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CORPS DE NOMBRES CUBIQUES CYCLIQUES AYANT UNE CAPITULATION HARMONIEUSEMENT ÉQUILIBRÉE

by

Bill Allombert and Daniel C. Mayer

Abstract. — It is proved that $c = 689\,347 = 31 \cdot 37 \cdot 601$ is the smallest conductor of a cyclic cubic number field K whose maximal unramified pro-3-extension $E = \mathrm{F}_3^{\infty}(K)$ possesses an automorphism group $G = \mathrm{Gal}(E/K)$ of order 6561 with coinciding relation and generator rank $d_2(G) = d_1(G) = 3$ and harmonically balanced transfer kernels $\varkappa(G) \in S_{13}$. The result depends on computations done under the assumption of the GRH.

Résumé. — Nous établissons que $c = 689\,347 = 31 \cdot 37 \cdot 601$ est le plus petit conducteur d'un corps cubique cyclique K dont la pro-3-extension maximale non-ramifiée $E = \mathrm{F}_3^{\infty}(K)$ admet un groupe d'automorphismes $G = \mathrm{Gal}(E/K)$ d'ordre 6561, avec égalité du rang des relations et des générateurs $d_2(G) = d_1(G) = 3$, et des noyaux de transfert harmonieusement équilibrés $\varkappa(G) \in S_{13}$. Le résultat dépend de calculs fait sous l'hypothèse de Riemann généralisée.

1. Introduction

Let p be a prime number. Finite p-groups G with *balanced presentation*, that is, with relation rank $d_2(G)$ equal to the generator rank $d_1(G)$,

$$d_2(G) = \dim_{\mathbb{F}_p} \mathrm{H}^2(G, \mathbb{F}_p) = \dim_{\mathbb{F}_p} \mathrm{H}^1(G, \mathbb{F}_p) = d_1(G), \tag{1}$$

have attracted the vigilance and interest of researchers since the beginning. Issai Schur dubbed such groups *closed*, but today it is more usual to call them *Schur groups*.

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In 1964, one year before his famous joint paper with Golod on infinite class field towers, Shafarevich [42] established an application of closed groups of the greatest importance and with fundamental impact on class field theory. Let K be an algebraic number field with p-class rank $\varrho = \mathrm{rank}_p \mathrm{Cl}(K)$, signature (r_1, r_2), and torsion free Dirichlet unit rank $r = r_1 + r_2 - 1$. Let $\theta \in \{0, 1\}$ be an indicator for the existence of a primitive p-th root of unity $\zeta = \exp(2\pi\sqrt{-1}/p)$ in K. Then the relation rank $d_2(G)$ of the Galois group $G = \mathrm{Gal}(\mathrm{F}_p^\infty(K)/K)$ of the maximal unramified pro-p-extension $\mathrm{F}_p^\infty(K)$ of K is bounded

$$\varrho \le d_2(G) \le \varrho + r + \theta, \tag{2}$$

according to Shafarevich [27, Thm. 5.1, p. 28]. In particular, imaginary quadratic fields $k = \mathbb{Q}(\sqrt{d})$ with negative discriminant $d < 0$, which have the simplest possible signature $(0, 1)$ (except $(1, 0)$ for the rational number field $\mathbb{Q}$) require a *Schur σ-group* [4] (with balanced presentation and a generator- and relator-inverting automorphism σ) $\mathrm{Gal}(\mathrm{F}_p^\infty(k)/k)$ for all p-class towers with an odd prime $p \ge 3$. Shafarevich himself immediately drew some conclusions [42, pp. 91–92] about the few non-abelian 3-class towers for $d = -4027$ and $d = -3299$ which were known at this early stage, due to Scholz and Taussky [40]. Unfortunately, his interpretations of these 3-class towers for 3-class groups with abelian type invariants $(3, 3)$ and $(9, 3)$ were incorrect (his claimed groups were only Schur but not Schur σ), and his important theorem on the bounds for $d_2(G)$ remained largely unnoticed for about thirty years.

In 1975, several events happened precipitately. On the one hand, Andozhskii and Tsvetkov [1, 2] found the first closed finite 3-groups G with elementary tricyclic commutator quotient $G/G' \simeq (3, 3, 3)$. But on the other hand, Koch and Venkov [21] proved that, for odd primes $p \ge 3$, an imaginary quadratic number field $k = \mathbb{Q}(\sqrt{d})$ with p-class group $\mathrm{Cl}_p(k)$ of p-rank $\varrho \ge 3$ possesses an unbounded p-class field tower. In particular, for $\mathrm{Cl}_3(k) \simeq (3, 3, 3)$ the Galois group $\mathrm{Gal}(F_3^\infty(k)/k)$ of the 3-class tower must be an infinite Schur σ-group and cannot be one of the Andozhskii–Tsvetkov groups (briefly *AT-groups*), which are finite and do not possess a σ-automorphism.

This is exactly the point where our present article sets in. After preparatory Sections 2, 3, 4, 5 with algebraic and arithmetic foundations, we investigate AT-groups G more closely in Section 6.1. We determine their order $\#G = 3^8 = 6561$ and position in the descendant tree [28], and we discover with surprise that all of them possess *harmonically balanced capitulation* (HBC), that is, their transfer kernels $\varkappa(G) = (\ker(V_j))_{j=1}^{13}$ are cyclic of order 3 and form a permutation in the symmetric group S_{13} of degree thirteen.

Then we try to realize AT-groups G as automorphism groups $G \simeq \mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$ of maximal unramified pro-3-extensions of suitable algebraic number fields K, necessarily different from imaginary quadratic fields, by the result of Koch and Venkov. Since real quadratic fields of type $(3, 3, 3)$ are firstly very sparse and secondly also require a σ-group with generator- and relator-inverting automorphism (though with looser bounds $d_1(G) \le d_2(G) \le d_1(G) + 1$, due to the signature $(2, 0)$), they disqualify for the realization of AT-groups. In Section 6.2, we employ the next simplest number fields with signature $(3, 0)$ and unit rank $r = 2$, but still absolutely Galois over $\mathbb{Q}$, that is, the cyclic cubic fields K, which quite frequently have elementary tricyclic 3-class groups $\mathrm{Cl}_3(K) \simeq (3, 3, 3)$, provided their conductor c is divisible by three or four primes, whence they arise as quartets $(K_1, \ldots, K_4)$ or octets $(K_1, \ldots, K_8)$ sharing a common discriminant $d = c^2$. This is known from earlier works by G. Gras, 1973 [18], who determined the 3-class rank in dependence on Graphs which describe cubic residue

conditions between the prime divisors of c, and by M. Ayadi, 1995, 2001 [7, 8], who introduced a classification of quartets into Categories.
Indeed, we find a rather sparse sequence of conductors $c = q_1q_2q_3$, exclusively with Graph 2 of Category I, $q_2 \leftarrow q_1 \rightarrow q_3$, in the sense of G. Gras and Ayadi, for which precisely one component $K := K_1$ has $\mathrm{Cl}_3(K) \simeq (3,3,3)$ and HBC, whereas the other three components K_2, K_3, K_4 of the quartet have elementary bicyclic $\mathrm{Cl}_3(K_i) \simeq (3,3)$ and $\varkappa(K_i) = (1243)$, called capitulation type G.16 [3, §6.2, Thm. 11]. It is published as sequence A359310 in the On-line Encyclopedia of Integer Sequences (OEIS).
A drawback of cyclic cubic fields K is the broad spread for the relation rank $d_2(G)$ of the 3-class tower group $G = \mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$,

$$3 = \varrho = d_1(G) \leq d_2(G) \leq \varrho + r + \theta = 3 + 2 + 0 = 5, \tag{3}$$

which enables the occurrence of numerous smaller groups G of order $\#G = 3^6 = 729$ with $d_2(G) = 5$ as parents of AT-groups, or of order $\#G = 3^7 = 2187$ with $d_2(G) = 4$ as siblings of AT-groups, instead of the desired minimal relation rank $d_2(G) = 3$. We had to wait for the 25-th term $c = \mathbf{689\,347}$ of the OEIS sequence A359310 [43] until a proper AT-group occurred, which we only found after several months of computations running on multiple high-end computing node.

2. Group theoretic foundations

The original motivation for the present paper was the question whether there exist finite 3-groups with three generators $G = \langle x, y, z\rangle$ and the following cohomological property.

Definition 2.1. — Let p be an odd prime number. A finite p-group G is called *closed*, or a *Schur group*, if its relation rank $d_2(G) = \dim_{\mathbb{F}_p} \mathrm{H}^2(G, \mathbb{F}_p)$ coincides with its generator rank $d_1(G) = \dim_{\mathbb{F}_p} \mathrm{H}^1(G, \mathbb{F}_p)$. A closed p-group is also said to possess a *balanced presentation*.

The relation rank $d_2(G)$ is essential for the clarification whether G can occur as the Galois group $\mathrm{Gal}(\mathrm{F}_p^\infty(K)/K)$ of the maximal unramified pro-p-extension of an algebraic number field K.
With respect to the action of automorphisms, further kinds of p-groups are distiguished.

Definition 2.2. — For an odd prime p, a finite p-group G is called a *σ-group*, if it admits an automorphism $\sigma \in \mathrm{Aut}(G)$ which acts as inversion $x \mapsto x^\sigma = x^{-1}$ on the first and second cohomology group, $\mathrm{H}^1(G, \mathbb{F}_p)$ and $\mathrm{H}^2(G, \mathbb{F}_p)$. A closed σ-group is called *Schur σ-group*.

A σ-group is also said to possess a generator- and relator-inverting (briefly, a GI- and RI-)automorphism. The action on $\mathrm{H}^1(G, \mathbb{F}_p)$ is essentially equivalent with the action on the commutator quotient G/G'. A similar concept is expressed in terms of the action of an entire subgroup of the automorphism group.

Definition 2.3. — Let $\Phi(G)$ be the Frattini subgroup, that is, the meet of all maximal subgroups of G, of a finite group G, then G is said to possess an *operation* by some finite group S if S is a subgroup of the automorphism group $\mathrm{Aut}(G/\Phi(G))$ of the Frattini quotient.

The operation is important to decide whether a given p-group is admissible as Galois group $\mathrm{Gal}(\mathrm{F}_p^n(K)/K)$ of some stage $\mathrm{F}_p^n(K)$ in the Hilbert p-class field tower of a number field K.

Since the intention of the present paper is the arithmetic realization of Andozhskii–Tsvetkov groups G as Galois groups $G \simeq \mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$ of maximal unramified pro-3-extensions of suitable number fields K, some collections of invariants are required which enable the unambiguous identification, and can be translated from group theory to number theory, and vice versa.
Let G be a finite 3-group with an elementary tricyclic commutator quotient $G/G' \simeq (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})$. This quotient is also called the abelianization G^{ab} of G and the elementary tricyclic 3-group is abreviated by $(3,3,3)$. Then, G has 13 maximal subgroups $H_1, \ldots, H_{13}$ with Artin transfer homomorphisms (Verlagerungen) $V_i : G/G' \to H_i/H_i'$, defined in [28, Def. 3.3, p. 69].

Definition 2.4. — The family of abelian quotient invariants (AQI) $\alpha(G) = (H_i/H_i')_{i=1}^{13}$ is called the *transfer target type* (TTT), the family of transfer kernels $\varkappa(G) = (\ker(V_i))_{i=1}^{13}$ is called the *transfer kernel type* (TKT) of the group G. Both combined form the *Artin pattern* $\mathrm{AP}(G) = (\alpha(G), \varkappa(G))$ of G. The *rank distribution* of G is the family $\rho(G) = (\mathrm{rank}_3(H_i/H_i'))_{i=1}^{13}$, where $\mathrm{rank}_3(X) = \dim_{\mathbb{F}_3}(X/X^3)$, for any finite 3-group X. Summarized:

$$\alpha(G) = (H_i/H_i')_{1\le i\le 13}, \quad \varkappa(G) = (\ker(V_i))_{1\le i\le 13}, \quad \rho(G) = (\mathrm{rank}_3(H_i/H_i'))_{1\le i\le 13}. \tag{4}$$

Frequently, the Artin pattern, or even only a part of it (α alone or $\varkappa$ alone), is able to identify a unique finite 3-group. Unfortunately, this is not the case for Andozhskii–Tsvetkov groups, where a database query yields a batch of a dozen possible groups with several distinct orders. Therefore, a set of more subtle invariants is necessary to reduce the number of hits. The drawback is the requirement to compute class groups for multiple fields of degree 27, requiring the use of HPC-class computer systems.
The following definition restricts to the special situation investigated in the present paper, namely rank distributions $\rho(G) \in \{(3^1, 2^{12}), (3^4, 2^9), (3^7, 2^6)\}$.

Definition 2.5. — For each $1 \le i \le 13$, let $H_{i,1}, \ldots, H_{i,n_i}$ be the maximal subgroups of the maximal subgroup H_i of G. Here, $n_i = 13$ if $\mathrm{rank}_3(H_i/H_i') = 3$, and $n_i = 4$ if $\mathrm{rank}_3(H_i/H_i') = 2$. The components of the family

$$\alpha_2(G) = ((H_{i,j}/H_{i,j}')_{1\le j\le n_i})_{1\le i\le 13} \tag{5}$$

are called *abelian quotient invariants of second order* (AQI2) of G.

Usually, the components of $\alpha(G)$ and $\alpha_2(G)$ are written in logarithmic form (with respect to the basis 3), for instance, $(22) \mathrel{\widehat{=}} (9,9)$ and $(211) \mathrel{\widehat{=}} (9,3,3)$.
Throughout the sequel, each finite group of small prime power or composite order is characterized with its unique designation by the pair $\langle \mathrm{order}, \mathrm{identifier}\rangle$ in angle brackets, taken from the SmallGroups database [11, 12], for instance $\langle 27, 5\rangle$ in the following section.

3. The elementary abelian 3-group of rank three

This group $\langle 27, 5\rangle$ can be viewed as a *vector space* O of dimension $\dim_{\mathbb{F}_3}(O) = 3$ over the finite field $\mathbb{F}_3$ with three elements. The vector space O possesses $13 = 3^2 + 3 + 1$ *lines*, that is, subgroups L_i of index $(O : L_i) = 3^2$, and 13 *planes*, that is, subgroups P_i of index $(O : P_i) = 3$, where $1 \le i \le 13$. Let x, y, z be fixed generators of $O = \langle x, y, z\rangle$, then the generators of the lines $L_i = \langle g_i \rangle$ will be arranged in the way shown in Table 1.

Identifiers for the planes $P_i = \langle h_i, k_i\rangle$ are introduced as shown in Table 2. The elements h_i, k_i can be viewed as generators of a transversal of the line $L_i = \langle g_i\rangle$, i.e., a system of coset representatives for L_i in O. Each set S_i contains the subscripts j of generators g_j contained in P_i.
In Table 3, it is also useful to list the *bundles* $\mathcal{B}_i$, of four planes each, containing an assigned line L_i, where we simply denote $L_i = \langle g_i\rangle$ by its subscript i, for the sake of brevity.

Table 1. Generators of thirteen lines L_i in O

i	1	2	3	4	5	6	7	8	9	10	11	12	13
g_i	x	y	z	xy	yz	zx	xy^2	yz^2	zx^2	x^2yz	xy^2z	xyz^2	xyz

Table 2. Identifiers and generators of thirteen planes P_i in O

i	1	2	3	4	5	6	
h_i	y	z	x	z	x	y	
k_i	z	x	y	xy^2	yz^2	zx^2	
S_i	$2,3,5,8$	$1,3,6,9$	$1,2,4,7$	$3,7,11,10$	$1,8,11,12$	$2,9,10,12$	
i	7	8	9	10	11	12	13
h_i	z	x	y	xy	yz	xy^2	zx
k_i	xy	yz	zx	yz	zx	yz^2	xy
S_i	$3,4,12,13$	$1,5,10,13$	$2,6,11,13$	$4,5,9,11$	$5,6,7,12$	$7,8,9,13$	$4,6,8,10$

Table 3. Thirteen bundles $\mathcal{B}_i$ of planes in the space O

i	1	2	3	4	5
$\mathcal{B}_i$	P_2, P_3, P_5, P_8	P_1, P_3, P_6, P_9	P_1, P_2, P_4, P_7	P_3, P_7, P_{10}, P_{13}	P_1, P_8, P_{10}, P_{11}
i	6	7	8	9	10
$\mathcal{B}_i$	P_2, P_9, P_{11}, P_{13}	P_3, P_4, P_{11}, P_{12}	P_1, P_5, P_{12}, P_{13}	P_2, P_6, P_{10}, P_{12}	P_4, P_6, P_8, P_{13}
i	11	12	13		
$\mathcal{B}_i$	P_4, P_5, P_9, P_{10}	P_5, P_6, P_7, P_{11}	P_7, P_8, P_9, P_{12}		

Remark 3.1. — For any finite 3-group G with $G/G' \simeq (3,3,3)$ elementary tricyclic, our implementation of the Artin transfer homomorphisms $V_i : G/G' \to H_i/H_i'$ from G to its 13 maximal subgroups $H_1, \ldots, H_{13}$ uses the interpretation of the commutator quotient G/G' as vector space O of dimension 3 over $\mathbb{F}_3$, of the 13 planes $P_i = \langle h_i, k_i\rangle$ as quotients H_i/G', of the 13 lines $L_i = \langle g_i\rangle$ as mutual transversals in pairs (P_i, L_i), the *outer transfer* mapping $g_i \cdot G' \mapsto g_i^3 \cdot H_i'$, and the *inner transfer* mappings $h_i \cdot G' \mapsto (h_i \cdot g_i^{-1})^3 \cdot g_i^3 \cdot H_i'$, $k_i \cdot G' \mapsto (k_i \cdot g_i^{-1})^3 \cdot g_i^3 \cdot H_i'$. This mapping law requires the artificial definition $P_{13} = \langle zx, xy\rangle$, $P_{12} = \langle xy^2, yz^2\rangle$, although a more natural definition would be $P_{12} = \langle zx, xy\rangle$, in view of $P_{10} = \langle xy, yz\rangle$, $P_{11} = \langle yz, zx\rangle$. The unnnatural definition is mandatory for the following reason in Lemma 3.2.

Lemma 3.2. — *The permutation automorphism $x \mapsto y \mapsto z \mapsto x$ has two fixed points, line $L_{13} = \langle xyz\rangle$ and plane $P_{12} = \langle xy^2, yz^2\rangle$, which do not form a pair of mutual transversals.*

Proof. — The permutation automorphism $x \mapsto y \mapsto z \mapsto x$ in form of a 3-cycle generates five orbits among each, the 13 lines $L_1, \ldots, L_{13}$, and the 13 planes $P_1, \ldots, P_{13}$. Four orbits consist of three elements. Those with subscripts $\{1,2,3\}$, $\{4,5,6\}$, $\{7,8,9\}$ form pairs of mutual transversals, in a natural way. However, a single orbit consists of a fixed point: the fixed line is $L_{13} = \langle g_{13} \rangle$ with $g_{13} = xyz$, the fixed plane is $P_{12} = \langle h_{12}, k_{12} \rangle$ with $h_{12} = xy^2$, $k_{12} = yz^2$. These fixed points do not form a pair of mutual transversals, since

$$h_{12} \cdot k_{12}^2 = xy^2 \cdot (yz^2)^2 = xy^2 \cdot y^2 z = xyz = g_{13} \quad \text{(exponents modulo 3)}.$$

Whereas the lines $L_{10} \mapsto L_{11} \mapsto L_{12} \mapsto L_{10}$ permute in the natural manner, the planes $P_{10} \mapsto P_{11} \mapsto P_{13} \mapsto P_{10}$ permute irregularly. □

4. Number theoretic foundations

As outlined in Section 2, the collections of invariants required for the unambiguous identification of Galois groups must now be translated from group theory to number theory by means of the Artin reciprocity law [5, 6], which is explained thoroughly by Miyake [35].
Let K be an algebraic number field with elementary tricyclic 3-class group $\mathrm{Cl}_3(K) = \mathrm{Syl}_3\mathrm{Cl}(K) \simeq (3,3,3)$. (Since the class group $\mathrm{Cl}(K) = \mathcal{I}_K/\mathcal{P}_K$ is the quotient of the commutative group $\mathcal{I}_K$ of fractional ideals of the maximal order $\mathcal{O}_K$ of K by the subgroup $\mathcal{P}_K$ of principal ideals, it is abelian and has a unique Sylow 3-subgroup.) Then, K has $13 = \frac{3^3-1}{3-1}$ unramified abelian extensions E_i/K of relative degree $[E_i : K] = 3$ with transfers (class extension homomorphisms) $T_i : \mathrm{Cl}_3(K) \to \mathrm{Cl}_3(E_i)$, $\mathfrak{a}\mathcal{P}_K \mapsto (\mathfrak{a}\mathcal{O}_{E_i})\mathcal{P}_{E_i}$.

Definition 4.1. — The family of abelian type invariants (ATI) of the 3-class groups $\alpha(K) = (\mathrm{Cl}_3(E_i))_{i=1}^{13}$ is called the *transfer target type* (TTT), and the family of the transfer kernels (capitulation kernels) $\varkappa(K) = (\ker(T_i))_{i=1}^{13}$ is called the *transfer kernel type* (TKT) or *capitulation type* of the field K. The pair $\mathrm{AP}(K) = (\alpha(K), \varkappa(K))$ is called the *Artin pattern* of K. The *rank distribution* of K is the family $\rho(K) = (\mathrm{rank}_3(\mathrm{Cl}_3(E_i)))_{i=1}^{13}$, where $\mathrm{rank}_3(X) = \dim_{\mathbb{F}_3}(X/X^3)$, for any finite 3-group X. Summarized:

$$(6) \qquad \alpha(K) = (\mathrm{Cl}_3(E_i))_{1\le i\le 13},\ \varkappa(K) = (\ker(T_i))_{1\le i\le 13},\ \rho(K) = (\mathrm{rank}_3(\mathrm{Cl}_3(E_i)))_{1\le i\le 13}.$$

The capitulation is called *harmonically balanced* (HBC) if each transfer kernel is a line $\ker(T_i) = L_{\pi(i)}$ in the space $O = \mathrm{Cl}_3(K)$, and $\pi \in S_{13}$ is a permutation in the symmetric group of degree 13.

If $G = \mathrm{Gal}(\mathrm{F}_3^2(K)/K)$ denotes the second 3-class group of K [24], then the translation of invariants between group theory and number theory is performed by $\alpha(G) = \alpha(K)$ and $\varkappa(G) = \varkappa(K)$, according to [25, §2.3, pp. 476–478] and Miyake [35].

Definition 4.2. — For each $1 \le i \le 13$, let $E_{i,1}, \ldots, E_{i,n_i}$ be the unramified cyclic cubic relative extensions of the extension E_i of K. Here, $n_i = 13$ if $\mathrm{rank}_3(\mathrm{Cl}_3(E_i)) = 3$, and $n_i = 4$ if $\mathrm{rank}_3(\mathrm{Cl}_3(E_i)) = 2$. The components of the family

$$(7) \qquad \alpha_2(K) = ((\mathrm{Cl}_3(E_{i,j}))_{1\le j\le n_i})_{1\le i\le 13}$$

are called *abelian type invariants of second order* (ATI2) of K.

Again, the Artin reciprocity law makes sure that $\alpha_2(G) = \alpha_2(K)$ for $G = \mathrm{Gal}(\mathrm{F}_3^2(K)/K)$. Usually, the components of $\alpha(K)$ and $\alpha_2(K)$ are written in logarithmic form (with respect to the basis 3), for instance, $(22) \widehat{=} (9,9)$ and $(211) \widehat{=} (9,3,3)$.

5. Quartets of cyclic cubic fields

The following details supplement our main results on cyclic cubic fields K with HBC in Section 6.2. In Section 6.5 they will be mandatory for the correct selection of representatives R_i in isomorphism classes among the unramified cyclic cubic relative extensions E_j/K, $1 \le j \le 13$, and among the unramified nonic but not necessarily Galois extensions $E_{j,\ell}/K$, $1 \le j \le 13$, $1 \le \ell \le n_j$, $n_j \in \{4, 13, 40\}$, of absolute degree 27, for a 3-class rank $\varrho_j \in \{2, 3, 4\}$ of E_j, respectively.

Theorem 5.1. — *Let K be a cyclic cubic number field with conductor $c = q_1q_2q_3$ divisible by exactly three distinct prime(power)s, $q_i \equiv +1 \pmod 3$, or $q_i = 3^2$. Then*

1. *K is member of a quartet $(K_1, \ldots, K_4)$ of four cyclic cubic fields sharing the common conductor c and the common discriminant $d = c^2$.*

2. *The absolute genus field $K^* = (K/\mathbb{Q})^*$ of K is unramified over K and abelian over $\mathbb{Q}$. More precisely, its Galois group $\mathrm{Gal}(K^*/\mathbb{Q}) \simeq (\mathbb{Z}/3\mathbb{Z})^3$ is elementary tricyclic. Its absolute degree is $[K^* : \mathbb{Q}] = 27$ and the relative degree is $[K^* : K] = 9$.*

3. *K^* contains 13 cyclic cubic subfields, three $k_{q_1}, k_{q_2}, k_{q_3}$ with prime(power) conductors, six (in three doublets) $k_{q_1q_2}, \widetilde{k}_{q_1q_2}, k_{q_1q_3}, \widetilde{k}_{q_1q_3}, k_{q_2q_3}, \widetilde{k}_{q_2q_3}$ with conductors divisible by two prime(power)s, and the four members $K_1, \ldots, K_4$ of the abovementioned quartet with conductor c.*

4. *The composita $L := k_{q_1q_2}k_{q_1q_3}k_{q_2q_3}$ and $\widetilde{L} := \widetilde{k}_{q_1q_2}\widetilde{k}_{q_1q_3}\widetilde{k}_{q_2q_3}$ satisfy the following **skew balance of degrees**: $[L : \mathbb{Q}] \cdot [\widetilde{L} : \mathbb{Q}] = 243$, with*

$$[L : \mathbb{Q}] = 9 \Longleftrightarrow [\widetilde{L} : \mathbb{Q}] = 27, \tag{8}$$

or vice versa.

Proof. — See [7, §4.1, p. 40, Proof of Prop. 4.6, p. 49, Prop. 4.1, p. 40]. □

Definition 5.2. — The selection of cyclic cubic subfields $k_{q_1q_2}, k_{q_1q_3}, k_{q_2q_3}$ with conductors q_1q_2, q_1q_3, q_2q_3 within the absolute genus field K^* of an assigned cyclic cubic field K with conductor $c = q_1q_2q_3$ is called *normalized*, if the absolute degree of their compositum $L = k_{q_1q_2}k_{q_1q_3}k_{q_2q_3}$ is $[L : \mathbb{Q}] = 9$. In this article, we always assume this normalization.

The following theorem corresponds to [3, §4.3, Thm. 8, Eqn. (22)–(24)].

Theorem 5.3. — *Under the assumptions of Theorem 5.1 and the mandatory normalization of* $k_{q_1q_2}, k_{q_1q_3}, k_{q_2q_3}$ *according to Definition 5.2, the remaining* 13 *bicyclic bicubic subfields* B_j*,* $1 \le j \le 13$*, of the absolute genus field* K^* *of* K *are given as composita by*

4 ***single capitulation targets***

$$(9)\quad \begin{aligned} B_1 &:= k_{q_1q_2}k_{q_1q_3} = K_1k_{q_1q_2}k_{q_1q_3}k_{q_2q_3},\\ B_2 &:= \widetilde{k}_{q_1q_3}\widetilde{k}_{q_2q_3} = K_2k_{q_1q_2}\widetilde{k}_{q_1q_3}\widetilde{k}_{q_2q_3},\\ B_3 &:= \widetilde{k}_{q_1q_2}\widetilde{k}_{q_1q_3} = K_3\widetilde{k}_{q_1q_2}\widetilde{k}_{q_1q_3}k_{q_2q_3},\\ B_4 &:= \widetilde{k}_{q_1q_2}\widetilde{k}_{q_2q_3} = K_4\widetilde{k}_{q_1q_2}k_{q_1q_3}\widetilde{k}_{q_2q_3}, \end{aligned}$$

6 ***double capitulation targets***

$$(10)\quad \begin{aligned} B_5 &:= k_{q_1}\widetilde{k}_{q_2q_3} = K_1K_3k_{q_1}\widetilde{k}_{q_2q_3},\\ B_6 &:= k_{q_2}\widetilde{k}_{q_1q_3} = K_1K_4k_{q_2}\widetilde{k}_{q_1q_3},\\ B_7 &:= k_{q_3}\widetilde{k}_{q_1q_2} = K_1K_2k_{q_3}\widetilde{k}_{q_1q_2},\\ B_8 &:= k_{q_1}k_{q_2q_3} = K_2K_4k_{q_1}k_{q_2q_3},\\ B_9 &:= k_{q_2}k_{q_1q_3} = K_2K_3k_{q_2}k_{q_1q_3},\\ B_{10} &:= k_{q_3}k_{q_1q_2} = K_3K_4k_{q_3}k_{q_1q_2}, \end{aligned}$$

and 3 ***sub genus fields***

$$(11)\quad \begin{aligned} B_{11} &:= k_{q_1q_2}\widetilde{k}_{q_1q_2} = k_{q_1}k_{q_2}k_{q_1q_2}\widetilde{k}_{q_1q_2},\\ B_{12} &:= k_{q_1q_3}\widetilde{k}_{q_1q_3} = k_{q_1}k_{q_3}k_{q_1q_3}\widetilde{k}_{q_1q_3},\\ B_{13} &:= k_{q_2q_3}\widetilde{k}_{q_2q_3} = k_{q_2}k_{q_3}k_{q_2q_3}\widetilde{k}_{q_2q_3}. \end{aligned}$$

The shape with two components suffices for the construction, but the shape with four components ostensively illuminates all cyclic cubic subfields of each bicyclic bicubic field B_j*.*

Proof. — See Ayadi's Thesis [7, Lem. 4.1, p. 42, and Fig. 10, p. 41]. □

The following corollary corresponds to [3, §4.3, Cor. 2].

Corollary 5.4. — *For each member of the quartet* $(K_1, \dots, K_4)$ *of cyclic cubic fields with conductor* $c = q_1q_2q_3$*, the rank* ϱ_i *of the* 3*-class group* $\mathrm{Cl}_3(K_i)$ *is bounded by* $2 \le \varrho_i \le 4$*, and four unramified cyclic cubic relative extensions of* K_i *are given in the following way:*

$$(12)\quad \begin{aligned} &(B_1, B_5, B_6, B_7) \text{ for } K_1,\\ &(B_2, B_7, B_8, B_9) \text{ for } K_2,\\ &(B_3, B_5, B_9, B_{10}) \text{ for } K_3,\\ &(B_4, B_6, B_8, B_{10}) \text{ for } K_4. \end{aligned}$$

If the rank of the 3*-class group* $\mathrm{Cl}_3(K_i)$ *of* K_i *is* $\varrho_i = 2$*, then the set of unramified extensions given in Equation* (12) *is complete and consists entirely of absolutely abelian extensions.*

Proof. — This is an immediate consequence of the constitution of the B_j in Theorem 5.3. □

The following theorem is a special case of [3, §6.2, Prop. 6, Tbl. 6, and Thm. 11].

Theorem 5.5. — *Let* K *be a cyclic cubic number field with conductor* $c = q_1q_2q_3$ *divisible by precisely three distinct prime(power)s,* $q_i \equiv +1 \pmod 3$*, or* $q_i = 3^2$*, such that only two* *cubic residue symbols* $\left(\frac{q_1}{q_2}\right)_3 = 1$ *and* $\left(\frac{q_1}{q_3}\right)_3 = 1$ *are trivial, that is,* K *belongs to* ***Graph 2*** *of* ***Category I***, $q_2 \leftarrow q_1 \rightarrow q_3$, *in the sense of G. Gras and Ayadi. Then:*

1. *The* 3*-class group* $\mathrm{Cl}_3(K)$ *has either rank* $\varrho = 3$, *for a single component, or it is elementary bicyclic* $\mathrm{Cl}_3(K) \simeq (\mathbb{Z}/3\mathbb{Z})^2$, *with* $\varrho = 2$, *for three components of the quartet* $(K_1, \dots, K_4)$. *The former is* K_1 *if* q_2 *splits in* $k_{q_1q_3}$ *and* q_3 *splits in* $k_{q_1q_2}$.

2. *If* $\varrho = 2$, *then* q_1 *is the unique minimal norm of a non-trivial primitive ambiguous principal ideal of* K, *called* ***Parry invariant*** *of* K *by Ayadi* [39, pp. 499–501].

3. *If* q_2 *splits in* $k_{q_1q_3}$, q_3 *splits in* $k_{q_1q_2}$, *and* K_1 *possesses an elementary tricyclic* 3*-class group* $\mathrm{Cl}_3(K_1) \simeq (\mathbb{Z}/3\mathbb{Z})^3$ *with* ***HBC***, *then the Parry invariant of* K_1 *is also* q_1 *and the remaining three fields* K_2, K_3, K_4 *with* $\varrho = 2$ *share the common capitulation type* $\varkappa(K_\mu) \sim (1243)$ *with two fixed points* $1, 2$ *and a transposition* (43), *called type* G.16, *and their second* 3*-class group* $\mathrm{Gal}(\mathrm{F}_3^2(K_\mu)/K_\mu)$ *is either* $\langle 729, 52\rangle$ *or one among the six groups* $\langle 2187, i\rangle$ *with* $294 \le i \le 299$. *Their* 3*-class field tower has either two or three stages, in the latter case with automorphism group* $\langle 6561, j\rangle$, $2039 \le j \le 2044$.

Proof. — Denote by $G := \mathrm{Gal}(K/\mathbb{Q}) = \langle\sigma\rangle$ the cyclic Galois group of K. Among the prime ideals of K, let $\mathfrak{P} = \mathfrak{P}^\sigma$ divide q_1, $\mathfrak{Q} = \mathfrak{Q}^\sigma$ divide q_2, and $\mathfrak{R} = \mathfrak{R}^\sigma$ divide q_3. If the rank ϱ of the 3-class group $\mathrm{Cl}_3(K)$ is $\varrho = 2$, then $\mathrm{Cl}_3(K) \simeq (\mathbb{Z}/3\mathbb{Z})^2$ [7, Prop. 4.3, p. 43]. If $\varrho = 2$, then $\mathfrak{P}$ generates the group $\mathcal{P}_K^G/\mathcal{P}_\mathbb{Q}$ of primitive ambiguous principal ideals of K [7, Rem. 4.2, p. 50], whereas $\mathfrak{Q}$ and $\mathfrak{R}$ are not principal [7, Rem. 4.8, p. 65], and their ideal classes $[\mathfrak{Q}] = \mathfrak{Q}\cdot\mathcal{P}_K$ and $[\mathfrak{R}] = \mathfrak{R}\cdot\mathcal{P}_K$ generate $\mathrm{Cl}_3(K) = \langle[\mathfrak{Q}], [\mathfrak{R}]\rangle$. If q_2 splits in $k_{q_1q_3}$ and q_3 splits in $k_{q_1q_2}$, then K_1 is the field with $\varrho = 3$ [7, Prop. 4.4, pp. 43–44], and K_2, K_3, K_4 have elementary bicyclic 3-class groups. According to [7, Tbl., p. 66], the kernels of the transfers $T_{\mu\nu} : \mathrm{Cl}_3(K_\mu) \to \mathrm{Cl}_3(B_\nu)$ from K_μ, $2 \le \mu \le 4$, to its four unramified cyclic cubic extensions B_ν, given in Corollary 5.4, are as follows:

$$\begin{aligned}
&\ker(T_{22}) = \langle[\mathfrak{Q}\mathfrak{R}^2]\rangle,\ \ker(T_{27}) = \langle[\mathfrak{R}]\rangle,\ \ker(T_{28}) = \langle[\mathfrak{Q}\mathfrak{R}]\rangle,\ \ker(T_{29}) = \langle[\mathfrak{Q}]\rangle;\\
&\ker(T_{33}) = \langle[\mathfrak{Q}\mathfrak{R}]\rangle,\ \ker(T_{35}) = \langle[\mathfrak{Q}\mathfrak{R}^2]\rangle,\ \ker(T_{39}) = \langle[\mathfrak{Q}]\rangle,\ \ker(T_{3,10}) = \langle[\mathfrak{R}]\rangle;\\
&\ker(T_{44}) = \langle[\mathfrak{Q}\mathfrak{R}^2]\rangle,\ \ker(T_{46}) = \langle[\mathfrak{Q}]\rangle,\ \ker(T_{48}) = \langle[\mathfrak{Q}\mathfrak{R}]\rangle,\ \ker(T_{4,10}) = \langle[\mathfrak{R}]\rangle.
\end{aligned}\tag{13}$$

For each row $2 \le \mu \le 4$, the transfer kernels form a permutation of the four cyclic subgroups of order 3 of $\mathrm{Cl}_3(K_\mu) = \langle[\mathfrak{Q}], [\mathfrak{R}]\rangle$, more precisely, each row has two fixed points, where the norm class group $N_{B_\nu/K_\mu}(\mathrm{Cl}_3(B_\nu))$ coincides with the transfer kernel $\ker(T_{\mu\nu})$, and a transposition, where the norm class group and the transfer kernel are twisted. This characterizes type G.16 unambiguously [25, Tbl. 6, p. 492], and the Galois group $\mathrm{Gal}(F_3^2(K_\mu)/K_\mu)$ of the maximal metabelian unramified 3-extension, for $2 \le \mu \le 4$, is either the metabelian 3-group $M = \langle 729, 52\rangle$ or one among its six immediate descendants $D = \langle 2187, i\rangle$ with $294 \le i \le 299$, all with coclass $\mathrm{cc} = 2$ and relation rank $d_2 = 3 < 5$, as required for $\varrho = 2$ by Formula (2). These metabelian groups are contained in the SmallGroups database [12]. The abelian type invariants of second order admit the distinction between M with $\alpha_2(M) = [(22; 211, 211, 211, 211), (21; 211, 21, 21, 21)^3]$ and its immediate descendants D with $\alpha_2(D) = [(22; 221, 211, 211, 211), (21; 221, 21, 21, 21)^3]$, but unfortunately not between each D and its unique terminal non-metabelian immediate descendant $D - \#1; 1$, which has identical ATI2. For these groups with soluble length $\mathrm{sl} = 3$, the identifier $\langle 6561, j\rangle$, $2039 \le j \le 2044$, must be taken from the supplementary package [22]. □

Anticipating several informations in Section 6.1 and in Table 6 of Section 6.5, we are able to refine item (3) of Theorem 5.5 with respect to the second 3-class group $\mathrm{Gal}(F_3^2(K_\mu)/K_\mu)$ for $2 \le \mu \le 4$.

Corollary 5.6. — *Let $(K_1, \dots, K_4)$ be a quartet of cyclic cubic fields with conductor $c = q_1q_2q_3$ belonging to Graph 2 of Category I, $q_2 \leftarrow q_1 \rightarrow q_3$. Suppose K_1 is the unique component with 3-class rank $\varrho_3(K_1) = 3$, and the other three components have elementary bicyclic 3-class groups $\mathrm{Cl}_3(K_\mu) \simeq (3,3)$, for $2 \le \mu \le 4$. If K_1 possesses an elementary tricyclic 3-class group $\mathrm{Cl}_3(K_1) \simeq (3,3,3)$ and HBC, then the second 3-class group $\mathrm{Gal}(F_3^2(K_\mu)/K_\mu)$ for $2 \le \mu \le 4$ is determined in dependence on the 3-genus field K^*:*

$$\begin{aligned}
&\mathrm{Gal}(F_3^2(K_\mu)/K_\mu) \simeq \langle 729, 52\rangle\\
&\quad\Longleftrightarrow \mathrm{Cl}_3(K^*) \simeq (9,3,3)\\
&\quad\Longleftrightarrow \mathrm{ord}(\mathrm{Gal}(F_3^2(K_1)/K_1)) = 729,\\
&\mathrm{Gal}(F_3^2(K_\mu)/K_\mu) \simeq \langle 2187, i\rangle \text{ with } 294 \le i \le 299\\
&\quad\Longleftrightarrow \mathrm{Cl}_3(K^*) \simeq (9,9,3)\\
&\quad\Longleftrightarrow \mathrm{ord}(\mathrm{Gal}(F_3^2(K_1)/K_1)) \in \{2187, 6561\}.
\end{aligned} \tag{14}$$

Proof. — The assumption of HBC for K_1 enforces one of the groups in Section 6.1, which all possess the Artin pattern in Table 4, for the second 3-class group $\mathrm{Gal}(F_3^2(K_1)/K_1)$.
According to Corollary 5.4, B_1, B_5, B_6, B_7 are the 4 absolutely bicyclic bicubic fields among the 13 unramified cubic relative extensions of K_1.
Table 6 shows that, independently of the scenarios with distinct rank distribution, always $\mathrm{Cl}_3(B_1) \simeq (9,3,3)$ and $\mathrm{Cl}_3(B_j) \simeq (9,9)$ for $j \in \{5,6,7\}$.
According to Ayadi [7, Table, p. 66], $\mathrm{Cl}_3(B_j) \simeq (9,3)$ for $j \in \{2,3,4,8,9,10\}$.
Again, Corollary 5.4 lists the unramified cubic relative extensions of K_μ for $2 \le \mu \le 4$, namely (B_2, B_7, B_8, B_9) for K_2, (B_3, B_5, B_9, B_{10}) for K_3, (B_4, B_6, B_8, B_{10}) for K_4. This determines the first component of the Artin pattern $\alpha(K_\mu) = (21, 21, 21, 22)$ for $2 \le \mu \le 4$, which enforces a group $\mathrm{Gal}(F_3^2(K_\mu)/K_\mu)$ of coclass 2.
Item (3) of Theorem 5.5 supplements the second component of the Artin pattern $\varkappa(K_\mu) = (1243)$, which unambiguously leads to either $\mathrm{Gal}(F_3^2(K_\mu)/K_\mu) \simeq M = \langle 729, 52\rangle$ or $\mathrm{Gal}(F_3^2(K_\mu)/K_\mu) \simeq D = \langle 2187, i\rangle$ with $294 \le i \le 299$.
The decision is possible by means of AQI, respectively ATI, of second order. For the groups $\mathrm{Gal}(F_3^2(K_\mu)/K_\mu)$, the patterns $\alpha_2(M)$ and $\alpha_2(D)$ were given at the end of the proof of Theorem 5.5. Only one element (211), repectively (221), occurs in all four components of the AQI2. This must be the 3-class group $\mathrm{Cl}_3(K^*) \in \{(9,3,3), (9,9,3)\}$ of the 3-genus field K^*.
Finally, Table 5 shows that only (211) occurs in the second order invariants of the candidate groups of order 729 for $\mathrm{Gal}(F_3^2(K_1)/K_1)$, whereas (221) is element of four second order invariants of all candidate groups of order 2187 and 6561.
This proves the assertion of Corollary 5.6. □

6. Closed Andozhskii–Tsvetkov groups

According to Koch and Venkov [21], *Schur σ-groups S* are known to be mandatory for realizations $S \simeq \mathrm{Gal}(\mathrm{F}_p^\infty(k)/k)$ by p-class field towers of *imaginary* quadratic fields k, with an odd

prime p. They possess a balanced presentation $d_1(S) = d_2(S)$ with coinciding generator rank $d_1(S) = \dim_{\mathbb{F}_p} \mathrm{H}^1(S, \mathbb{F}_p)$ and relation rank $d_2(S) = \dim_{\mathbb{F}_p} \mathrm{H}^2(S, \mathbb{F}_p)$, and an automorphism $\sigma \in \mathrm{Aut}(S)$ acting as inversion $x \mapsto x^{-1}$ on the commutator quotient $S/[S, S]$. Schoof [41] has proved the supplementary requirement that σ must induce the inversion on both, $\mathrm{H}^1(S, \mathbb{F}_p)$ and $\mathrm{H}^2(S, \mathbb{F}_p)$. However, in the older literature, for instance Shafarevich [42, §6, pp. 88–91], there also appear *Schur groups* with balanced presentation, but without a generator- and relator-inverting σ-automorphism, and they are called *closed*, according to the original terminology by Schur. In the present article, we are interested in finite closed 3-groups G discovered by Andozhskii and Tsvetkov (briefly *AT groups*) [1, 2]. These authors only announce the order $\#G = 3^8 = 6561$ of the smallest closed groups with three generators, without giving any presentations or other details.
Therefore, our first task is to find the algebraic invariants of AT-groups and their position in the descendant tree of finite 3-groups G with commutator quotient $G/G' \simeq (\mathbb{Z}/3\mathbb{Z})^3$.
In Section 6.1, we identify the 17 *closed AT groups* as the smallest 3-groups of type $(3, 3, 3)$ with balanced presentation. Their order is either $3^8 = 6561$ or $3^9 = 19683$. We start by proving their *existence* and determining their *number* (Theorem 6.1). Then we compute their *invariants* (Corollaries 6.2 and 6.5). In Section 6.2, we show that three or four of them can be realized as Galois groups of the 3-class tower of *cyclic cubic fields*.

6.1. Identification of closed Andozhskii–Tsvetkov groups. — A database query for groups G in the *SmallGroups library* [11, 12], by the search criteria $G/G' \simeq (3, 3, 3)$ and $d_2(G) = d_1(G)$, yields a void result set, since the order of 3-groups is limited by $3^7 = 2187$. The same query in the extension `data3to8` [22] of the SmallGroups database with all 3-groups of order $3^8 = 6561$ produces 14 hits. This justifies the following theorem.

Theorem 6.1. — *Among the finite* 3*-groups* G *with commutator quotient* $G/G' \simeq (3, 3, 3)$*, there exist precisely* **14** ***metabelian closed*** *groups* S *of order* $\#S = 3^8$ *with identifiers*

$$S \simeq \langle 6561, 217700 + i\rangle \text{ where } 1 \le i \le 6 \text{ or } 10 \le i \le 17, \tag{15}$$

and **3** ***non-metabelian closed*** *groups* S *of order* $\#S = 3^9 = 19683$ *with identifiers*

$$S \simeq \langle 6561, 217700 + i\rangle - \#1; 1 \text{ where } i \in \{7, 8, 9\}. \tag{16}$$

They possess a trivial Schur multiplier $M(S) = \mathrm{H}_2(S, \mathbb{Q}/\mathbb{Z}) = 0$ *and a balanced presentation* $d_1(S) = d_2(S)$ *with coinciding generator rank* $d_1(S) = \dim_{\mathbb{F}_p} \mathrm{H}^1(S, \mathbb{F}_p)$ *and relation rank* $d_2(S) = \dim_{\mathbb{F}_p} \mathrm{H}^2(S, \mathbb{F}_p)$*. The class is* $\mathrm{Cl}(S) = 3$ *for soluble length* $\mathrm{sl}(S) = 2$ *and* $\mathrm{Cl}(S) = 4$ *for* $\mathrm{sl}(S) = 3$*. They possess harmonically balanced transfer kernels* $\varkappa(S) \in S_{13}$*, but no* σ*-automophism (i.e., they are* ***Schur groups*** *but not Schur* σ*-groups). There do not exist any closed groups* G *with* $G/G' \simeq (3, 3, 3)$ *and order* $\#G \le 3^7$*.*

Proof. — By a search in the SmallGroups database [12] with supplementary package [22], extended to order 3^9 by the p-group generation algorithm [20, 36, 37], the finite closed Andozhskii–Tsvetkov 3-groups S are identified. There are no hits of order $\#S \le 3^7$, 14 hits of order $\#S = 3^8$, and only three hits of order $\#S = 3^9$. The non-metabelian groups are characterized by their relative identifiers defined in the ANUPQ package [17]. See Figure 1 and the HBC in Table 4. □

The discovery that all AT-groups in Theorem 6.1 have a harmonically balanced capitulation (HBC) suggests to ask whether there exist other finite 3-groups G with elementary tricyclic

commutator quotient and HBC. A database query in the SmallGroups library [12], with search criteria $G/G' \simeq (3,3,3)$, $\#G \le 3^6$, and the much looser condition that all 13 transfer kernels are cyclic of order 3, produces no hits with $\#G \le 3^5 = 243$, but 6 hits with $\#G = 3^6 = 729$ which astonishingly even have HBC. It turns out that the AT-groups in Theorem 6.1 are descendants of 5 among these 6 groups, with the following relative ANUPQ identifiers [17]:
$\langle 6561, 217700+i\rangle = \langle 729, 133\rangle - \#2; i$, $i \in \{1,2,3\}$;
$\langle 6561, 217700+i\rangle = \langle 729, 134\rangle - \#2; (i-3)$, $i \in \{4,5,6\}$;
$\langle 6561, 217700+i\rangle = \langle 729, 135\rangle - \#2; (i-6)$, $i \in \{7,8,9\}$;
$\langle 6561, 217700+i\rangle = \langle 729, 136\rangle - \#2; (i-9)$, $i \in \{10,11,12\}$; and
$\langle 6561, 217700+i\rangle = \langle 729, 137\rangle - \#2; (i-12)$, $i \in \{13,14,15,16,17\}$.
The group $\langle 729, 132\rangle$ has also HBC but does not possess any closed descendants.

Corollary 6.2. — *Each of the* 17 *closed* AT*-groups* $S = \langle 6561, 217700+i\rangle$*,* $1 \le i \le 17$*,* $i \notin \{7,8,9\}$*, and* $S = \langle 6561, 217700+i\rangle - \#1; 1$*,* $i \in \{7,8,9\}$*, in Theorem 6.1 shares a common Artin pattern* $(\varkappa, \alpha)$ *with its ancestor* $A \simeq \langle 729, 130+j\rangle$*,* $3 \le j \le 7$*, as given in Table 4, where*

$$j = \begin{cases} 3 & \text{for } 1 \le i \le 3, \\ 4 & \text{for } 4 \le i \le 6, \\ 5 & \text{for } 7 \le i \le 9, \\ 6 & \text{for } 10 \le i \le 12, \\ 7 & \text{for } 13 \le i \le 17. \end{cases} \tag{17}$$

There do not exist any groups G *with* $G/G' \simeq (3,3,3)$*,* HBC*, and order* $\#G \le 3^5$*.*

Proof. — According to the *theorem on the antitony* of the Artin pattern [28, §§5.1–5.4, pp. 78–87], it suffices to calculate the *stable* transfer kernels of the five ancestors A of the 17 closed groups in Theorem 6.1. They are of order $\#A = 3^6$ and have much simpler power-commutator-presentations $A = \langle x,y,z \mid x^3 = R_x,\ y^3 = R_y,\ z^3 = R_z\rangle$, in terms of relator words R_x, R_y, R_z containing main commutators $u = [y,x]$, $v = [z,x]$, $w = [z,y]$, as given in Table 4. Additional to the defining database query, the transfer kernel type (TKT) is *harmonically balanced*, that is, a permutation in the symmetric group S_{13} of degree 13. □

Table 4 has a layout with double rows. It shows invariants of the six groups $A = \langle 729, \mathrm{id}\rangle$, $132 \le \mathrm{id} \le 137$, in Corollary 6.2. The first row contains the identifier id, the transfer kernel type (TKT) $\varkappa(A)$, according to Tables 1 and 2, the numbers N_1, N_2, N_3 of immediate descendants with step sizes $s \in \{1,2,3\}$, and the operator group on the Frattini quotient $A/\Phi(A)$. The second row contains the nuclear rank $\nu(A)$, the relation rank $\mu(A) = d_2(A)$, the transfer target type (TTT) $\alpha(A)$, the rank distribution $\rho(A)$, and the relator words R_x, R_y, R_z in the pc-presentation given in the proof of Corollary 6.2.

Remark 6.3. — Whereas rank distribution $\rho = (3^7, 2^6)$ occurs only for $\langle 729, 133\rangle$, and $\rho = (3^1, 2^{12})$ only for $\langle 729, 136\rangle$, the distribution $\rho = (3^4, 2^9)$ is more frequent and occurs for the remaining four groups. Although each group in Table 4 has immediate descendants of (at least) two step sizes, the tree terminates with metabelian groups of order $3^7 = 2187$ and $3^8 = 6561$ below $\langle 729, 130+j\rangle$ with $j \in \{3,4,6,7\}$. For $\langle 729, 135\rangle$, the tree terminates with the non-metabelian groups of order $3^9 = 19683$ in Theorem 6.1. The descendant tree of $\langle 729, 132\rangle$ with three step sizes is infinite, due to periodic trifurcations. None of the descendants D is

Table 4. TKT $\varkappa$, TTT α, ranks ρ, and operation of groups $A = \langle 729, \mathrm{id}\rangle$

id ν, μ	$\varkappa$ (above) α (below)													N_i ρ	Operation R_x, R_y, R_z
132	1	2	3	7	8	6	4	5	9	13	10	11	12	$3, 6, 4$	$\langle 24, 12\rangle$
$3, 6$	22	22	22	211	211	22	211	211	22	22	22	22	22	$3^4, 2^9$	w, v, u
133	9	2	3	10	8	1	12	5	6	4	11	7	13	$4, 3$	$\langle 6, 2\rangle$
$2, 5$	22	22	22	211	211	22	211	211	211	211	22	211	22	$3^7, 2^6$	uw, v, u
134	1	8	3	11	5	6	12	2	9	4	7	10	13	$2, 3$	$\langle 4, 1\rangle$
$2, 5$	22	22	22	211	22	22	211	22	22	211	211	22	22	$3^4, 2^9$	w, uv, u
135	9	8	3	7	5	1	13	2	6	12	10	11	4	$2, 3$	$\langle 3, 1\rangle$
$2, 5$	22	22	22	211	22	22	211	22	211	22	22	22	211	$3^4, 2^9$	uw, uv, u
136	11	8	3	12	5	10	1	2	7	6	13	4	9	$4, 3$	$\langle 6, 2\rangle$
$2, 5$	22	22	22	22	22	211	22	22	22	22	22	22	22	$3^1, 2^{12}$	vw, uv, u
137	13	8	3	7	2	12	9	5	4	1	10	11	6	$3, 5$	$\langle 24, 3\rangle$
$2, 5$	22	22	22	211	22	211	22	211	22	22	22	22	211	$3^4, 2^9$	uvw, uv^2, u

closed, since the minimal relation rank is $d_2(D) = 4 > 3 = d_1(D)$. Further, the derived length $\mathrm{dl}(D)$ increases unboundedly. (It is 3 for order 3^{10}, 4 for order 3^{22}, and 5 for order 3^{46}.)

Definition 6.4. — The transfer kernel (capitulation kernel) $\ker(V_i)$ of an Artin transfer homomorphism $V_i : G/G' \to H_i/H_i'$ [28] from a 3-group G with $G/G' \simeq (3,3,3)$ to one of its 13 maximal subgroups H_i, $1 \le i \le 13$, is called of *Taussky type* A, if the meet $\ker(V_i) \cap H_i > 1$ is non-trivial, and of *Taussky type* B, if $\ker(V_i) \cap H_i = 1$ is trivial [44].

Corollary 6.5. — *Artin patterns* $(\alpha(A), \varkappa(A))$ *of the six groups* $A = \langle 729, \mathrm{id}\rangle$, $132 \le \mathrm{id} \le 137$, *share the common property that the Taussky type of the transfer kernels* $\varkappa(A)_i = \ker(V_i)$ *is determined uniquely by the AQI* H_i/H_i' *of the corresponding maximal subgroups* H_i*:*

$$\alpha(A)_i = H_i/H_i' \simeq (211) \iff \varkappa(A)_i \cap H_i > 1, \ \textit{Taussky type } \mathrm{A}, \tag{18}$$
$$\alpha(A)_i = H_i/H_i' \simeq (22) \iff \varkappa(A)_i \cap H_i = 1, \ \textit{Taussky type } \mathrm{B},$$

for all $1 \le i \le 13$. *Here, the abelian quotient invariants are written in logarithmic form.*

Proof. — For all $1 \le i, j \le 13$, this law follows by comparing the 1-dimensional transfer kernels (lines) $L_{\pi(i)} = \ker(V_i)$ with $\pi \in S_{13}$ in Table 4 to the 2-dimensional subspaces (planes) $P_j = H_j$ of the space $O = A/A' \simeq (3,3,3)$ and the sets S_i in Table 2. Here, $\ker(V_i) \cap H_i > 1$ is equivalent to $L_{\pi(i)} < P_i$, because all transfer kernels are cyclic of order 3. Exemplarily, for the group $A = \langle 729, 136\rangle$ with (accumulated) rank distribution $\rho(A) = (3^1, 2^{12})$, the transfer kernel $\ker(V_6) = L_{10}$ is contained in P_6, since $10 \in S_6 = \{2, 9, 10, 12\}$. In contrast, $\ker(V_7) = L_1$ has trivial intersection with P_7, since $1 \notin S_7 = \{3, 4, 12, 13\}$. So the Taussky type of $\ker(V_6)$, respectively $\ker(V_7)$, is A, respectively B, corresponding to the transfer target $H_6/H_6' \simeq (211)$, respectively $H_7/H_7' \simeq (22)$. □

In order to draw the descendant tree with root $\langle 27, 5\rangle \simeq (3,3,3)$ containing all closed AT-groups in Theorem 6.1, we must supplement the *siblings* G of order $\#G = 3^7 = 2187$ of the metabelian AT-groups $S = \langle 6561, 217700 + i\rangle$, $1 \le i \le 17$, $i \notin \{7, 8, 9\}$, and of the non-metabelian AT-groups $S = \langle 6561, 217700 + i\rangle - \#1;1$, $i \in \{7, 8, 9\}$, which are immediate descendants of step size $s = 1$ of the ancestors $A = \langle 729, 130+j\rangle$, $3 \le j \le 7$, in Corollary 6.2.

Corollary 6.6. — *The siblings of the closed AT-groups are given by* $G \simeq$
$\langle 2187, 4660 + k\rangle = \langle 729, 133\rangle - \#1; k$, $1 \le k \le 4$,
$\langle 2187, 4660 + k\rangle = \langle 729, 134\rangle - \#1; (k-4)$, $5 \le k \le 6$,
$\langle 2187, 4660 + k\rangle = \langle 729, 135\rangle - \#1; (k-6)$, $7 \le k \le 8$,
$\langle 2187, 4660 + k\rangle = \langle 729, 136\rangle - \#1; (k-8)$, $9 \le k \le 12$,
$\langle 2187, 4660 + k\rangle = \langle 729, 137\rangle - \#1; (k-12)$, $13 \le k \le 15$.
All of them are terminal metabelian groups with relation rank $d_2(G) = 4$ *and HBC. Each of them shares a common Artin pattern* $(\alpha, \varkappa)$ *with its ancestor, as given in Table 4.*

Proof. — According to the antitony principle, the cyclic transfer kernels $\varkappa(G)$ of order 3 cannot shrink further, and, correspondingly, the transfer targets $\alpha(G)$ cannot expand. □

The position of AT-groups in the descendant tree of 3-groups G with elementary tricyclic commutator quotient $G/[G,G] \simeq (3,3,3)$ is illuminated in Figure 1. This is a rooted tree diagram showing a graph $\mathcal{G} = (V, E)$ with groups as vertices $D \in V$ and quotient relations $\pi(D) = D/\gamma_c(D)$ by the last non-trivial lower central (with c denoting the nilpotency class) as directed edges $(\pi : D \to \pi(D)) \in E$ between immediate descendants D and parents $\pi(D)$. On the left hand side there is a scale with increasing orders 3^e. Thus the identifiers of the groups can be abbreviated by the ⟨identifier⟩ in angle brackets, instead of the pair ⟨order, identifier⟩, taken from the SmallGroups database [12]. Metabelian groups are shown as circles, non-metabelian groups as squares. The symbol $*n$ indicates a batch of n siblings, drawn with a single vertex (to save space). Relative identifiers [17] are $\#s;n$.

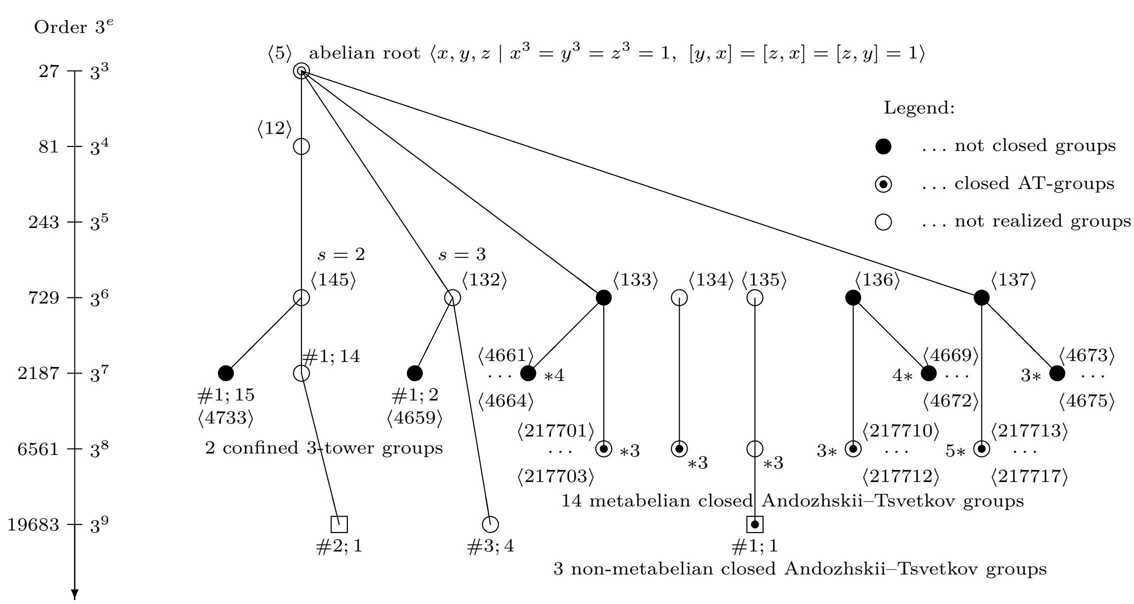

FIGURE 1. Tree of 3-groups G with $G/G' \simeq (3,3,3)$

Remark 6.7. — Incidentally, we point out that related but not closed 3-groups (see $j = 2$ in Table 4) are realized by numerous 3-class field towers over *totally complex S_3-fields* K, which are unramified extensions of imaginary quadratic fields $k = \mathbb{Q}(\sqrt{d})$ with 3-class group $\mathrm{Cl}_3(k) \simeq (3,3)$, capitulation type H.4, $\varkappa(k) \sim (4111)$, and three abelian type invariants of rank 3 in $\alpha(k) \sim (111,111,111,21)$. The latter 3-class towers are *confined*, since $\mathrm{F}_3^3(k) = \mathrm{F}_3^3(K) = \mathrm{F}_3^4(k)$ [32, Thm. 6.1, p. 678]. The group $\langle 729,145\rangle$ does not possess HBC, since its TKT contains 3 distinct planes P_i, meeting in three distinct lines L_j, and 10 times the full vectorspace O.

6.2. Realization of groups with HBC by algebraic number fields. — Decisive for the kind of algebraic number fields K which are able to realize the AT-groups S in Theorem 6.1 and their ancestors A in the Corollaries 6.2 and 6.5 as Galois groups $\mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$ of maximal unramified pro-3-extensions (that is, 3-class field towers) is the *operation* on the Frattini quotient, listed for the ancestors A in Table 4. Since $\langle 4,1\rangle \simeq \mathbb{Z}/4\mathbb{Z}$, the group $\langle 729,134\rangle$ requires *cyclic quartic fields*, and cannot be realized by cyclic cubic fields. In contrast, all the other operator groups admit *cyclic cubic fields*: $\langle 729,135\rangle$ with operation by $\langle 3,1\rangle \simeq \mathbb{Z}/3\mathbb{Z}$, and $\langle 729,133\rangle$, $\langle 729,136\rangle$ both with operator group $\langle 6,2\rangle \simeq \mathbb{Z}/6\mathbb{Z} \simeq (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})$, which also enables cyclic sextic fields. The exceptional group $\langle 729,132\rangle$ without closed descendants and with operation by $\langle 24,12\rangle \simeq S_4$, the symmetric group of degree 4, also admits S_3-fields, cyclic quartic fields, bicyclic biquadratic fields, and dihedral fields of degree 8, because S_4 contains $\langle 12,3\rangle \simeq A_4$, the alternating group of degree 4, $\langle 6,1\rangle \simeq S_3$, the symmetric group of degree 3, $\langle 4,1\rangle$, $\langle 4,2\rangle \simeq (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$, and $\langle 8,3\rangle \simeq D_8$, the dihedral group of order 8. Finally, $\langle 729,137\rangle$ with operator group $\langle 24,3\rangle \simeq SL(2,3)$, the special linear group of dimension 2 over $\mathbb{F}_3$, enables cyclic sextic fields, quaternion fields of degree 8, and cyclic quartic fields, since $SL(2,3)$ contains $\langle 6,2\rangle$, $\langle 8,4\rangle \simeq Q_8$, the quaternion group of order 8, and $\langle 4,1\rangle$.

6.3. Second order invariants of candidate groups with HBC. — In Table 5, we present all possible candidates for metabelian 3-groups G with HBC, characterized uniquely by their absolute identifier $\langle 3^{\mathrm{lo}},\mathrm{id}\rangle$ in the SmallGroups database [12, 22] with order 3^{lo}, logarithmic order $6 \le \mathrm{lo} \le 8$ and numerical identifier id.
Crucial invariants of these groups are the nuclear rank ν, the p-multiplicator rank $\mu = d_2$, which coincides with the *relation rank*, and the number of descendants N_s and of capable descendants C_s for all possible step sizes $1 \le s \le \nu$. The most important invariant, however, indispensable for the unambiguous identification ($\varkappa$ and α are insufficient), and demanding extreme computational challenge for the number theoretic verification, is the *Artin pattern of second order* [33],

$$\alpha_2 := [H_j/H_j'; (H_{j,\ell}/H_{j,\ell}')_{\ell=1}^{n_j}]_{j=1}^{13}, \qquad (19)$$

consisting of *logarithmic abelian quotient invariants* (AQI) of all maximal subgroups H_j, $1 \le j \le 13$, and second maximal subgroups $H_{j,\ell}$, $1 \le j \le 13$, with $1 \le \ell \le 4$, $n_j = 4$, when $H_j/H_j' \simeq (22) \mathrel{\widehat{=}} (9,9)$, and $1 \le \ell \le 13$, $n_j = 13$, when $H_j/H_j' \simeq (211) \mathrel{\widehat{=}} (9,3,3)$.
The group $\langle 729,132\rangle$ is forbidden for cyclic cubic fields, because of its relation rank 6. The group $\langle 729,134\rangle$ and all its descendants are forbidden for cyclic cubic fields, because they have a wrong action by $\langle 4,1\rangle$, which is only allowed for *cyclic quartic fields.* Among the groups of order $3^6 = 729$, two with identifiers id $\in \{135,137\}$ cannot be distinguished by second order invariants α_2. Similarly for groups of order $3^7 = 2187$, two with identifiers

TABLE 5. Second order invariants and propagation of 3-groups with HBC

lo	id	$\alpha^{(2)}$	ν	μ	$(N_s/C_s)_{s=1}^{\nu}$
6	**136**	$[(21^2);(21^2)^4(21)^9]^1[(2^2);(21^2)^4]^{12}$	2	5	(4/0;3/0)
6	132	$[(21^2);(21^2)^4(21)^9]^4[(2^2);(21^2)^4]^9$	3	6	(3/0;6/1;4/4)
6	134	$[(21^2);(21^2)^4(21)^9]^4[(2^2);(21^2)^4]^9$	2	5	(2/0;3/0)
6	135	$[(21^2);(21^2)^4(21)^9]^4[(2^2);(21^2)^4]^9$	2	5	(2/0;3/3)
6	**137**	$[(21^2);(21^2)^4(21)^9]^4[(2^2);(21^2)^4]^9$	2	5	(3/0;5/0)
6	**133**	$[(21^2);(21^2)^4(21)^9]^7[(2^2);(21^2)^4]^6$	2	5	(4/0;3/0)
7	4669	$[(21^2);(21^2)^4(21)^9]^1[(2^2);(31^2)(21^2)^3]^4[(2^2);(21^2)^4]^8$	0	4	—
7	**4670**	$[(21^2);(2^21)(21^2)^{12}]^1[(2^2);(2^21)(21^2)^3]^3[(2^2);(21^2)^4]^9$	0	4	—
7	4671	$[(21^2);(2^21)(21^2)^3(2^2)^9]^1[(2^2);(2^21)(21^2)^3]^3[(2^2);(21^2)^4]^9$	0	4	—
7	4672	$[(21^2);(2^21)(21^2)^3(2^2)^9]^1[(2^2);(2^21)(21^2)^3]^3[(2^2);(21^2)^4]^9$	0	4	—
7	**4673**	$[(21^2);(21^2)^4(21)^9]^3[(21^2);(2^21)(21^2)^3(2^2)^9]^1[(2^2);(2^21)(21^2)^3]^3[(2^2);(21^2)^4]^6$	0	4	—
7	4674	$[(21^2);(21^2)^4(21)^9]^3[(21^2);(2^21)(21^2)^{12}]^1[(2^2);(2^21)(21^2)^3]^3[(2^2);(21^2)^4]^6$	0	4	—
7	**4675**	$[(21^2);(21^2)^4(21)^9]^3[(21^2);(2^21)(21^2)^3(2^2)^9]^1[(2^2);(2^21)(21^2)^3]^3[(2^2);(21^2)^4]^6$	0	4	—
7	4661	$[(21^2);(21^2)^4(21)^9]^6[(21^2);(2^21)(21^2)^{12}]^1[(2^2);(2^21)(21^2)^3]^3[(2^2);(21^2)^4]^3$	0	4	—
7	**4662**	$[(21^2);(21^2)^4(21)^9]^6[(21^2);(2^21)(21^2)^3(2^2)^9]^1[(2^2);(2^21)(21^2)^3]^3[(2^2);(21^2)^4]^3$	0	4	—
7	**4663**	$[(21^2);(21^2)^4(21)^9]^6[(21^2);(2^21)(21^2)^3(2^2)^9]^1[(2^2);(2^21)(21^2)^3]^3[(2^2);(21^2)^4]^3$	0	4	—
7	4664	$[(21^2);(21^2)^4(21)^9]^5[(21^2);(31^2)(21^2)^3(31)^9]^2[(2^2);(31^2)(21^2)^3]^2[(2^2);(21^2)^4]^4$	0	4	—
8	**217710**	$[(21^2);(2^21)(21^2)^{12}]^1[(2^2);(31^2)^3(2^21)]^1[(2^2);(2^21)(21^2)^3]^2[(2^2);(31^2)(21^2)^3]^9$	0	3	—
8	217711	$[(21^2);(2^21)(21^2)^3(2^2)^9]^1[(2^2);(31^2)^3(2^21)]^1[(2^2);(2^21)(21^2)^3]^2[(2^2);(31^2)(21^2)^3]^9$	0	3	—
8	217712	$[(21^2);(2^21)(21^2)^3(2^2)^9]^1[(2^2);(31^2)^3(2^21)]^1[(2^2);(2^21)(21^2)^3]^2[(2^2);(31^2)(21^2)^3]^9$	0	3	—
8	**217713**	$[(21^2);(2^21)(21^2)^3(2^2)^9]^4[(2^2);(2^21)(21^2)^3]^8[(2^2);(21^2)^4]^1$	0	3	—
8	217714	$[(21^2);(2^21)(21^2)^3(2^2)^9]^2[(21^2);(2^21)(21^2)^{12}]^2[(2^2);(2^21)(21^2)^3]^8[(2^2);(21^2)^4]^1$	0	3	—
8	217715	$[(21^2);(2^21)(21^2)^3(2^2)^9]^3[(21^2);(2^21)(21^2)^{12}]^1[(2^2);(2^21)(21^2)^3]^8[(2^2);(21^2)^4]^1$	0	3	—
8	217716	$[(21^2);(2^21)(21^2)^3(2^2)^9]^1[(21^2);(2^21)(21^2)^{12}]^3[(2^2);(2^21)(21^2)^3]^8[(2^2);(21^2)^4]^1$	0	3	—
8	**217717**	$[(21^2);(2^21)(21^2)^3(2^2)^9]^4[(2^2);(2^21)(21^2)^3]^8[(2^2);(21^2)^4]^1$	0	3	—
8	217701	$[(21^2);(31^2)(21^2)^3(31)^9]^6[(21^2);(2^21)(21^2)^{12}]^1$ $[(2^2);(2^21)(21^2)^3]^2[(2^2);(31^2)(21^2)^3]^3[(2^2);(31^2)(2^21)^3]^1$	0	3	—
8	**217702**	$[(21^2);(31^2)(21^2)^3(31)^9]^6[(21^2);(2^21)(21^2)^3(2^2)^9]^1$ $[(2^2);(2^21)(21^2)^3]^2[(2^2);(31^2)(21^2)^3]^3[(2^2);(31^2)(2^21)^3]^1$	0	3	—
8	**217703**	$[(21^2);(31^2)(21^2)^3(31)^9]^6[(21^2);(2^21)(21^2)^3(2^2)^9]^1$ $[(2^2);(2^21)(21^2)^3]^2[(2^2);(31^2)(21^2)^3]^3[(2^2);(31^2)(2^21)^3]^1$	0	3	—

id $\in \{4662, 4663\}$ and two with identifiers id $\in \{4673, 4675\}$ cannot be separated by α_2. Similarly for groups of order $3^8 = 6561$, two with identifiers id $\in \{217702, 217703\}$ and two with identifiers id $\in \{217713, 217717\}$ cannot be separated by α_2.
So far, three or four AT-groups of order $3^8 = 6561$ have been **realized** as 3-class tower groups of cyclic cubic fields. Either $\langle 6561, 217702\rangle$ or $\langle 6561, 217703\rangle$ by the conductor $c = \mathbf{1\,406\,551}$, either $\langle 6561, 217713\rangle$ or $\langle 6561, 217717\rangle$ by the conductors $c = \mathbf{689\,347}$ and $c = \mathbf{869\,611}$, and $\langle 6561, 217710\rangle$ unambiguously by $c = \mathbf{753\,787}$ and $c = \mathbf{796\,779}$.

6.4. Realization as 3-class field tower groups. — Since the groups in Theorem 6.1 are non-σ groups, they cannot be realized by any quadratic field, neither imaginary nor real. Therefore, we investigated the possible Galois actions (Table 4) on the five ancestors $A = \mathrm{SmallGroup}(729, 130 + j)$. It turned out that the unique non-metabelian case $j = 5$ can only be realized by *cyclic cubic* fields, $j = 4$ by cyclic quartic fields, and $j \in \{3, 6, 7\}$ by *cyclic cubic or sextic* fields. We show that certain metabelian descendants S for $j \in \{3, 6, 7\}$ can actually be realized as Galois groups $\mathrm{Gal}(\mathrm{F}_3^\infty(K)/K) \simeq S$ of maximal unramified pro-3-extensions of cyclic cubic fields K with 53 conductors c in the OEIS sequence A359310 [43], from **59 031** to **1 406 551**, i.e., with 3-class group $\mathrm{Cl}_3(K) \simeq (3, 3, 3)$ and HBC.

Theorem 6.8. — *If a number field* $K/\mathbb{Q}$ *with* 3*-class group* $\mathrm{Cl}_3(K) \simeq (3, 3, 3)$ *possesses the Artin pattern* $(\varkappa(K), \alpha(K))$ *with* $\varkappa(K) \sim (9, 2, 3, 10, 8, 1, 12, 5, 6, 4, 11, 7, 13)$ *as harmonically balanced capitulation type and* $\alpha(K) \sim ((22)^3, (211)^2, 22, (211)^4, 22, 211, 22)$ *as abelian type invariants, i.e.,* $\rho = (3^7, 2^6)$*, then* $K/\mathbb{Q}$ *must be cyclic cubic or sextic, and has a metabelian*

3-class field tower with automorphism group $\mathrm{Gal}(\mathrm{F}_3^\infty(K)/K) \simeq$

$$\langle 6561, 217700+i\rangle,\ 1 \le i \le 3,\ \textit{or}\ \langle 2187, 4660+k\rangle,\ 1 \le k \le 4,\ \textit{or}\ \langle 729, 133\rangle. \tag{20}$$

Theorem 6.9. — *If a number field $K/\mathbb{Q}$ with 3-class group $\mathrm{Cl}_3(K) \simeq (3,3,3)$ possesses the Artin pattern $(\varkappa(K), \alpha(K))$ with harmonically balanced capitulation type $\varkappa(K) \sim (11,8,3,12,5,10,1,2,7,6,13,4,9)$ and abelian type invariants $\alpha(K) \sim ((22)^5, 211, (22)^7)$, i.e., $\rho = (3^1, 2^{12})$, then $K/\mathbb{Q}$ must be cyclic cubic or sextic, and has a metabelian 3-class field tower with automorphism group* $\mathrm{Gal}(\mathrm{F}_3^\infty(K)/K) \simeq$

$$\langle 6561, 217700+i\rangle,\ 10 \le i \le 12,\ \textit{or}\ \langle 2187, 4669+k\rangle,\ 0 \le k \le 3,\ \textit{or}\ \langle 729, 136\rangle. \tag{21}$$

Proof. — Theorems 6.8 and 6.9 are immediate consequences of the Tables 4 and 5. □

Conjecture 6.10. — *If a number field $K/\mathbb{Q}$ with 3-class group $\mathrm{Cl}_3(K) \simeq (3,3,3)$ possesses the Artin pattern $(\varkappa(K), \alpha(K))$ with $\varkappa(K) \sim (13,8,3,7,2,12,9,5,4,1,10,11,6)$ as harmonically balanced capitulation type and $\alpha(K) \sim ((22)^3, 211, 22, 211, 22, 211, (22)^4, 211)$ as abelian type invariants, i.e., $\rho = (3^4, 2^9)$, then $K/\mathbb{Q}$ must be cyclic cubic or sextic, and has a metabelian 3-class field tower with automorphism group* $\mathrm{Gal}(\mathrm{F}_3^\infty(K)/K) \simeq$

$$\langle 6561, 217700+i\rangle,\ 13 \le i \le 15,\ \textit{or}\ \langle 2187, 4673+k\rangle,\ 0 \le k \le 2,\ \textit{or}\ \langle 729, 137\rangle. \tag{22}$$

For this situation, the Tables 4 and 5 admit descendants of $\langle 3^6, 132\rangle$ and $\langle 3^6, 135\rangle$ as additional candidates. But, so far, experience provides evidence that no such realizations occur. Therefore we conjecture that this tendency will continue.

6.5. Galois structure of unramified cubic and nonic extensions. — The fundamental facts, on which the *Galois structure* of the lattice of intermediate fields $\mathbb{Q} < F < \mathrm{F}_3^1(K)$ of the Hilbert 3-class field $\mathrm{F}_3^1(K)$ of a cyclic cubic number field K with 3-class group $\mathrm{Cl}_3(K) \simeq (3,3,3)$ and conductor $c = q_1q_2q_3$ with precisely three prime (power) divisors $q_i \equiv +1 \pmod 3$, or $q_i = 3^2$, is based, can be summarized as follows (see Figure 2):

- The *absolute 3-genus field* $K^* = (K/\mathbb{Q})^*$ of K is the maximal unramified 3-extension of K which is abelian over the rational field $\mathbb{Q}$. In the situation with conductor $c = q_1q_2q_3$, its absolute Galois group is $\mathrm{Gal}(K^*/\mathbb{Q}) \simeq (3,3,3)$, whence it also called the *elementary 3-extension* of K, and possesses 13 cyclic cubic subfields $K_1, \ldots, K_{13}$, one of them K, and 13 bicyclic bicubic subfields $B_1, \ldots, B_{13}$. The former consist of three singlets with partial conductors q_1, q_2, q_3, three doublets with partial conductors q_1q_2, q_1q_3, q_2q_3, and a quartet with complete conductor $c = q_1q_2q_3$. The exact constitution of the latter was analyzed by Ayadi [7]: Three of them are sub-genus fields $B_{11} = k^*_{q_1q_2}, B_{12} = k^*_{q_1q_3}, B_{13} = k^*_{q_2q_3}$ with conductors q_1q_2, q_1q_3, q_2q_3. Among the remaining ten, four contain K, namely, in Ayadi's notation [7, Lem. 4.1, p. 42],
  $B_1 = Kk_{q_1q_2}k_{q_1q_3}k_{q_2q_3}$, $B_5 = KK_3k_{q_1}\widetilde{k}_{q_2q_3}$, $B_6 = KK_4k_{q_2}\widetilde{k}_{q_1q_3}$, $B_7 = KK_2k_{q_3}\widetilde{k}_{q_1q_2}$.
- The *Hilbert 3-class field* $\mathrm{F}_3^1(K)$ of K is the maximal abelian unramified 3-extension of K. By Artin's reciprocity law [5], its relative Galois group $\mathrm{Gal}(\mathrm{F}_3^1(K)/K)$ is isomorphic to the 3-class group $\mathrm{Cl}_3(K) \simeq (3,3,3)$ of K. Among the 13 cyclic cubic relative extensions $K < E_1, \ldots, E_{13} < \mathrm{F}_3^1(K)$, only four are abelian, namely the bicyclic bicubic fields

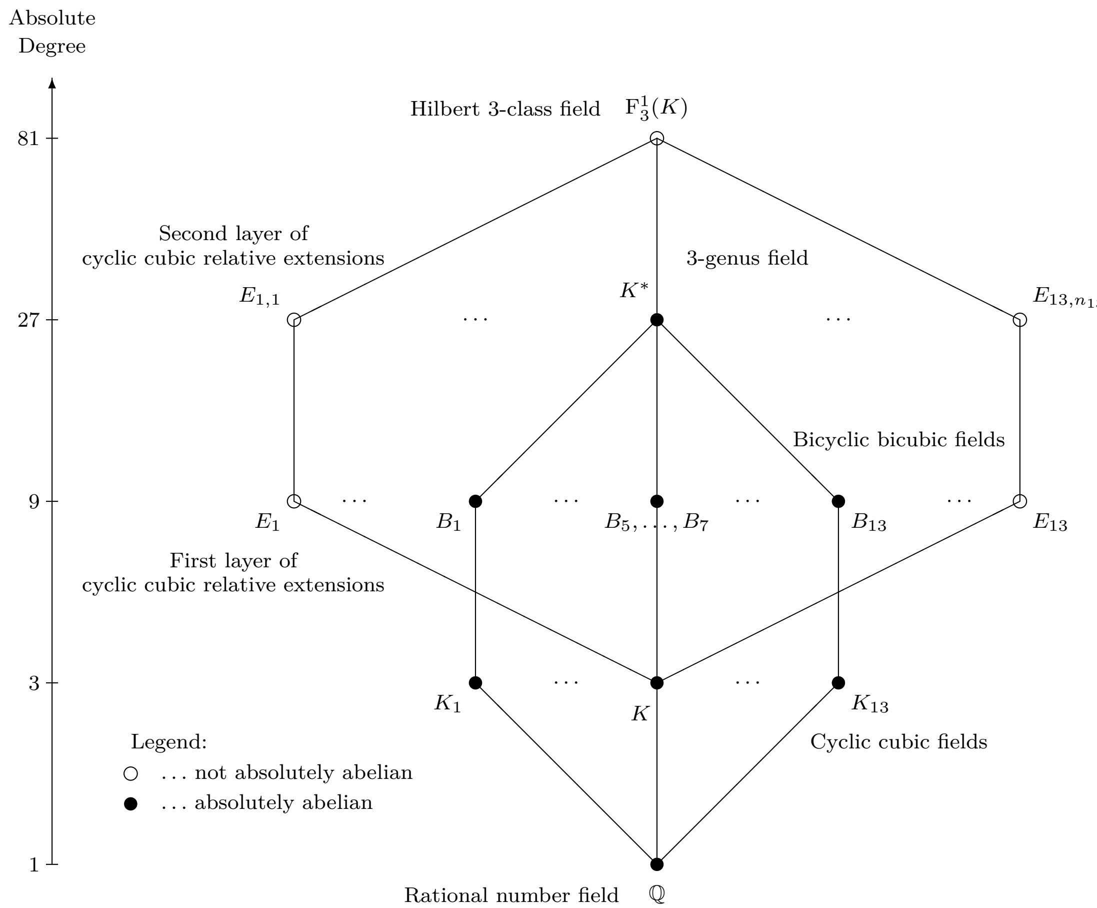

Figure 2. Lattice of subfields

B_1, B_5, B_6, B_7 with absolute Galois group $\langle 9, 2\rangle$. The remaining nine extensions are non-Galois with Galois group $\mathrm{Gal}(\bar{E}_i/\mathbb{Q}) \simeq \langle 27, 3\rangle$, $i \in \{1, 2, 3\}$ of the splitting field $\bar{E}_i/E_i$, arranged in three triplets,

$$(E_1, E_1', E_1''),\ (E_2, E_2', E_2''),\ (E_3, E_3', E_3''), \tag{23}$$

of three conjugate isomorphic fields each.

Three possible scenarios for the rank distribution ρ and abelian quotient invariants of the first order α are summarized in Table 6, a number theoretic refinement of Table 4.
Now we turn to the correct selection of representatives R_i in isomorphism classes among the unramified cyclic cubic relative extensions E_j/K, $1 \le j \le 13$, and among the unramified nonic but not necessarily Galois extensions $E_{j,\ell}/K$, $1 \le j \le 13$, $1 \le \ell \le n_j$, $n_j \in \{4, 13\}$, of absolute degree 27, for a 3-class rank $\varrho_j \in \{2, 3\}$ of E_j, respectively.
An unsophisticated way to determine the Artin pattern α_2 of second order of a cyclic cubic field K would be to construct the entire collection of the following extensions $E_{j,\ell}$.
Recall that we have *three possible scenarios*, according to Table 4:

Table 6. Three scenarios for the rank distribution

Scenario	ρ	α	Extensions				
			B_1	B_5, B_6, B_7	E_1, E_1', E_1''	E_2, E_2', E_2''	E_3, E_3', E_3''
(1)	$3^1, 2^{12}$	$(211)^1, (22)^{12}$	211	22, 22, 22	22, 22, 22	22, 22, 22	22, 22, 22
			B_1	E_1, E_1', E_1''	B_5, B_6, B_7	E_2, E_2', E_2''	E_3, E_3', E_3''
(2)	$3^4, 2^9$	$(211)^4, (22)^9$	211	211, 211, 211	22, 22, 22	22, 22, 22	22, 22, 22
			B_1	E_1, E_1', E_1''	E_2, E_2', E_2''	B_5, B_6, B_7	E_3, E_3', E_3''
(3)	$3^7, 2^6$	$(211)^7, (22)^6$	211	211, 211, 211	211, 211, 211	22, 22, 22	22, 22, 22

1. the rank distribution $3^1, 2^{12}$, equivalently the Taussky types $\mathrm{A}^1, \mathrm{B}^{12}$, with $1 \cdot 13 + 12 \cdot 4 = 13 + 48 = 61$ unramified nonic but not necessarily Galois extensions $E_{j,\ell}/K$;

2. the rank distribution $3^4, 2^9$, equivalently the Taussky types $\mathrm{A}^4, \mathrm{B}^9$, with $4 \cdot 13 + 9 \cdot 4 = 52 + 36 = 88$ unramified nonic but not necessarily Galois extensions $E_{j,\ell}/K$;

3. the rank distribution $3^7, 2^6$, equivalently the Taussky types $\mathrm{A}^7, \mathrm{B}^6$, with $7 \cdot 13 + 6 \cdot 4 = 91 + 24 = 115$ unramified nonic but not necessarily Galois extensions $E_{j,\ell}/K$.

We avoid the computation of 61, respectively 88, respectively 115, extensions $E_{j,\ell}$ and their 3-class groups $\mathrm{Cl}_3(E_{j,\ell})$ by using *isomorphisms to representatives* $R_i \simeq E_{j,\ell}$.
Firstly, we only need *seven* extensions (Table 7), the four abelian B_1, B_5, B_6, B_7 and three non-Galois E_j, one of each triplet of three isomorphic fields, in the *first layer* of unramified cyclic cubic relative extensions $E_1, \ldots, E_{13}$ of K, which are of absolute degree 9. Their 3-class groups $[\mathrm{Cl}_3(E_j)]_{j=1}^{13}$ constitute the Artin pattern $\alpha(K) = \alpha_1(K)$ of *first order* of K.
(In the column #, the symbol n/m denotes n conjugacy classes with m members each.)

Table 7. Isomorphisms and representatives among extensions of degree 9

Sc.	ρ		#	Rep.	Abelian	#	Rep.	Non-Galois		Census
(1)	$3^1, 2^{12}$	1	1	B_1	21^2, $\langle 9, 2\rangle$				Rep.	$1 + 3 + 3 = 7$
		12	3	B_5, B_6, B_7	2^2, $\langle 9, 2\rangle$	3/3	E_1, E_2, E_3	2^2, $\langle 27, 3\rangle$	Tot.	$1 + 3 + 9 = 13$
(2)	$3^4, 2^9$	4	1	B_1	21^2, $\langle 9, 2\rangle$	1/3	E_1	21^2, $\langle 27, 3\rangle$	Rep.	$1 + 1 + 3 + 2 = 7$
		9	3	B_5, B_6, B_7	2^2, $\langle 9, 2\rangle$	2/3	E_2, E_3	2^2, $\langle 27, 3\rangle$	Tot.	$1 + 3 + 3 + 6 = 13$
(3)	$3^7, 2^6$	7	1	B_1	21^2, $\langle 9, 2\rangle$	2/3	E_1, E_2	21^2, $\langle 27, 3\rangle$	Rep.	$1 + 2 + 3 + 1 = 7$
		6	3	B_5, B_6, B_7	2^2, $\langle 9, 2\rangle$	1/3	E_3	2^2, $\langle 27, 3\rangle$	Tot.	$1 + 6 + 3 + 3 = 13$

Secondly, among the fields of absolute degree 27 in the *second layer*, we can restrict our class group computations to

1. *eight* for the first scenario with rank distribution $(3^1, 2^{12})$,

2. *eleven* for the second scenario with rank distribution $(3^4, 2^9)$, and

3. *fourteen* for the third scenario with rank distribution $(3^7, 2^6)$ (Table 8).

For each scenario, only the 3-genus field $K^* =: R_1$ is abelian with absolute Galois group $\langle 27, 5\rangle$, the other fields may be Galois with group $\langle 27, 3\rangle$ or non-Galois with various groups of the splitting field: $\langle 729, 411\rangle$ for $\rho = (3^4, 2^9)$ or $\rho = (3^7, 2^6)$, and $\langle 81, 12\rangle$, $\langle 243, 58\rangle$ for all scenarios. Their 3-class groups $[(\mathrm{Cl}_3(E_{j,\ell}))_{\ell=1}^{n_j}]_{j=1}^{13}$ constitute the Artin pattern $\alpha_2(K)$ of *second order* of K.

Table 8. Isomorphisms and representatives among extensions of degree 27

Sc.	ρ		#	Rep.	Abelian	#	Rep.	Galois	#	Rep.	Non-Galois
(1)	$3^1, 2^{12}$	1	1	R_1	$\langle 27,5\rangle$	3	R_2, R_3, R_4	$\langle 27,3\rangle$	1/9	R_8	$\langle 243,58\rangle$
		3	3	R_1	$\langle 27,5\rangle$				3/3	R_5, R_6, R_7	$\langle 81,12\rangle$
		9				9	R_2, R_3, R_4	$\langle 27,3\rangle$	9/3	R_5, R_6, R_7	$\langle 81,12\rangle$
	Subtotal		4			12			45	Total	$4+12+45=61$
(2)	$3^4, 2^9$	1	1	R_1	$\langle 27,5\rangle$	3	R_2, R_3, R_4	$\langle 27,3\rangle$	1/9	R_8	$\langle 243,58\rangle$
		3				3	R_2	$\langle 27,3\rangle$	3/3	R_5, R_6, R_7	$\langle 81,12\rangle$
									3/9	R_9, R_{10}, R_{11}	$\langle 729,411\rangle$
		3	3	R_1	$\langle 27,5\rangle$				3/3	R_5, R_6, R_7	$\langle 81,12\rangle$
		6				6	R_3, R_4	$\langle 27,3\rangle$	6/3	R_5, R_6, R_7	$\langle 81,12\rangle$
	Subtotal		4			12			72	Total	$4+12+72=88$
(3)	$3^7, 2^6$	1	1	R_1	$\langle 27,5\rangle$	3	R_2, R_3, R_4	$\langle 27,3\rangle$	1/9	R_8	$\langle 243,58\rangle$
		3				3	R_2	$\langle 27,3\rangle$	3/3	R_5, R_6, R_7	$\langle 81,12\rangle$
									3/9	R_9, R_{10}, R_{11}	$\langle 729,411\rangle$
		3				3	R_3	$\langle 27,3\rangle$	3/3	R_5, R_6, R_7	$\langle 81,12\rangle$
									3/9	R_{12}, R_{13}, R_{14}	$\langle 729,411\rangle$
		3	3	R_1	$\langle 27,5\rangle$				3/3	R_5, R_6, R_7	$\langle 81,12\rangle$
		3				3	R_4	$\langle 27,3\rangle$	3/3	R_5, R_6, R_7	$\langle 81,12\rangle$
	Subtotal		4			12			99	Total	$4+12+99=115$

7. Computations

The computations were performed using the PARI/GP [38] computer algebra system. The two most important steps are the computations of class groups and unit groups (performed by the GP function `bnfinit`) and the computations of class fields (performed by the GP function `bnrclassfield`). A call to `bnrclassfield` for a field uses Kummer theory and requires calls to `bnfinit` for both the field and its extension by the third roots of unity (in our case). So the actual computation consists of these steps, starting from a suitable cyclic cubic field:

1. compute the class group and unit group of the cubic field
2. compute the class group and unit group of the cubic field extended with the 3-root of unity
3. compute class fields of degree 9
4. for each class field,
   (a) compute its class group and unit group
   (b) compute the class group and unit group of the field extended with the 3-root of unity
   (c) compute class fields of degree 27
   (d) compute the class groups of each class fields.

As a result, for each cyclic cubic field, we need to call `bnfinit` on one field of degree 3, one field of degree 6, 7 fields of degree 9, 7 fields of degree 18, and either 8, 11 or 14 non-isomorphic fields of degree 27.

The function `bnfinit` is an implementation of Buchmann subexponential algorithm for class groups and unit groups [13] by Cohen–Diaz–Olivier [15], [14]. It is based on searching relations between ideals in a set of prime ideals that generates the class group, and is correct under the assumption of the Riemann hypothesis for all Hecke L-functions attached to non-trivial characters of the ideal class group [10].

This computation was done with a specially-tuned, parallel version of this function. The set of primes is chosen by applying Grenié–Molteni [19] improvement to Belabas–Diaz–Friedman [9] criterion to find a small set of primes generating the class group (under GRH). It proceeds by searching in parallel for smooth elements in the ideals obtained by applying LLL-reduction to the ideals $\mathfrak{p}^6\mathfrak{q}$ for all pairs of ideals $(\mathfrak{p},\mathfrak{q})$ in the set, and looking for small vectors for the T_2 quadratic form (the sum of the square of the absolute value of the conjugates). The tuning parameters decide how far the program will search in each such ideal. It was regularly increased to account for the increase in the fields discriminant over the course of the computation. While the units are not required for the fields of degree 27, we still used the compact units representation of units because precision increases would be parallelised and were less expensive than with the logarithmic embedding representation. The program was run on several 128-core CPU with 1TB of RAM over the course of several months, using the internal POSIX threads parallel engine of PARI/GP.

Independently of the GRH assumption, any computer calculation can be incorrect due to hardware or software errors. To alleviate this, we checked the internal coherency of the data and their agreement with the theory, which was always perfect. Further we rechecked the computations leading to groups of order 6561, using different computers and different tuning parameters, including increasing the set of primes. Due to this, even if the Riemann hypothesis does not hold for some of the relevant L-functions, the result would likely still be correct.

8. Historical remarks

In April 2002, the second author used the *Voronoi algorithm* [45], implemented in Delphi (Object Pascal), and the *Euler product* method in order to compute the 15851 cyclic cubic fields K with conductors $c_{K/\mathbb{Q}} < 10^5$ and their class numbers, which cover the range $1 \leq h_K \leq 1953$. Among the fields, 4785 occur as singlets, 7726 in doublets, 3132 in quartets, and 208 in octets. Twenty years later, in July 2022, the second author confirmed these results, extended by the class group structures $\mathrm{Cl}(K)$ under the GRH. The cyclic cubic fields K were constructed as *ray class fields* over the rational number field, using Fieker's class field theoretic routines [16] in MAGMA [23]. Additionally, he constructed the 13 *unramified cyclic cubic relative extensions* E_j/K of absolute degree 9, whenever the 3-class group of K was $\mathrm{Cl}_3(K) \simeq (3,3,3)$, which was the primary goal for the reconstruction [34] in view of the intended realization of Andozhskii–Tsvetkov (AT-)groups with HBC.

In January 2023, both authors started a computationally extremely challenging search for cyclic cubic fields K with HBC and conductors $10^5 < c < 3 \cdot 10^6$. Since the 3-class tower group $\mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$ can only be identified by *AQI of second order* (Table 5), the first author constructed the unramified cyclic cubic relative extensions $E_{j,\ell}/E_j$ of absolute degree 27 and the class groups $\mathrm{Cl}(E_{j,\ell})$, $1 \leq j \leq 13$, $1 \leq \ell \leq n_j$, under the GRH, forming the *Artin pattern* α_2 *of second order*,

$$\alpha_2(K) := [\mathrm{Cl}(E_j); (\mathrm{Cl}(E_{j,\ell}))_{\ell=1}^{n_j}]_{j=1}^{13}. \tag{24}$$

Here, $1 \leq \ell \leq 4$, $n_j = 4$, when $\mathrm{Cl}_3(E_j) \simeq (2^2) \mathrel{\hat=} (9,9)$, and $1 \leq \ell \leq 13$, $n_j = 13$, when $\mathrm{Cl}_3(E_j) \simeq (21^2) \mathrel{\hat=} (9,3,3)$. Since this was impossible even on bigger workstations, due to the required RAM storage and CPU time, the first author had to employ super computers with 128 cores and 1TB RAM, which enabled highly parallel processes with PARI/GP [38] under the GRH. *Prototypes* with minimal conductors c in **boldface** font, realizing the vertex as 3-class field tower group, are visualized in surrounding ovals in Figure 3, which is similar to Figure 1, but restricted to AT-groups and their ancestors and siblings.

9. Invariants of realizing cyclic cubic fields

In Table 9, we present all 37 conductors $c < 10^6$, and a few $c > 10^6$, of the OEIS sequence A359310 [43].

The table gives the prime factors of c (admitting also the prime power $3^2 = 9$) with graphs $q_2 \leftarrow q_1 \rightarrow q_3$. They give rise to quartets of cyclic cubic number fields (K_1, K_2, K_3, K_4) with 3-class groups $\mathrm{Cl}_3(K_1) \simeq (3,3,3)$ and $\mathrm{Cl}_3(K_i) \simeq (3,3)$ for $2 \le i \le 4$. The *rank distribution* of the first order Artin pattern of $K := K_1$ with respect to the 13 unramified cyclic cubic extensions E_j/K is given by

$$\rho(K) := (\ \mathrm{rank}_3(\mathrm{Cl}_3(E_j))\ )_{j=1}^{13}, \tag{25}$$

denoted with symbolic exponents which indicate iteration. Finally, the unique or ambiguous candidate for the metabelian 3-class tower group $\mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$ is given by its absolute identifier in the SmallGroups database [11, 12, 22] (a vertical bar | means “or”).

10. Conclusion

Generally, our investigation of the 3-class field tower of cyclic cubic fields K with *elementary tricyclic* 3-class group $\mathrm{Cl}_3(K) \simeq (3,3,3)$ is a striking novelty [34]. Similar attempts with imaginary quadratic fields of type $(3,3,3)$, where all capitulation kernels are of order $\#L = 3$ (lines), but not harmonically balanced, successfully yielded the Artin pattern $\mathrm{AP} = (\alpha, \varkappa)$ by means of arithmetic computations [26, §7.2, Tbl. 2–4, pp. 308–311] but were doomed to group theoretic failure, since the order of relevant groups is at least 3^{31} and the complexity of descendant trees became unmanageable [26, §7.4, p. 312], [27, §10, p. 54], [31, Thm. 8.2, p. 174], [29, §8, pp. 98–99], [30, §2, Example, p. 6]. Therefore, we are delighted that cyclic cubic fields of type $(3,3,3)$ impose much less severe requirements on the second 3-class group $\mathfrak{M} = \mathrm{Gal}(\mathrm{F}_3^2(K)/K)$, since capitulation kernels of order $\#P = 9$ (planes) and even $\#O = 27$ (full space) are admissible [34]. However, in the present work our attention is devoted to cyclic cubic fields with *harmonically balanced capitulation* (HBC), where all transfer kernels are of order $\#L = 3$ (lines), but relevant groups set in at order $3^6 = 729$ already (Table 4). Our foremost target was the realization of *closed Andozhskii–Tsvetkov groups* (AT-groups) S with coinciding generator- and relation-rank $d_2(S) = d_1(S)$.

The main results concerning the realization of closed AT-groups S of order $\#S = 3^8 = 6561$ Theorem 6.1, their ancestors A of order $\#A = 3^6 = 729$ Corollary 6.2, and their siblings G of order $\#G = 3^7 = 2187$ Corollary 6.6 by Galois groups of maximal unramified pro-3-extensions of cyclic cubic fields are illustrated in Figure 3, where *prototypes* of cyclic cubic number fields K with minimal conductors c in **boldface** font are visualized in ovals surrounding the group $\mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$ with HBC.

Table 9. Conductors of cyclic cubic number fields with HBC

No.	c	$q_2 \leftarrow q_1 \rightarrow q_3$	ρ	$\mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$	Thm./Cnj.
1	59 031	$3^2 \leftarrow 937 \rightarrow 7$	$(3^1, 2^{12})$	$\langle 2187, 4670\rangle$	6.9
2	209 853	$3^2 \leftarrow 3331 \rightarrow 7$	$(3^7, 2^6)$	$\langle 729, 133\rangle$	6.8
3	247 437	$3^2 \leftarrow 19 \rightarrow 1447$	$(3^1, 2^{12})$	$\langle 729, 136\rangle$	6.9
4	263 017	$109 \leftarrow 19 \rightarrow 127$	$(3^7, 2^6)$	$\langle 729, 133\rangle$	6.8
5	271 737	$3^2 \leftarrow 109 \rightarrow 277$	$(3^4, 2^9)$	$\langle 729, 137\rangle$	6.10
6	329 841	$67 \leftarrow 3^2 \rightarrow 547$	$(3^1, 2^{12})$	$\langle 729, 136\rangle$	6.9
7	377 923	$7 \leftarrow 13 \rightarrow 4153$	$(3^1, 2^{12})$	$\langle 729, 136\rangle$	6.9
8	407 851	$37 \leftarrow 73 \rightarrow 151$	$(3^7, 2^6)$	$\langle 729, 133\rangle$	6.8
9	412 909	$7 \leftarrow 967 \rightarrow 61$	$(3^1, 2^{12})$	$\langle 729, 136\rangle$	6.9
10	415 597	$7 \leftarrow 13 \rightarrow 4567$	$(3^1, 2^{12})$	$\langle 2187, 4670\rangle$	6.9
11	416 241	$3^2 \leftarrow 6607 \rightarrow 7$	$(3^1, 2^{12})$	$\langle 729, 136\rangle$	6.9
12	416 727	$3^2 \leftarrow 19 \rightarrow 2437$	$(3^4, 2^9)$	$\langle 2187, 4673 \vert 4675\rangle$	6.10
13	462 573	$103 \leftarrow 3^2 \rightarrow 499$	$(3^7, 2^6)$	$\langle 2187, 4662 \vert 4663\rangle$	6.8
14	474 561	$67 \leftarrow 3^2 \rightarrow 787$	$(3^4, 2^9)$	$\langle 729, 137\rangle$	6.10
15	487 921	$7 \leftarrow 43 \rightarrow 1621$	$(3^4, 2^9)$	$\langle 2187, 4673 \vert 4675\rangle$	6.10
16	493 839	$3^2 \leftarrow 37 \rightarrow 1483$	$(3^1, 2^{12})$	$\langle 2187, 4670\rangle$	6.9
17	547 353	$61 \leftarrow 3^2 \rightarrow 997$	$(3^4, 2^9)$	$\langle 2187, 4673 \vert 4675\rangle$	6.10
18	586 963	$163 \leftarrow 13 \rightarrow 277$	$(3^1, 2^{12})$	$\langle 729, 136\rangle$	6.9
19	612 747	$103 \leftarrow 3^2 \rightarrow 661$	$(3^7, 2^6)$	$\langle 729, 133\rangle$	6.8
20	613 711	$73 \leftarrow 7 \rightarrow 1201$	$(3^4, 2^9)$	$\langle 729, 137\rangle$	6.10
21	615 663	$67 \leftarrow 3^2 \rightarrow 1021$	$(3^4, 2^9)$	$\langle 729, 137\rangle$	6.10
22	622 063	$13 \leftarrow 109 \rightarrow 439$	$(3^1, 2^{12})$	$\langle 2187, 4670\rangle$	6.9
23	648 427	$13 \leftarrow 31 \rightarrow 1609$	$(3^4, 2^9)$	$\langle 729, 137\rangle$	6.10
24	651 829	$37 \leftarrow 223 \rightarrow 79$	$(3^4, 2^9)$	$\langle 729, 137\rangle$	6.10
25	**689 347**	$37 \leftarrow 31 \rightarrow 601$	$(3^4, 2^9)$	$\langle \mathbf{6561}, 217713 \vert 217717\rangle$	6.10
26	690 631	$19 \leftarrow 163 \rightarrow 223$	$(3^4, 2^9)$	$\langle 729, 137\rangle$	6.10
27	**753 787**	$97 \leftarrow 19 \rightarrow 409$	$(3^1, 2^{12})$	$\langle \mathbf{6561}, 217710\rangle$	6.9
28	**796 779**	$3^2 \leftarrow 397 \rightarrow 223$	$(3^1, 2^{12})$	$\langle \mathbf{6561}, 217710\rangle$	6.9
29	811 069	$7 \leftarrow 1063 \rightarrow 109$	$(3^1, 2^{12})$	$\langle 729, 136\rangle$	6.9
30	818 217	$3^2 \leftarrow 397 \rightarrow 229$	$(3^1, 2^{12})$	$\langle 729, 136\rangle$	6.9
31	**869 611**	$19 \leftarrow 37 \rightarrow 1237$	$(3^4, 2^9)$	$\langle \mathbf{6561}, 217713 \vert 217717\rangle$	6.10
32	914 263	$7 \leftarrow 211 \rightarrow 619$	$(3^7, 2^6)$	$\langle 729, 133\rangle$	6.8
33	915 439	$19 \leftarrow 7 \rightarrow 6883$	$(3^4, 2^9)$	$\langle 2187, 4673 \vert 4675\rangle$	6.10
34	922 167	$3^2 \leftarrow 1297 \rightarrow 79$	$(3^1, 2^{12})$	$\langle 729, 136\rangle$	6.9
35	936 747	$3^2 \leftarrow 14869 \rightarrow 7$	$(3^7, 2^6)$	$\langle 2187, 4662 \vert 4663\rangle$	6.8
36	977 409	$3^2 \leftarrow 487 \rightarrow 223$	$(3^4, 2^9)$	$\langle 729, 137\rangle$	6.10
37	997 087	$7 \leftarrow 13 \rightarrow 10957$	$(3^4, 2^9)$	$\langle 729, 137\rangle$	6.10
40	1 083 607	$7 \leftarrow 547 \rightarrow 283$	$(3^7, 2^6)$	$\langle 729, 133\rangle$	6.8
43	1 181 971	$19 \leftarrow 7 \rightarrow 8887$	$(3^7, 2^6)$	$\langle 729, 133\rangle$	6.8
49	1 295 329	$7 \leftarrow 211 \rightarrow 877$	$(3^7, 2^6)$	$\langle 729, 133\rangle$	6.8
50	1 323 007	$331 \leftarrow 7 \rightarrow 571$	$(3^7, 2^6)$	$\langle 729, 133\rangle$	6.8
53	**1 406 551**	$181 \leftarrow 19 \rightarrow 409$	$(3^7, 2^6)$	$\langle \mathbf{6561}, 217702 \vert 217703\rangle$	6.8

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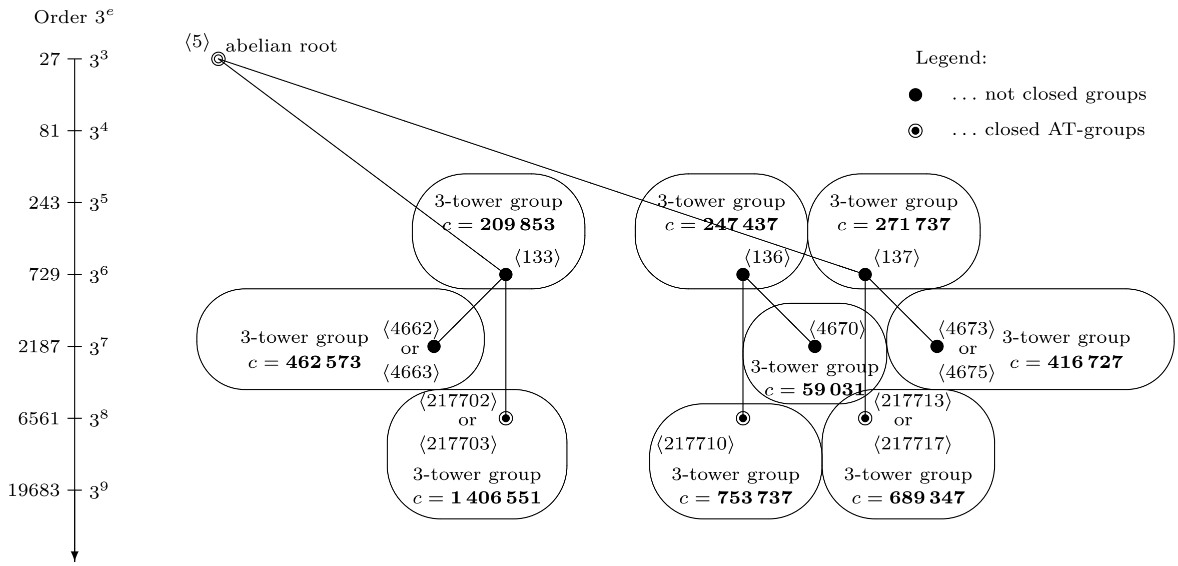

FIGURE 3. Realization of groups G with HBC as $\mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$

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