right)$.

Remark 4.4. — 1. Theorem 4.1 is in fact an equidistribution result, since it is completely standard to pass from a counting formulation to an equidistribution formulation; see, for example, the proof of Theorem A.

2. Equisitribution of Iwasawa components, and especially of the N-component, has been considered in a number of papers, e.g. [16, 19, 25, 27, 29]. All of the above in rank one real Lie groups; in higher rank, we mention [20]. We are unaware of equidistribution results of the Iwasawa components of lattice elements in the S-arithmetic setting.

3. For more equidistribution and counting results in the S-arithmetic or adelic settings, we refer to [3, 6, 9, 17].

5. Proof of Well-roundedness in $\mathrm{SL}_2(\mathbb{Q}_p)$

The goal of this final subsection is to prove Proposition 3.7, i.e. the well-roundedness of the sets $(Q_p)_{\Theta_p}(A_p)_{t_1,t_2}(N_p)_{\alpha+p^{\psi}\mathbb{Z}_p}$ in $\mathrm{SL}_2(\mathbb{Q}_p)$. The main step is a measurement of how the Bruhat components are modified by a small left or right perturbation. To state this proposition, we need an additional notation: for any $g \in \mathrm{SL}_2(\mathbb{Q}_p)$, we denote by $\| \mathrm{Ad}_g \|_{\mathrm{op}}$ the operator norm of Ad_g acting on $\mathrm{Mat}_2(\mathbb{Q}_p)$. Note that this operator norm takes values in $p^{\mathbb{Z}}$, as the max norm on $\mathbb{Q}_p^2$.

***Proposition 5.1* (Effective Bruhat-Iwasawa decomposition).** — *Let $g = qan \in \mathrm{SL}_2^+(\mathbb{Q}_p)$ with $a = \begin{bmatrix} p^{-t} & 0 \\ o & p^t \end{bmatrix}$. Let $c(a,n) = \|\mathrm{Ad}_n\|_{\mathrm{op}} \max(p^{-t}, 1) \in p^{\mathbb{Z}}$. The function c is bounded when n is restricted to a bounded set and t is bounded from below Moreover, we have*

$$\mathcal{O}_\epsilon^{G_p} qan \mathcal{O}_\epsilon^{G_p} \in q\mathcal{O}_{c(a,n)\cdot\epsilon}^{Q_p} a \mathcal{O}_{c(a,n)\cdot\epsilon}^{N_p} n$$

when $\epsilon \in p^{-\mathbb{N}}$ is small enough.

The proof of Proposition 5.1 requires the following three Lemmas:

Lemma 5.2. — *For $g \in \mathrm{SL}_2(\mathbb{Q}_p)$,*

$$g\mathcal{O}_{p^{-N}} g^{-1} \subseteq \mathcal{O}_{\|\mathrm{Ad}_g\|_{\mathrm{op}} p^{-N}}$$

where $\|\mathrm{Ad}_g\|_{\mathrm{op}} \in p^{\mathbb{Z}}$ is the operator norm of conjugation by g.

Proof. — Indeed,

$$\begin{aligned} g\mathcal{O}_{p^{-N}} g^{-1} &\subseteq g\left(\mathrm{I}_2 + p^N \mathrm{Mat}_2(\mathbb{Z}_p)\right) g^{-1} \\ &= \left(\mathrm{I}_2 + g \cdot p^N \mathrm{Mat}_2(\mathbb{Z}_p) \cdot g^{-1}\right) \\ &\subseteq \left(\mathrm{I}_2 + \|\mathrm{Ad}_g\|_{\mathrm{op}} p^N \mathrm{Mat}_2(\mathbb{Z}_p)\right) = \mathcal{O}_{\|\mathrm{Ad}_g\|_{\mathrm{op}} p^{-N}}. \end{aligned}$$

It is clear that $\|\mathrm{Ad}_g\|_{\mathrm{op}}$ is a power of p, since, as an operator norm, it is the maximum of norms of (p-adic) vectors. ☐

Lemma 5.3. — *For every $k_p \in K_p$, $\left\|\mathrm{Ad}_{k_p}\right\|_{\mathrm{op}} = 1$.*

Proof. — We know that $K_p \cdot \mathbb{S}_p^1 = \mathbb{S}_p^1$, and that this action preserves the p-adic norm. For $T \in \mathrm{Mat}_2(\mathbb{Q}_p)$, we need to show that $\left\|\mathrm{Ad}_{k_p}(T)\right\| = \|T\|$, where the norm on $\mathrm{Mat}_2(\mathbb{Q}_p)$ is the operator norm. Indeed

$$\left\|\mathrm{Ad}_{k_p}(T)\right\| = \sup_{x \in \mathbb{S}_p^1} \left\|\mathrm{Ad}_{k_p}(T) \cdot x\right\|_p = \sup_{x \in \mathbb{S}_p^1} \left\|k_p^{-1} T k_p \cdot x\right\|_p = \sup_{y \in \mathbb{S}_p^1} \|Ty\|_p = \|T\|.$$

Then Ad_{k_p} is norm preserving, and therefore has operator norm 1. $\square$

Lemma 5.4. — *For any $N \geq 1$,*

$$\mathcal{O}_{p^{-N}}^{G_p} = \mathcal{O}_{p^{-N}}^{Q_p} \mathcal{O}_{p^{-N}}^{N_p},$$

$$\mathcal{O}_{p^{-N}}^{Q_p} = \mathcal{O}_{p^{-N}}^{N_p^-} \mathcal{O}_{p^{-N}}^{M_p}.$$

Proof. — We note that when $N \geq 1$

$$\mathcal{O}_{p^{-N}}^{Q_p} = \left\{ \begin{pmatrix} 1 + p^N a & 0 \\ p^N b & \left(1 + p^N a\right)^{-1} \end{pmatrix} : a, b \in \mathbb{Z}_p \right\},$$

$$\mathcal{O}_{p^{-N}}^{N_p} = \left\{ \begin{pmatrix} 1 & p^N \mathbb{Z}_p \\ 0 & 1 \end{pmatrix} \right\}$$

and

$$\mathcal{O}_{p^{-N}}^{M_p} = \left\{ \begin{pmatrix} 1 + p^N \alpha & 0 \\ 0 & \left(1 + p^N \alpha\right)^{-1} \end{pmatrix} : \alpha \in \mathbb{Z}_p \right\}.$$

The inclusions $\supseteq$ in the statement of the lemma are trivial. For the opposite direction, observe that

$$\mathcal{O}_{p^{-N}}^{G_p} \ni \begin{pmatrix} 1 + p^N a & p^N c \\ p^N d & 1 + p^N b \end{pmatrix} = \begin{pmatrix} 1 + p^N x & 0 \\ p^N y & \left(1 + p^N x\right)^{-1} \end{pmatrix} \begin{pmatrix} 1 & p^N z \\ 0 & 1 \end{pmatrix}$$

for

$$x = a, \quad y = d, \quad z = \tfrac{c}{1 + p^N a},$$

as $\left(1 + p^N x\right)^{-1} = 1 - \frac{P^N x}{1 + p^N x}$ and the determinant of the left-hand matrix is 1. Similarly,

$$\mathcal{O}_{p^{-N}}^{Q_p} \ni \begin{pmatrix} 1 + p^N a & 0 \\ p^N b & \left(1 + p^N a\right)^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ p^N z & 1 \end{pmatrix} \begin{pmatrix} 1 + p^N x & 0 \\ 0 & \left(1 + p^N x\right)^{-1} \end{pmatrix}$$

for

$$x = a, \quad z = \tfrac{b}{1 + p^N a}.$$

$\square$

We now turn to prove Proposition 5.1.

Proof of Proposition 5.1. — Let $g = qan$ with $q \in Q$, $a = \begin{pmatrix} p^{-t} & 0 \\ 0 & p^{t} \end{pmatrix} \in A_p$ and $n = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \in N_p$. We will use the fact that

$$a^{-1} n_x a = \begin{pmatrix} p^{t} & 0 \\ 0 & p^{-t} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^{-t} & 0 \\ 0 & p^{t} \end{pmatrix} = \begin{pmatrix} 1 & xp^{2t} \\ 0 & 1 \end{pmatrix} = n_{xp^{2t}}$$

(8)

$$a n_y^- a^{-1} = \begin{pmatrix} p^{-t} & 0 \\ 0 & p^{t} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} p^{t} & 0 \\ 0 & p^{-t} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ yp^{2t} & 1 \end{pmatrix} = n^-_{yp^{2t}}$$

where $n_x = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \in N_p$, $n_y^- = \left(\begin{smallmatrix} 1 & 0 \\ y & 1 \end{smallmatrix}\right) \in N_p^-$ and $a = \left(\begin{smallmatrix} p^{-t} & 0 \\ o & p^{t} \end{smallmatrix}\right)$.

Step 1: Left perturbations. — Set $\epsilon = p^{-N}$. Since $\|\mathrm{Ad}_q\|_{\mathrm{op}} = 1$ (Lemma 5.3), then by Lemmas 5.2 and 5.4 we have that,

$$\mathcal{O}_\epsilon^{G_p} qan = q\left(q^{-1}\mathcal{O}_\epsilon^{G_p} q\right) an \subseteq q\mathcal{O}_\epsilon^{G_p} an = q\mathcal{O}_\epsilon^{Q_p}\mathcal{O}_\epsilon^{N_p} an.$$

According to (8),

$$= q\mathcal{O}_\epsilon^{Q_p} a \cdot a^{-1}\mathcal{O}_\epsilon^{N_p} an = q\mathcal{O}_\epsilon^{Q_p} \cdot a \cdot \mathcal{O}^{N_p}_{p^{-2t}\epsilon} n.$$

Step 2: Right perturbations. — By letting $C(n) = \|\mathrm{Ad}_n\|_{\mathrm{op}} \in p^{\mathbb{Z}}$, we have according to Lemma 5.2 that

$$qan\mathcal{O}_\epsilon^{G_p} = qa\left(n\mathcal{O}_\epsilon^{G_p} n^{-1}\right) n \subseteq qa\mathcal{O}^{G_p}_{C(n)\epsilon} n;$$

By Lemma 5.4 and to (8),

$$\begin{aligned}
&\subseteq qa \cdot \mathcal{O}^{N_p^-}_{C(n)\epsilon}\mathcal{O}^{M_p}_{C(n)\epsilon}\mathcal{O}^{N_p}_{C(n)\epsilon} \cdot n \\
&= qa \cdot \mathcal{O}^{N_p^-}_{C(n)\epsilon} a^{-1} \cdot a\mathcal{O}^{M_p}_{C(n)\epsilon}\mathcal{O}^{N_p}_{C(n)\epsilon} \cdot n \\
&= q\mathcal{O}^{N_p^-}_{p^{-2t}C(n)\epsilon} a \cdot \mathcal{O}^{M_p}_{C(n)\epsilon}\mathcal{O}^{N_p}_{C(n)\epsilon} \cdot n.
\end{aligned}$$

Since A and M commute,

$$= q\mathcal{O}^{N_p^-}_{p^{-2t}C(n)\epsilon}\mathcal{O}^{M_p}_{C(n)\epsilon} \cdot a\mathcal{O}^{N_p}_{C(n)\epsilon} n.$$

By letting $C(a) = \max\left\{p^{-2t}, 1\right\}$ we have

$$\subseteq q\mathcal{O}^{N_p^-}_{C(a)C(n)\epsilon}\mathcal{O}^{M_p}_{C(a)C(n)\epsilon} \cdot a\mathcal{O}^{N_p}_{C(n)\epsilon} n.$$

and then by Lemma 5.4

$$\subseteq q\mathcal{O}^{Q_p}_{C(a)C(n)\epsilon}\mathcal{O}^{N_p}_{C(a)C(n)\epsilon} a\mathcal{O}^{N_p}_{C(n)\epsilon} n.$$

Finally, by (8)

$$\begin{aligned}
&= q\mathcal{O}^{Q_p}_{C(a)C(n)\epsilon} a \cdot a^{-1}\mathcal{O}^{N_p}_{C(a)C(n)\epsilon} a\mathcal{O}^{N_p}_{C(n)\epsilon} n \\
&= q\mathcal{O}^{Q_p}_{C(a)C(n)\epsilon} a\mathcal{O}^{N_p}_{p^{-2k}C(a)C(n)\epsilon}\mathcal{O}^{N_p}_{C(n)\epsilon} n \\
&\subseteq q\mathcal{O}^{Q_p}_{C(a)^2C(n)\epsilon} a\mathcal{O}^{N_p}_{C(a)^2C(n)\epsilon} n.
\end{aligned}$$

Combining the effect of both left and right perturbations, we obtain that

$$\mathcal{O}_{\epsilon}^{G_p} qan \mathcal{O}_{\epsilon}^{G_p} \in q \mathcal{O}_{c(a,n)\cdot\epsilon}^{Q_p} a \mathcal{O}_{c(a,n)\cdot\epsilon}^{N_p} n$$

where $c(a,n) = C(a)^2 C(n)$ is a power of p (since $C(a)$ and $C(n)$ are) that is bounded when t is bounded from below and n is restricted to a bounded set. Require that $\epsilon < c(a,n)^{-1}$ to obtain that $c(a,n)\epsilon \in p^{-\mathbb{N}}$. $\square$

We can now prove Proposition 3.7. The proof essentially relies on the ultrametric nature of $\mathbb{Q}_p$: a small enough perturbation of a ball is the ball itself. The first claim (9) is the translation of this phenomenon in our setting.

Proof of Proposition 3.7. — We first claim that for $N \geq 0$ large enough and $c\epsilon \leq p^{-N}$ we have:

$$\begin{cases} \mathcal{O}_{c\epsilon}^{N_p} \cdot (N_p)_{\alpha+p^{\psi}\mathbb{Z}_p} \subseteq (N_p)_{\alpha+p^{\psi}\mathbb{Z}_p} & \text{and } (N_p)_{\alpha+p^{\psi}\mathbb{Z}_p} \cdot \mathcal{O}_{c\epsilon}^{N_p} \subseteq (N_p)_{\alpha+p^{\psi}\mathbb{Z}_p} \\ \mathcal{O}_{c\epsilon}^{Q_p} (Q_p)_{\Theta_p} \subseteq (Q_p)_{\Theta_p} & \text{and } (Q_p)_{\Theta_p} \cdot \mathcal{O}_{c\epsilon}^{Q_p} \subseteq (Q_p)_{\Theta_p}. \end{cases} \tag{9}$$

The inclusions in the first row are a trivial computation. For the inclusions in the second row, write $\Theta_p = \Theta_p\left(\check{v}, p^k\right)$ where $k \geq 0$ and $\check{v} \in \mathbb{S}_p^{1,+}$. Let $N \geq 0$ such that $\mathcal{O}_{c\epsilon} = \mathcal{O}_{p^N}$, and assume that $N \geq k$. Observe that

$$\mathcal{O}_{c\epsilon}^{Q_p} = \mathcal{O}_{p^N}^{Q_p} = \left\{ \begin{pmatrix} 1 + p^N\mathbb{Z}_p & 0 \\ p^N\mathbb{Z}_p & * \end{pmatrix} \right\}.$$

By letting $\check{v} = \binom{u_1}{p^{\ell}u_2} \in \mathbb{S}_p^{1,+}$ where $u_1, u_2 \in \mathbb{Z}_p^{\times}$ and $\ell \geq 0$, then

$$(Q_p)_{\Theta_p(\check{v},p^k)} = \left\{ \left(\check{v} + p^k\mathbb{Z}_p^2 \quad 0 \right) \right\} = \left\{ \begin{pmatrix} u_1 + p^k\mathbb{Z}_p & 0 \\ p^{\ell}u_2 + p^k\mathbb{Z}_p & * \end{pmatrix} \right\}.$$

Take $\begin{pmatrix} u_1+p^k\alpha & 0 \\ p^{\ell}u_2+p^k\beta & (u_1+p^k\alpha)^{-1} \end{pmatrix} \in (Q_p)_{\Theta_p(\check{v},p^k)}$ and $\begin{pmatrix} 1+p^N\gamma & 0 \\ p^N\delta & (1+p^N\gamma)^{-1} \end{pmatrix} \in \mathcal{O}_{p^{-N}}^{Q_p}$ (here $\alpha, \beta, \gamma \in \mathbb{Z}_p$). Now,

$$\begin{pmatrix} 1 + p^N\gamma & 0 \\ p^N\delta & \left(1 + p^N\gamma\right)^{-1} \end{pmatrix} \cdot \begin{pmatrix} u_1 + p^k\alpha & 0 \\ p^{\ell}u_2 + p^k\beta & \left(u_1 + p^k\alpha\right)^{-1} \end{pmatrix} \overset{N\geq k}{\in} \left\{ \begin{pmatrix} u_1 + p^k\mathbb{Z}_p & 0 \\ p^{\ell}u_2 + p^k\mathbb{Z}_p & * \end{pmatrix} \right\} = (Q_p)_{\Theta_p(\check{v},p^k)},$$

$$\begin{pmatrix} u_1 + p^k\alpha & 0 \\ p^{\ell}u_2 + p^k\beta & \left(u_1 + p^k\alpha\right)^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 + p^N\gamma & 0 \\ p^N\delta & \left(1 + p^N\gamma\right)^{-1} \end{pmatrix} \overset{N\geq k}{\in} \begin{pmatrix} u_1 + p^k\mathbb{Z}_p & 0 \\ p^{\ell}u_2 + p^k\mathbb{Z}_p & * \end{pmatrix} = (Q_p)_{\Theta_p(\check{v},p^k)}.$$

Having proved the inclusions in (9), the statement of Proposition 3.7 follows: according to Proposition 5.1, when $g_p = qan \in G_p^+$ lies in $Q_{\Theta_p^+} A_{t_1,t_2} N_{\alpha+p^{\psi}\mathbb{Z}_p}$, then

$$\mathcal{O}_{\epsilon}^{G_p} qan \mathcal{O}_{\epsilon}^{G_p} \subseteq q \mathcal{O}_{c\epsilon}^{Q_p} a \mathcal{O}_{c\epsilon}^{N_p} n$$

(where $c = c(a,n)$); but then according to (9), this is contained in $Q_{\Theta_p^+} A_{t_1,t_2} N_{\alpha+p^\psi \mathbb{Z}_p}$. Thus

$$\left(Q_{\Theta_p^+} A_{t_1,t_2} N_{\alpha+p^\psi \mathbb{Z}_p}\right)^{+\epsilon} \subseteq Q_{\Theta_p^+} A_{t_1,t_2} N_{\alpha+p^\psi \mathbb{Z}_p},$$

and the opposite inclusion is obvious. Similarly,

$$Q_{\Theta_p^+} A_{t_1,t_2} N_{\alpha+p^\psi \mathbb{Z}_p} \subseteq \left(Q_{\Theta_p^+} A_{t_1,t_2} N_{\alpha+p^\psi \mathbb{Z}_p}\right)^{-\epsilon}$$

and the opposite inclusion is obvious. Then

$$\left(Q_{\Theta_p^+} A_{t_1,t_2} N_{\alpha+p^\psi \mathbb{Z}_p}\right)^{-\epsilon} = Q_{\Theta_p^+} A_{t_1,t_2} N_{\alpha+p^\psi \mathbb{Z}_p} = \left(Q_{\Theta_p^+} A_{t_1,t_2} N_{\alpha+p^\psi \mathbb{Z}_p}\right)^{+\epsilon},$$

meaning that the well-roundedness condition holds trivially. □

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