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VARIOUS PRODUCTS OF REPRESENTATIVE SERIES

by

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Abstract. — For factorizing and decomposing the representative (or rational) series, with coefficients in a commutative ring A containing $\mathbb{Q}$, we examine various products (such as concatenation, shuffle, quasi-shuffle) defined on the free monoid which are such that their associated bialgebras are isomorphic to the Sweedler's dual, for A being a field K.

Résumé. — Pour factoriser et décomposer la série représentative (ou rationnelle) à coefficients dans un anneau commutatif A contenant $\mathbb{Q}$, nous examinons divers produits (tels que concaténation, shuffle, quasi-shuffle) définis sur le monoïde libre qui sont tels que leurs bialgèbres associées sont isomorphes au dual de Sweedler, pour A étant un corps K.

1. Introduction

Formal series in noncommutative variables have been introduced for the first time by M.P. Schützenberger [48] to study problems related to theoretical computer science, such as language theory and the theory of automata. He generalizes the theorem of S.C. Kleene [36] to noncommutative formal series [47]. He then showed the fundamental role played, for the study of noncommutative formal series, by matrix representations of free monoids. In particular irreducible representations, allowed him to find fine results on the growth of coefficients [49]. After M.P. Schützenberger, we must cite the work of M. Fliess [22], G. Jacob [32] and C. Reutenauer [43]. They developed sets of fundamental tools for combinatorial studies of free monoids, linked to the theory of automata. Specifically, M. Fliess, using the Hankel matrices, characterizes noncommutative rational series (a series is rational if and only if the rank of its Hankel matrix is finite) and series with positive coefficients. Furthermore, using matrix representations, G. Jacob generalized the notion of loops in finite automata (star lemma) which enables to solve problems of decidability of the finiteness of the coefficients of rational series. C. Reutenauer characterized rational series by their syntactic algebra (a series is rational if and only if the dimension of its syntactic algebra is finite) and defined the notion of varieties of formal series in the sense of S. Eilenberg [21]. These works are widely reproduced in books by J. Berstel and C. Reutenauer [1], by W. Kuich and A. Salomaa [38] and by A. Salomaa and M. Soittola [46].

Key words and phrases. — noncommutative rational series, representative series, Sweedler's dual.

As N. Chomsky and M.P. Schützenberger showed that the algebraic languages are the supports of algebraic series [12] as being solutions of a system of propre algebraic equations with integer coefficients. M. Nivat [42] and then M. Fliess [22] showed that the study of noncommutative algebraic series mainly depends on the study of rational transductions[1] of algebraic series. Introducing the notion of regulated transduction (to establish the converse of the Shamir's theorem), G. Jacob showed that the image by regulated rational (resp. algebraic) transduction of a noncommutative rational (resp. algebraic) series is a noncommutative rational (resp. algebraic) series, opening a way to the study of nice families of noncommutative formal series. W. Kuich further developed this method in an approach by cycle-free push-down automata to study noncommutative algebraic series [37]. Noncommutative series therefore benefited from knowledge of language theories and automata. In return, the techniques and the results in [22, 32, 43] enabled new visions in these fields [1]. These developments made the algebra of formal series a preferred tool for the syntactic study of operator algebras [45]. Furthermore, algebra of formal series proved to be a particularly well-suited tool for implementations of effective calculations in modern computer algebra systems [34, 35].
In particular, since the input-output behaviors of nonlinear dynamical systems (or causal functionals) was encoded by noncommutative series (see [26, 33] for an introduction), the noncommutative symbolic computation (a generalization of the Heaviside's calculus) became efficient for dealing with special functions[2] (hypergeometric function, hyperlogarithm, polylogarithm [18, 28, 29, 30, 39]) in the study of (nonlinear) differential equations in control theory [23, 24, 31, 33, 44] and in quantum electrodynamics (QED) [5, 6, 18, 30, 31]. Let us consider, for instance, the following nonlinear dynamical system

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}z}q(z) = A_0(q)u_0(z) + \cdots + A_m(q)u_m(z), \\ q(z_0) = \eta, \\ y(z) = f(q(z)), \end{cases} \tag{1}$$

where

1. y is the output,
2. the vector state $q = (q_1, \ldots, q_n)$ belongs to a complex holomorphic manifold $\mathcal{M}$ of dimension n,
3. the observation f is defined within a fixed connected neighbourhood[3] U of the initial state η,
4. the vector fields $(A_i)_{i=0,\ldots,m}$ are defined, with respect to the coordinates, by

$$A_i = \sum_{j=1}^{n} A_i^j(q)\frac{\partial}{\partial q_j}, \quad \text{with} \quad A_i^j(q) \in \mathcal{H}(U), \tag{2}$$

[1]It will be pratical for transforming rational series over an alphabet X to rational series over another alphabet Y with coeffients in a ring A (see Examples 4.6–4.7 and Remark 4.8 below).
[2]In [29], an overview of main results already obtained [5, 17, 28, 29] concerning polylogarithms, harmonic sums and polyzetas which are indexed by words as well as by rational series, using the present algebraic framework.
[3]In this introductive description, the points are loosely identified with their coordinates through some chart $\varphi_U : U \to \mathbb{C}^n$ likewise, in [44], the space of holomorphic functions $\mathcal{H}(U)$ is described by $\mathbb{C}^{\mathrm{cv}}[\![q_1, \ldots, q_n]\!]$.

5. the inputs[4] $(u_i)_{i=0,\ldots,m}$, as well as their inverses $(u_i^{-1})_{i=0,\ldots,m}$, belong to the subring $\mathcal{C}_0$ of the ring of holomorphic functions $\mathcal{H}(\Omega)$ with the neutral element $1_{\mathcal{H}(\Omega)}$ over the simply connected manifold Ω.

It is convenient (and possible) to separate the contribution of the vector fields $(A_i)_{i=0,\ldots,m}$ and that of the differential forms $(\omega_i)_{i=0,\ldots,m}$, defined by the inputs, i.e. $\omega_i(z) = u_i(z)dz$, through the encoding alphabet $X = \{x_i\}_{i=0,\ldots,m}$ which generates the free monoid X^* with the neutral element 1_{X^*}. Indeed, the output y (depending on z_0) can be computed by

$$y(z) = \sum_{w\in X^*} \alpha_{z_0}^z(w)\mathcal{Y}(w)f_{|_\eta}, \tag{3}$$

as the pairing (under suitable convergence conditions [18, 24, 44]) between the Chen series of $(\omega_i)_{i=0,\ldots,m}$ along the path $z_0 \rightsquigarrow z$ over Ω [8, 11, 25] and the generating series of (1) [24], defined as follows

$$C_{z_0\rightsquigarrow z} := \sum_{w\in X^*} \alpha_{z_0}^z(w)\, w \in \mathcal{H}(\Omega)\langle\langle X\rangle\rangle, \tag{4}$$

$$\sigma f := \sum_{w\in X^*} \mathcal{Y}(w)f\, w \in \mathcal{H}(U)\langle\langle X\rangle\rangle, \tag{5}$$

where, in (3)–(5), the iterated integral $\alpha_{z_0}^z(w)$ and the differential operator $\mathcal{Y}(w)$, are computed, from the word $w \in X^*$, recursively as follows

$$\begin{cases} \alpha_{z_0}^z(w) = 1_{\mathcal{H}(\Omega)} & \text{and} \quad \mathcal{Y}(w) = \mathrm{Id}, \qquad\qquad \text{for } w = 1_{X^*}, \\ \alpha_{z_0}^z(w) = \displaystyle\int_{z_0}^z \omega_i(s)\alpha_{z_0}^s(v) & \text{and} \quad \mathcal{Y}(w) = A_i \circ \mathcal{Y}(v), \quad \text{for } w = x_iv,\ x_i \in X,\ v \in X^*. \end{cases} \tag{6}$$

There is a large literature concerning Chen series (see [8, 11, 25] and their bibliographies, see also [6] for our study). In the present work, applications are focussing on σf as being rational series for which (1) is transformed into the following form [23, 24]

$$\frac{\mathrm{d}}{\mathrm{d}z}q(z) = M_0(q)u_0(z) + \cdots + M_m(q)u_m(z), q(z_0) = \eta, y(z) = \nu q(z) \tag{7}$$

where $\{M_i\}_{1\le i\le n}$ and ν are matrices in, respectively, $\mathcal{M}_{n,n}(A)$ and $\mathcal{M}_{1,n}(A)$ and the vectors fields (2) can be expressed as follows

$$A_i = \sum_{j=1}^n \left(\sum_{k=1}^n M_{k,j}q_k\right)\frac{\partial}{\partial q_j}. \tag{8}$$

Letting μ be the morphism from X^* to $\mathcal{M}_{n,n}(A)$ mapping x_i to M_i, the generating series $\sigma f_{|_\eta}$ in (3) admits (ν, μ, η) as linear representation, of rank n (see Section 4 below) such that, for any $w \in X^*$,

$$\langle \sigma f_{|_\eta} \,|\, w\rangle = \mathcal{Y}(w)f_{|_\eta} = \nu\mu(w)\eta. \tag{9}$$

***Example 1.1* (Hypergeometric equation).** — Let t_0, t_1, t_2 be parameters and

$$z(1-z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}y(z) + [t_2 - (t_0 + t_1 + 1)z]\frac{\mathrm{d}}{\mathrm{d}z}y(z) - t_0t_1y(z) = 0.$$

[4] For any $0 \le i \le m$, if $u_i \in \mathbb{C}(z)$ then y can be expressed using elementary or hypergeometric functions, hyperlogarithms and, in particular, polylogarithms (see [13, 18, 39]).

Let $q_1(z) = -y(z)$ and $q_2(z) = (1-z)d/dz\ y(z)$. One has

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix} q_1(z) \\ q_2(z) \end{pmatrix} = (M_0 u_0(z) + M_1 u_1(z)) \begin{pmatrix} q_1(z) \\ q_2(z) \end{pmatrix},$$

where $u_i \in \mathbb{C}(z)$ and $M_i \in \mathcal{M}_{2,2}(\mathbb{C}[t_0, t_1, t_2])$ $(i = 0, 1)$ and

$$u_0(z) = \frac{1}{z},\ u_1(z) = \frac{1}{1-z} \quad \text{and} \quad M_0 = -\begin{pmatrix} 0 & 0 \\ t_0 t_1 & t_2 \end{pmatrix},\ M_1 = -\begin{pmatrix} 0 & 1 \\ 0 & t_2 - t_0 - t_1 \end{pmatrix}.$$

Or equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}z} q(z) = A_0(q) u_0(z) + A_1(q) u_1(z) \quad \text{and} \quad y(z) = -q_1(z),$$

where A_0 and A_1 are the following parametrized linear vector fields

$$A_0 = -(t_0 t_1 q_1 + t_2 q_2)\frac{\partial}{\partial q_2} \quad \text{and} \quad A_1 = -q_1 \frac{\partial}{\partial q_1} - (t_2 - t_0 - t_1) q_2 \frac{\partial}{\partial q_2}.$$

acting by

$$\frac{\partial}{\partial q_1}(q) = \frac{\partial}{\partial q_1}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \frac{\partial}{\partial q_2}(q) = \frac{\partial}{\partial q_2}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In this work, using Lazard and Schützenberger monoidal factorizations [40, 52] and extending results of [13, 18, 29] (already obtained over $\mathbb{C}$), the representative series, with coefficients in the commutative ring A containing $\mathbb{Q}$ (see examples in (4)–(5)) are factorized and decomposed, as function on monoids within their associated bialgebras [7, 9, 10]. The organization of the paper is following.

1. In Section 2, we will examine combinatorial aspects of various products and coproducts (concatenation, shuffle, quasi-shuffle) for which group like and primitive elements will be characterized, using Proposition 2.6.

2. In Section 3, applying Theorems 3.1–3.2, pairs of dual bases (see (72)–(73) and (75)–(76)) of shuffle and quasi-shuffle A-bialgebras will be constructed.

3. After that, in Section 4, to determine the output y in (3), the noncommutative generating series[5] σf in (5) will be computed (see Propositions 4.5–4.16, Theorems 4.3–4.21 and Corollaries 4.9–4.19). It will be also computed, according to the (commutative or nilpotent or solvable) Lie algebra generated by the matrices $\{M_i\}_{1\le i\le n}$ [50] in the case where A is an algebraically closed fields of characteristic zero K (see Theorem 4.23).

2. Various products and coproducts

In the sequel, as already said in Section 1, A denotes a commutative ring containing $\mathbb{Q}$ and $\mathcal{X}$ denotes a finite alphabet (as $X = \{x_0, x_1\}$) or infinite alphabet (as $Y = \{y_k\}_{k\ge1}$) genetating the free monoid $(\mathcal{X}^*, 1_{\mathcal{X}^*})$, for the concatenation product (denoted by conc and omitted when there is no ambiguity). One also denotes

(10) $$\mathcal{X}^+ = \mathcal{X}^*\mathcal{X} = \mathcal{X}^* \setminus \{1_{\mathcal{X}^*}\}.$$

[5]See [26] for various approximations of $\sigma f_{|\eta}$, in (3), by noncommutative rational series.

An element of $\mathcal{X}^*$ (resp. $\mathcal{X}$) is called *word* (resp. *letter*). In particular, the empty word is the neutral element of the monoid, i.e. $1_{\mathcal{X}^*}$. The *length* of a word w in $\mathcal{X}^*$ is denoted by $|w|$. A *series* S is a map $\mathcal{X}^* \longrightarrow A$, mapping w to $\langle S \,|\, w\rangle$ (called *coefficient* of w in S), and its *graph* is described as follows

$$S = \sum_{w\in\mathcal{X}^*} \langle S \,|\, w\rangle w. \tag{11}$$

The constant term of S is $\langle S \,|\, 1_{\mathcal{X}^*}\rangle$. If $\langle S \,|\, 1_{\mathcal{X}^*}\rangle = 0$ then S is said to be *propre*. The image of S is denoted by $\mathrm{Im}(S)$. The *support* of S is the following language

$$\mathrm{supp}(S) := \{w \in \mathcal{X}^* \,|\, \langle S \,|\, w\rangle \neq 0\}. \tag{12}$$

If $\mathrm{supp}(S)$ is finite then S is a *polynomial*. On defines the *degree* of the polynomial P as follows

$$\deg(P) := \max\{|w|\}_{w\in\mathrm{supp}(S)}. \tag{13}$$

P is *homogenous* in degree n, if it is linear combination of words of length n. The *characteristic series* of $L \subset \mathcal{X}^*$ (i.e. L is a *language*) is defined by (see [1])

$$\mathrm{char}(L) = \sum_{w\in L} w. \tag{14}$$

In particular, one still denotes, for convenience, the *characteristic series* of $\mathcal{X}$ (resp. $\mathcal{X}^*$) by $\mathcal{X}$ (resp. $\mathcal{X}^*$). The set of noncommutative formal series, over $\mathcal{X}$ with coefficients in A, is denoted by $A\langle\langle\mathcal{X}\rangle\rangle$:

$$A\langle\langle\mathcal{X}\rangle\rangle = A^{\mathcal{X}^*} \tag{15}$$

The sum of $S, R \in A\langle\langle\mathcal{X}\rangle\rangle$ and the mutiplication S by $\lambda \in A$ are given by

$$\forall\, w \in \mathcal{X}^*, \quad \langle S + T \,|\, w\rangle = \langle S \,|\, w\rangle + \langle T \,|\, w\rangle, \langle \lambda S \,|\, w\rangle = \lambda\langle S \,|\, w\rangle. \tag{16}$$

The set of noncommutative polynomials, over $\mathcal{X}$ with coefficients in A, is denoted by $A\langle\mathcal{X}\rangle$ and is an A-module. It admits $\{w\}_{w\in X^*}$ as linear basis:

$$A\langle X\rangle \cong A[X^*]. \tag{17}$$

By the following *pairing*[6]

$$A\langle\langle\mathcal{X}\rangle\rangle \otimes A\langle\mathcal{X}\rangle \longrightarrow A, \tag{18}$$

$$T \otimes P \longmapsto \langle T \,|\, P\rangle := \sum_{w\in\mathcal{X}^*} \langle T \,|\, w\rangle\langle P \,|\, w\rangle, \tag{19}$$

there is a natural duality between $A\langle\mathcal{X}\rangle$ and $A\langle\langle\mathcal{X}\rangle\rangle$, i.e. [1]

$$A\langle\langle\mathcal{X}\rangle\rangle = A\langle\mathcal{X}\rangle^{\vee}. \tag{20}$$

From (19), using the Kronecker delta, it follows that

$$\forall\, u, v \in \mathcal{X}^+, \quad \langle u \,|\, v\rangle = \delta_{u,v}. \tag{21}$$

Let $A\langle\langle\mathcal{X}\rangle\rangle$ be equipped the ultrametric distance defined by [1]

$$\forall\, S, T \in A\langle\langle\mathcal{X}\rangle\rangle, \quad d(S, T) = 2^{-\omega(S-T)}. \tag{22}$$

[6]This sum is finite because P is a polynomial.

where $\omega(S)$ is the valuation of $S \in A\langle\langle\mathcal{X}\rangle\rangle$ defined by [1]

$$\omega(S) := \begin{cases} +\infty & \text{if } S = 0, \\ \inf\{|w|\}_{w\in\mathrm{supp}(S)} & \text{if } S \neq 0, \end{cases} \tag{23}$$

For the discrete topology defined in (22), $A\langle\langle\mathcal{X}\rangle\rangle$ is a complete topological ring and $A\langle\mathcal{X}\rangle$ is a dense subring of $A\langle\langle\mathcal{X}\rangle\rangle$, i.e. $\widehat{A\langle\mathcal{X}\rangle} = A\langle\langle\mathcal{X}\rangle\rangle$ [1].
Let $\mathcal{L}ie_A\langle\mathcal{X}\rangle$ be the smallest A-submodule of $A\langle\mathcal{X}\rangle$, contaning $\mathcal{X}$ and being closed for by the Lie bracket defined, for any P and $Q \in A\langle\mathcal{X}\rangle$, by [40, 45, 52]

$$[P, Q] = PQ - QP. \tag{24}$$

This bracket is anticommutative and satisfies the Jacobi identity [40, 45, 52]. Any element P of $\mathcal{L}ie_A\langle\mathcal{X}\rangle$ is called *Lie polynomial* and it is propre, i.e. $\langle P \,|\, 1_{\mathcal{X}^*}\rangle = 0$. It is also proved that $\mathcal{L}ie_A\langle\mathcal{X}\rangle$ is the free Lie algebra over A [40, 45, 52]. A series $S \in A\langle\langle\mathcal{X}\rangle\rangle$ is a *Lie series* if it is uniquely expressed as follows [45]

$$S = \sum_{k\geq 1} P_k, \tag{25}$$

where each P_k is a Lie polynomial, homogenous of weight k [40, 45, 52]. The set of Lie series over $\mathcal{X}$, with coefficients in A, is denoted by $\mathcal{L}ie_A\langle\langle\mathcal{X}\rangle\rangle$. One also defines the bracket, of two Lie series

$$S = \sum_{k\geq 1} P_k \quad \text{and} \quad R = \sum_{l\geq 1} Q_l \tag{26}$$

as follows

$$[S, R] = \sum_{k,l\geq 1} [P_k, Q_l]. \tag{27}$$

With this bracket, $\mathcal{L}ie_A\langle\langle\mathcal{X}\rangle\rangle$ is a Lie algebra over A.
As algebras the A-module $A\langle\mathcal{X}\rangle$ is equipped

1. The associative noncommutative and unital concatenation, i.e. the following bilinear map

$$\mathtt{conc} : A\langle\mathcal{X}\rangle \otimes A\langle\mathcal{X}\rangle \longrightarrow A\langle\mathcal{X}\rangle \tag{28}$$

or, equivalently, by the coproduct (with respected to the pairing in (19))

$$\Delta_{\mathtt{conc}} : A\langle\mathcal{X}\rangle \longrightarrow A\langle\mathcal{X}\rangle \otimes A\langle\mathcal{X}\rangle, \tag{29}$$

$$x \longmapsto 1_{\mathcal{X}^*} \otimes x + x \otimes 1_{\mathcal{X}^*} \tag{30}$$

such that, for any $u, v, w \in \mathcal{X}^*$ as follows

$$\langle \mathtt{conc}(u, v) \,|\, w\rangle = \langle uv \,|\, \Delta_{\mathtt{conc}} w\rangle. \tag{31}$$

$\Delta_{\mathtt{conc}}$ is a morphism for concatenation and then, for any $w \in \mathcal{X}^*$, one has

$$\Delta_{\mathtt{conc}} w = \sum_{u,v\in\mathcal{X}^*, uv=w} u \otimes v \tag{32}$$

2. The associative commutative and unital shuffle product, i.e. the following bilinear map

$$\sqcup\!\sqcup : A\langle\mathcal{X}\rangle \otimes A\langle\mathcal{X}\rangle \longrightarrow A\langle\mathcal{X}\rangle \tag{33}$$

defined, for any $x, y \in \mathcal{X}$ and $u, v, w \in \mathcal{X}^*$, by the following recursion

$$w \sqcup\!\sqcup 1_{\mathcal{X}^*} = 1_{\mathcal{X}^*} \sqcup\!\sqcup w = w \quad \text{and} \quad xu \sqcup\!\sqcup yv = x(u \sqcup\!\sqcup yv) + y(xu \sqcup\!\sqcup v), \tag{34}$$

or, equivalently, by the coproduct (with respected to the pairing in (19))

$$\Delta_{\sqcup\!\sqcup} : A\langle\mathcal{X}\rangle \longrightarrow A\langle\mathcal{X}\rangle \otimes A\langle\mathcal{X}\rangle, \tag{35}$$

$$x \longmapsto 1_{\mathcal{X}^*} \otimes x + x \otimes 1_{\mathcal{X}^*} \tag{36}$$

and such that, for any $u, v, w \in \mathcal{X}^*$,

$$\langle u \sqcup\!\sqcup v \,|\, w\rangle = \langle uv \,|\, \Delta_{\sqcup\!\sqcup}(w)\rangle. \tag{37}$$

It is a morphism for concatenation, i.e.

$$\forall\, u, v \in \mathcal{X}^*, \quad \Delta_{\sqcup\!\sqcup}(uv) = (\Delta_{\sqcup\!\sqcup} u)(\Delta_{\sqcup\!\sqcup} v). \tag{38}$$

3. Additionally, the A-module $A\langle Y\rangle$ is also equipped with the associative commutative and unital quasi-shuffle product, i.e. the following bilinear map

$$\uplus : A\langle Y\rangle \otimes A\langle Y\rangle \longrightarrow A\langle Y\rangle \tag{39}$$

defined, for any $u, v \in Y^*$ and $y_i, y_j \in Y$, by

$$u \uplus 1_{Y^*} = 1_{Y^*} \uplus u = u, \tag{40}$$

$$y_i u \uplus y_j v = y_i(u \uplus y_j v) + y_j(y_i u \uplus v) + y_{i+j}(u \uplus v), \tag{41}$$

or, equivalently, by the coproduct (with respected to the pairing in (19))

$$\Delta_{\uplus} : A\langle Y\rangle \longrightarrow A\langle Y\rangle \otimes A\langle Y\rangle, \tag{42}$$

$$y_k \longmapsto y_k \otimes 1_{Y^*} + 1_{Y^*} \otimes y_k + \sum_{i+j=k} y_i \otimes y_j \tag{43}$$

and such that, for any $u, v, w \in Y^*$,

$$\langle u \uplus v \,|\, w\rangle = \langle uv \,|\, \Delta_{\uplus}(w)\rangle. \tag{44}$$

It is also a `conc`-morphism, i.e.

$$\forall\, u, v \in Y^*, \quad \Delta_{\uplus}(uv) = (\Delta_{\uplus} u)(\Delta_{\uplus} v). \tag{45}$$

Now, let us extend the above products (i.e. `conc`, $\sqcup\!\sqcup$ and $\uplus$ in, respectively, (28), (33) and (39))

$$\texttt{conc}, \sqcup\!\sqcup : A\langle\langle\mathcal{X}\rangle\rangle \otimes A\langle\langle\mathcal{X}\rangle\rangle \longrightarrow A\langle\langle\mathcal{X}\rangle\rangle, \tag{46}$$

$$\uplus : A\langle\langle Y\rangle\rangle \otimes A\langle\langle Y\rangle\rangle \longrightarrow A\langle\langle Y\rangle\rangle \tag{47}$$

as follows (see also Remark 11 below)

$$\forall\, S,R\in A\langle\langle\mathcal{X}\rangle\rangle,\qquad SR=\sum_{w\in\mathcal{X}^*}\Bigg(\sum_{\substack{u,v\in\mathcal{X}^*\\ uv=w}}\langle S\,|\,u\rangle\langle R\,|\,v\rangle\Bigg)w, \tag{48}$$

$$\forall\, S,R\in A\langle\langle\mathcal{X}\rangle\rangle,\quad S\sqcup\!\sqcup R=\sum_{u,v\in\mathcal{X}^*}\langle S\,|\,u\rangle\langle R\,|\,v\rangle u\sqcup\!\sqcup v, \tag{49}$$

$$\forall\, S,R\in A\langle\langle Y\rangle\rangle,\quad S \uplus R=\sum_{u,v\in Y^*}\langle S\,|\,u\rangle\langle R\,|\,v\rangle u \uplus v, \tag{50}$$

and their coproducts ($\Delta_{\texttt{conc}}, \Delta_{\sqcup\!\sqcup}$ and $\Delta_{\uplus}$ in, respectively, (29), (35) and (42))

$$\Delta_{\texttt{conc}}, \Delta_{\sqcup\!\sqcup} : A\langle\langle\mathcal{X}\rangle\rangle \longrightarrow A\langle\langle\mathcal{X}^*\otimes\mathcal{X}^*\rangle\rangle, \tag{51}$$

$$\Delta_{\uplus} : A\langle\langle Y\rangle\rangle \longrightarrow A\langle\langle Y^*\otimes Y^*\rangle\rangle \tag{52}$$

as follows (see also Remark 11 below)

$$\forall\, S\in A\langle\langle\mathcal{X}\rangle\rangle,\quad \Delta_{\texttt{conc}}S=\sum_{w\in\mathcal{X}^*}\langle S\,|\,w\rangle\Delta_{\texttt{conc}}w\in A\langle\langle\mathcal{X}^*\otimes\mathcal{X}^*\rangle\rangle, \tag{53}$$

$$\forall\, S\in A\langle\langle\mathcal{X}\rangle\rangle,\qquad \Delta_{\sqcup\!\sqcup}S=\sum_{w\in\mathcal{X}^*}\langle S\,|\,w\rangle\Delta_{\sqcup\!\sqcup}w\in A\langle\langle\mathcal{X}^*\otimes\mathcal{X}^*\rangle\rangle, \tag{54}$$

$$\forall\, S\in A\langle\langle Y\rangle\rangle,\qquad \Delta_{\uplus}S=\sum_{w\in Y^*}\langle S\,|\,w\rangle\Delta_{\uplus}w\in A\langle\langle Y^*\otimes Y^*\rangle\rangle. \tag{55}$$

Remark 2.1. —

1. If $A=K$, is a field, then $K\langle\langle\mathcal{X}\rangle\rangle\otimes K\langle\langle\mathcal{X}\rangle\rangle$ embeds (injectively) in $K\langle\langle\mathcal{X}^*\otimes\mathcal{X}^*\rangle\rangle\cong[K\langle\langle\mathcal{X}\rangle\rangle]\langle\langle\mathcal{X}\rangle\rangle$. Indeed, $K\langle\langle\mathcal{X}\rangle\rangle\otimes K\langle\langle\mathcal{X}\rangle\rangle$ contains the elements of the form $\sum_{i\in I}G_i\otimes D_i$, for some finite set I and $(G_i,D_i)\in K\langle\langle\mathcal{X}\rangle\rangle\times K\langle\langle\mathcal{X}\rangle\rangle$. But, for $S=\sum_{i\geq 0}u^i\otimes v^i$, for non empty words u and v, S belongs to $K\langle\langle\mathcal{X}^*\otimes\mathcal{X}^*\rangle\rangle$ and S does not belong to $K\langle\langle\mathcal{X}\rangle\rangle\otimes K\langle\langle\mathcal{X}\rangle\rangle$.

2. Over the algebras of polynomials, $A\langle\mathcal{X}\rangle$ (resp. $A\langle Y\rangle$), the coproducts $\Delta_{\texttt{conc}}$ and $\Delta_{\sqcup\!\sqcup}$ (resp. $\Delta_{\uplus}$) are well defined [45] (resp. [27]). But in the algebras of series, $A\langle\langle\mathcal{X}\rangle\rangle$, these are less studied so we are doing in the remainder of this section (step by step and not in the general way[7]) for the concepts of group like and primitive series (see Definitions 2.3–2.4 and Proposition 2.6 below) which are classical in the theory of bialgebras.

Definition 2.2. — Let $S\in A\langle\langle\mathcal{X}\rangle\rangle$ (resp. $A\langle\langle\mathcal{X}\rangle\rangle\otimes A\langle\langle\mathcal{X}\rangle\rangle$). If $\langle S\,|\,1_{\mathcal{X}^*}\rangle=0$ (resp. $\langle S\,|\,1_{\mathcal{X}^*}\otimes 1_{\mathcal{X}^*}\rangle=0$) then one defines the Kleene star of S as the infinite sum [1]

$$S^*:=1+S+S^2+\cdots.$$

In the same way, one also defines the diagonal series as follows

$$\mathcal{M}_{\mathcal{X}}:=\sum_{t\in\mathcal{X}}t\otimes t\quad\text{and then}\quad\mathcal{D}_{\mathcal{X}}:=\mathcal{M}_{\mathcal{X}}^*=\sum_{w\in\mathcal{X}^*}w\otimes w.$$

[7] It will be done in future works.

For any $S \in A\langle\langle \mathcal{X} \rangle\rangle$ (resp. $A\langle\langle \mathcal{X} \rangle\rangle \otimes A\langle\langle \mathcal{X} \rangle\rangle$) such that $\langle S \mid 1_{\mathcal{X}^*} \rangle = 0$ (resp. $\langle S \mid 1_{\mathcal{X}^*} \otimes 1_{\mathcal{X}^*} \rangle = 0$), the Kleene star S^* is the unique[8] left solution and right solution of the following star equations (T is unknown series)

$$\nabla T = ST \quad \text{and} \quad \nabla T = TS, \tag{56}$$

where

$$\nabla T := T - 1_{\mathcal{X}^*} \quad (\text{resp. } \nabla T := T - 1_{\mathcal{X}^*} \otimes 1_{\mathcal{X}^*}). \tag{57}$$

Similarly, the diagonal series $\mathcal{D}_{\mathcal{X}}$ is the unique left solution and right solution of the following equations

$$\nabla T = \mathcal{M}_{\mathcal{X}} T \quad \text{and} \quad \nabla T = T\mathcal{M}_{\mathcal{X}}. \tag{58}$$

With the extented definitions (48)–(50) and in (53)–(55), one defines the following notions, considered as classic in the theory of Hopf algebras (see [10]).

Definition 2.3. — A series $S \in A\langle\langle Y \rangle\rangle$ (resp. $A\langle\langle \mathcal{X} \rangle\rangle$) is

1. $\uplus$ (resp. $\sqcup\!\sqcup$ and $\mathtt{conc}$)-character of $(A\langle Y \rangle, \mathtt{conc}, 1_{Y^*})$ (resp. $(A\langle \mathcal{X} \rangle, \mathtt{conc}, 1_{\mathcal{X}^*})$ if and only if, for any u and $v \in Y^*$ (resp. $\mathcal{X}^*$), $\langle S \mid 1_{Y^*} \rangle = 1_A$ (resp. $\langle S \mid 1_{\mathcal{X}^*} \rangle = 1_A$) and $\langle S \mid u \uplus v \rangle = \langle S \mid u \rangle \langle S \mid v \rangle$ (resp. $\langle S \mid u \sqcup\!\sqcup v \rangle = \langle S \mid u \rangle \langle S \mid v \rangle$ and $\langle S \mid uv \rangle = \langle S \mid u \rangle \langle S \mid v \rangle$).

2. $\uplus$ (resp. $\sqcup\!\sqcup$ and $\mathtt{conc}$)-infinitesimal character of $(A\langle Y \rangle, \mathtt{conc}, 1_{Y^*})$ (resp. $(A\langle \mathcal{X} \rangle, \mathtt{conc}, 1_{\mathcal{X}^*})$ if and only if, for any u and $v \in Y^*$ (resp. $\mathcal{X}^*$), $\langle S \mid u \uplus v \rangle = \langle S \mid u \rangle \langle v \mid 1_{Y^*} \rangle + \langle u \mid 1_{Y^*} \rangle \langle S \mid v \rangle$ (resp. $\langle S \mid u \sqcup\!\sqcup v \rangle = \langle S \mid u \rangle \langle v \mid 1_{Y^*} \rangle + \langle u \mid 1_{Y^*} \rangle \langle S \mid v \rangle$ and $\langle S \mid uv \rangle = \langle S \mid u \rangle \langle v \mid 1_{Y^*} \rangle + \langle u \mid 1_{Y^*} \rangle \langle S \mid v \rangle$).

Definition 2.4. — A series $S \in A\langle\langle Y \rangle\rangle$ (resp. $A\langle\langle \mathcal{X} \rangle\rangle$) is

1. group like for $\Delta_{\uplus}$ (resp. $\Delta_{\sqcup\!\sqcup}$ and $\Delta_{\mathtt{conc}}$), if and only if $\langle S \mid 1_{Y^*} \rangle = 1_A$ (resp. $\langle S \mid 1_{X^*} \rangle = 1_A$) and $\Delta_{\uplus} S = S \otimes S$ (resp. $\Delta_{\sqcup\!\sqcup} S = S \otimes S$ and $\Delta_{\mathtt{conc}} S = S \otimes S$). Let $\mathcal{G}^{Y}_{\uplus}$ (resp. $\mathcal{G}^{\mathcal{X}}_{\sqcup\!\sqcup}$ and $\mathcal{G}^{\mathcal{X}}_{\mathtt{conc}}$) denote the set of group like series for $\Delta_{\uplus}$ (resp. $\Delta_{\sqcup\!\sqcup}$ and $\Delta_{\mathtt{conc}}$).

2. primitive for $\Delta_{\uplus}$ (resp. $\Delta_{\sqcup\!\sqcup}$ and $\Delta_{\mathtt{conc}}$), if and only if $\Delta_{\uplus} S = 1_{Y^*} \otimes S + S \otimes 1_{Y^*}$ (resp. $\Delta_{\sqcup\!\sqcup} S = 1_{\mathcal{X}^*} \otimes S + S \otimes 1_{\mathcal{X}^*}$ and $\Delta_{\mathtt{conc}} S = 1_{\mathcal{X}^*} \otimes S + S \otimes 1_{\mathcal{X}^*}$). Let $\mathcal{P}^{Y}_{\uplus}$ (resp. $\mathcal{P}^{\mathcal{X}}_{\sqcup\!\sqcup}$ and $\mathcal{P}^{\mathcal{X}}_{\mathtt{conc}}$) denote the set of primitive series for $\Delta_{\uplus}$ (resp. $\Delta_{\sqcup\!\sqcup}$ and $\Delta_{\mathtt{conc}}$).

Remark 2.5. —

1. By (29) and (35), any letter $x \in \mathcal{X}$ is primitive for $\Delta_{\mathtt{conc}}$ and $\Delta_{\sqcup\!\sqcup}$. By (42), the letter y_1 is primitive for $\Delta_{\uplus}$.

2. Since $\Delta_{\uplus}$ (resp. $\Delta_{\sqcup\!\sqcup}$ and $\Delta_{\mathtt{conc}}$) is a morphism of algebras then
   (a) $\mathcal{G}^{Y}_{\uplus}$ (resp. $\mathcal{G}^{\mathcal{X}}_{\sqcup\!\sqcup}$ and $\mathcal{G}^{\mathcal{X}}_{\mathtt{conc}}$) is a group.
   (b) $\mathcal{P}^{Y}_{\uplus}$ (resp. $\mathcal{P}^{\mathcal{X}}_{\sqcup\!\sqcup}$ and $\mathcal{P}^{\mathcal{X}}_{\mathtt{conc}}$) is a Lie algebra.

The following proposition is an extension of the Friedrichs criterion, initially established by Ree for shuffle [45] and extended to quasi-shuffle in [27].

[8]Solutions obtained by convergent iteration process for a discrete topology.

Proposition 2.6. — *Let $S \in A\langle\langle \mathcal{X} \rangle\rangle$. Then the series S is*

1. *group like for $\Delta_{⧢}$ (resp. $\Delta_{⊔⊔}$ and $\Delta_{\texttt{conc}}$), if and only if S is a $⧢$ (resp. $⊔⊔$ and $\texttt{conc}$)-character of $(A\langle Y\rangle, \texttt{conc}, 1_{Y^*})$ (resp. $(A\langle \mathcal{X}\rangle, \texttt{conc}, 1_{\mathcal{X}^*})$).*

2. *primitive for $\Delta_{⧢}$ (resp. $\Delta_{⊔⊔}$ and $\Delta_{\texttt{conc}}$), if and only if S is an infinitesimal character of $(A\langle Y\rangle, \texttt{conc}, 1_{Y^*})$ (resp. $(A\langle \mathcal{X}\rangle, \texttt{conc}, 1_{\mathcal{X}^*})$).*

Proof. — As already said, similarly to $\Delta_{⧢}$ [27] and $\Delta_{⊔⊔}$ [45], one has

$$\Delta_{\texttt{conc}} S = S \otimes 1_{Y^*} + 1_{Y^*} \otimes S - \langle S \,|\, 1_{Y^*}\rangle 1_{Y^*} \otimes 1_{Y^*} + \sum_{u,v \in Y^+} \langle S \,|\, uv\rangle u \otimes v,$$

$$\Delta_{\texttt{conc}} S = \sum_{u,v \in Y^*} \langle S \,|\, uv\rangle u \otimes v \quad \text{and} \quad S \otimes S = \sum_{u,v \in Y^*} \langle S \,|\, u\rangle \langle S \,|\, v\rangle u \otimes v.$$

Then, by Definitions 2.3–2.4, it follows that

1. $\Delta_{\texttt{conc}} S = S \otimes 1_{Y^*} + 1_{Y^*} \otimes S \iff \langle S \,|\, 1_{Y^*}\rangle = 0$ and $\langle S \,|\, uv\rangle = 0$, for $u, v \in Y^+$.
2. $\Delta_{\texttt{conc}} S = S \otimes S \iff \langle S \,|\, 1_{Y^*}\rangle = 1$ and $\langle S \,|\, uv\rangle = \langle S \,|\, u\rangle\langle S \,|\, v\rangle$, for $u, v \in Y^*$. □

3. Various bialgebras

An important class of problems in the theory of Hopf algebras is the question of primitive elements (see, for examples, Defintions 2.3–2.4 and Proposition 2.6) and the aim of[9] the CQMM theorem [3, 7, 9, 10, 41] is to provide necessary and sufficient conditions for a bialgebra to be an enveloping algebra [3, 14]. Indeed,

Theorem 3.1 **(Cartier–Quillen–Milnor–Moore theorem,** [4]**).** — *Let A be a unitary commutative associative $\mathbb{Q}$-algebra and $\mathcal{B}$ be a (general[10]) co-commutative A-bialgebra. Let the envelopping algebra generated by the primitive elements of $\mathcal{B}$ for $\Delta_\times$, be denoted by $\mathcal{U}(\mathcal{P}_\times)$. Then the following assertions are equivalent:*

1. *There is an increasing sequence $\{\mathcal{B}_n\}$, $\mathcal{B}_0 = A.1_{\mathcal{B}} \subset \mathcal{B}_1 \subset \cdots \subset \mathcal{B}_n \subset \cdots$, satisfying*
   (a) $\mathcal{B} = \bigcup_{n\geq 0} \mathcal{B}_n$,
   (b) $\forall\, p, q \in \mathbb{N}$, $\mathcal{B}_p\mathcal{B}_q \subset \mathcal{B}_{p+q}$,
   (c) $\forall\, n \in \mathbb{N}$, $\Delta_\times(\mathcal{B}_n) \subset \sum_{p+q=n} \mathcal{B}_p \otimes \mathcal{B}_q$.

2. *The envelopping algebra $\mathcal{U}(\mathcal{P}_\times)$ is isomorphic to the bialgebra $(\mathcal{B}, \texttt{conc}, 1_{\mathcal{B}}, \Delta_\times)$.*

Now, let $m_i \in \mathbb{N}^{(I)}, i \in I$, be the elementary multiindex defined by

(59) $$m_i(j) = \delta_{i,j}, j \in I$$

and let $\{b_i\}_{i\in I}$ be a basis of $\mathcal{P}_\times$. By the following multiindex notation

(60) $$\forall\, \alpha \in \mathbb{N}^{(I)}, \quad \mathrm{supp}(\alpha) \subset \{i_1, \ldots, i_n\},\ b^\alpha = b_{i_1}^{\alpha(i_1)} \ldots b_{i_n}^{\alpha(i_n)},$$

[9]CQMM is an abbreviation of P. Cartier, D. Quillen, J. Milnor and J. Moore.
[10]Applications below concern $(A\langle \mathcal{X}\rangle, \texttt{conc}, 1_{\mathcal{X}^*}, \Delta_{⊔⊔})$ and $(A\langle Y\rangle, \texttt{conc}, 1_{Y^*}, \Delta_{⧢})$.

the[11] PBW basis $\{b^\alpha\}_{\alpha\in\mathbb{N}^{(I)}}$ of the enveloping algebra $\mathcal{U}(\mathcal{P}_\times)$ and the dual basis $\{\check{b}^\alpha\}_{\alpha\in\mathbb{N}^{(I)}}$ of its dual $\mathcal{U}(\mathcal{P}_\times)^\vee$ are constructed as follows [3, 14]

$$\langle b^\beta \mid \check{b}^\alpha\rangle = \delta_{\alpha,\beta}. \tag{61}$$

It follows that [3, 14]

$$\check{b}^\alpha \times \check{b}^\beta = \frac{(\alpha+\beta)!}{\alpha!\beta!}\check{b}^{\alpha+\beta}, \quad \text{where } \forall\, \alpha\in\mathbb{N}^{(I)},\ \alpha! = \prod_{i\in I}\alpha_i!, \tag{62}$$

$$\check{b}_{\alpha(i_1)m_{i_1}+\cdots+\alpha(i_k)m_{i_k}} = \frac{\check{b}_{m_{i_1}}^{\times\alpha(i_1)}\times\cdots\times\check{b}_{m_{i_k}}^{\times\alpha(i_k)}}{\alpha(i_1)!\ldots\alpha(i_k)!} \tag{63}$$

and the following infinite product identity holds, within $\mathrm{End}(\mathcal{U}(\mathcal{P}_\times))$,

$$\mathrm{Id}_{\mathcal{U}(\mathcal{P}_\times)} = \prod_{i\in I} e^{\check{b}^{e_i}\otimes b^{e_i}}. \tag{64}$$

As applications of the CQMM theorem and the factorization in (64), let us consider the shuffle and quasi-shuffle bialgebras

$$\mathcal{H}_{⧢}(\mathcal{X}) := (A\langle\mathcal{X}\rangle, \mathtt{conc}, 1_{\mathcal{X}^*}, \Delta_{⧢}), \tag{65}$$

$$\mathcal{H}_{\uplus}(Y) := (A\langle Y\rangle, \mathtt{conc}, 1_{Y^*}, \Delta_{\uplus}), \tag{66}$$

and their respective duals

$$\mathcal{H}^\vee_{⧢}(\mathcal{X}) := (A\langle\mathcal{X}\rangle, ⧢, 1_{\mathcal{X}^*}, \Delta_{\mathtt{conc}}), \tag{67}$$

$$\mathcal{H}^\vee_{\uplus}(Y) := (A\langle Y\rangle, \uplus, 1_{Y^*}, \Delta_{\mathtt{conc}}). \tag{68}$$

By Theorem 3.1, the enveloping algebras $\mathcal{U}(\mathcal{P}^{\mathcal{X}}_{⧢})$ and $\mathcal{U}(\mathcal{P}^{Y}_{\uplus})$ are isomorphic to the A-modules associated to the bialgebras $\mathcal{H}_{⧢}(\mathcal{X})$ and $\mathcal{H}_{\uplus}(Y)$, respectively.
In (64), when the noncommutative polynomials $\{b_i\}_{i\in I}$ are totally ordered then the commutative polynomials $\{\check{b}^\alpha\}_{\alpha\in\mathbb{N}^{(I)}}$ can be also chosen such that these two orderings are compatible and (62) is then also ordered (for example, the bases in (72)–(73) and (75)–(76) and their product in (74) and (77) below, with decreasing lexicographic ordering over $\mathcal{L}yn\,\mathcal{X}$).
For that, let $\pi_1 : A\langle Y\rangle \longrightarrow A\langle Y\rangle$ denote the Eulerian projector defined, for any $w = y_{i_i}\ldots y_{i_r}\in Y^*$, by [29]

$$\pi_1(w) = w + \sum_{k\geq 2}\frac{(-1)^{k-1}}{k}\sum_{u_1,\ldots,u_k\in Y^+}\langle w \mid u_1 \uplus\cdots\uplus u_k\rangle u_1\ldots u_k. \tag{69}$$

In particular, for any $k\geq 1$,

$$\pi_1(y_k) = y_k + \sum_{l\geq 2}\frac{(-1)^{l-1}}{l}\sum_{\substack{j_1,\ldots,j_l\geq 1\\ j_1+\cdots+j_l=k}} y_{j_1}\ldots y_{j_l}. \tag{70}$$

We are now in the position to state the following

[11] PBW is an abbreviation of H. Poincaré, G. Birkhoff and E. Witt.

Theorem 3.2 **([29]).** — *Let A be a $\mathbb{Q}$-algebra, then the endomorphism of algebras φ_{π_1} : $(A\langle Y\rangle, \mathtt{conc}, 1_{Y^*}) \longrightarrow (A\langle Y\rangle, \mathtt{conc}, 1_{Y^*})$ maps y_k to $\pi_1(y_k)$ (see (70)). Then φ_{π_1} is an automorphism of $A\langle Y\rangle$ realizing an isomorphism of bialgebras between $\mathcal{H}_{\sqcup\!\sqcup}(Y)$ and $\mathcal{H}_{\uplus}(Y)$. Moreover, the following diagram commutes*

$$\begin{array}{ccc} A\langle Y\rangle & \xhookrightarrow{\Delta_{\sqcup\!\sqcup}} & A\langle Y\rangle \otimes A\langle Y\rangle \\ {\scriptstyle\varphi_{\pi_1}}\downarrow & & \downarrow{\scriptstyle\varphi_{\pi_1}\otimes\varphi_{\pi_1}} \\ A\langle Y\rangle & \xhookrightarrow{\Delta_{\uplus}} & A\langle Y\rangle \otimes A\langle Y\rangle. \end{array}$$

Remark 3.3. —

1. Any letter in Y is a Lie polynomial and primitive, for $\Delta_{\sqcup\!\sqcup}$. Moreover, both $\mathcal{L}ie_A\langle Y\rangle$ and $\mathcal{P}^Y_{\sqcup\!\sqcup}$ are, by definition, closed by linear combinaisons and by the Lie bracket. Then, as consequence of (65),
$$\mathcal{U}(\mathcal{P}^Y_{\sqcup\!\sqcup}) = \mathcal{U}(\mathcal{L}ie_A\langle Y\rangle) \cong \mathcal{H}_{\sqcup\!\sqcup}(Y).$$
2. By (42), since $\Delta_{\uplus} y_k \neq y_k \otimes 1_{Y^*} + 1_{Y^*} \otimes y_k$, for $k > 1$, then $\mathcal{P}^Y_{\uplus} \neq \mathcal{L}ie_A\langle Y\rangle$. Then let $Y' := \{y'_k\}_{k\geq 1}$ such that $y'_k := \pi_1(y_k)$, for $k \geq 1$ (see (70)). On the one hand, by the previous item, $\mathcal{L}ie_A\langle Y'\rangle = \mathcal{P}^{Y'}_{\sqcup\!\sqcup}$ and on the other hand, by Theorem 3.2, $\mathcal{P}^Y_{\uplus} \cong \mathcal{P}^{Y'}_{\sqcup\!\sqcup} \cong \operatorname{Im}\pi_1$.

Now, let $\mathcal{X}$ be equipped the following usual total orders

(71) $$x_0 \prec x_1 \quad \text{and} \quad y_1 \succ \cdots \succ y_n \succ y_{n+1} \succ \cdots,$$

for which, any word $w \in \mathcal{X}^+$ is a Lyndon word if it is strictly smaller in lexicographic order (induced by (71)) than all of its rotations [40, 52]. Or equivalently, w is a Lyndon word if and only if it is lexicographically strictly smaller than any of its proper suffixes that is, for any $u, v \in \mathcal{X}^+$ such that $w = uv$, one has $w < v$ [40, 52]. The set of Lyndon words over $\mathcal{X}$ is denoted by $\mathcal{L}yn\,\mathcal{X}$.
Any pair of Lyndon words (l_1, l_2) is called the standard factorization of $l \in \mathcal{L}yn\,\mathcal{X}$, and is denoted by $st(l)$, if $l = l_1 l_2$ and l_2 is the longest nontrivial proper right factor of l or, equivalently, its smallest such (for the lexicographic ordering, see [40] for proofs).
According to Radford's theorem (see [45]), $\mathcal{L}yn\,\mathcal{X}$ forms a pure transcendence basis of the A-shuffle algebra $(A\langle\mathcal{X}\rangle, \sqcup\!\sqcup, 1_{\mathcal{X}^*})$, i.e. $\mathcal{L}yn\,\mathcal{X}$ is a algebraic basis of the A-shuffle algebra and each Lyndon word is trancendent over A.
In the case when A is a $\mathbb{Q}$-algebra, one classically endows $A\langle\mathcal{X}\rangle$ with the graded[12] linear basis $\{P_w\}_{w\in\mathcal{X}^*}$, expanded by decreasing PBW theorem [3] after any basis $\{P_l\}_{l\in\mathcal{L}yn\,\mathcal{X}}$ of $\mathcal{L}ie_A\langle\mathcal{X}\rangle$, homogeneous in weight, and its graded dual basis $\{S_w\}_{w\in\mathcal{X}^*}$ (containing the pure transcendence basis $\{S_l\}_{l\in\mathcal{L}yn\,\mathcal{X}}$ of the A-shuffle algebra) [14, 45]. These dual bases of polynomials $\{P_w\}_{w\in\mathcal{X}^*}$ and $\{S_w\}_{w\in\mathcal{X}^*}$, homogeneous in weight, can be constructed recursively as

[12]For $\mathcal{X} = X$ or $= Y$ the corresponding monoids are equipped with length functions, for X we consider the length of words and for Y the length is given by the weight $\ell(y_{i_1} \ldots y_{i_n}) = i_1 + \cdots + i_n$. This naturally induces a grading of $A\langle\mathcal{X}\rangle$ and $\mathcal{L}ie_A\langle\mathcal{X}\rangle$ in free modules of finite rank. For general $\mathcal{X}$, we consider the fine grading [45] i.e. the grading by all partial degrees which, as well, induces a grading of $A\langle\mathcal{X}\rangle$ and $\mathcal{L}ie_A\langle\mathcal{X}\rangle$ in free modules of finite rank.

follows [40, 52]

$$(72)\quad \begin{cases} P_x = x, & \text{for } x \in \mathcal{X}, \\ P_l = [P_{l_1}, P_{l_2}], & \text{for } l = yl' \in \mathcal{L}yn\,\mathcal{X} \setminus \mathcal{X},\ st(l) = (l_1, l_2), \\ P_w = P_{l_1}^{i_1} \dots P_{l_k}^{i_k}, & \text{for } w = l_1^{i_1} \dots l_k^{i_k},\ \text{with } l_1, \dots, l_k \in \mathcal{L}yn\,\mathcal{X},\ l_1 \succ \dots \succ l_k. \end{cases}$$

and then by duality [45],

$$(73)\quad \begin{cases} S_x = x & \text{for } x \in \mathcal{X}, \\ S_l = yS_{l'}, & \text{for } l = yl' \in \mathcal{L}yn\,\mathcal{X} \setminus \mathcal{X},\ st(l) = (l_1, l_2), \\ S_w = \dfrac{S_{l_1}^{\sqcup\!\sqcup i_1} \sqcup\!\sqcup \dots \sqcup\!\sqcup S_{l_k}^{\sqcup\!\sqcup i_k}}{i_1! \dots i_k!}, & \text{for } w = l_1^{i_1} \dots l_k^{i_k},\ \text{with } l_1, \dots, l_k \in \mathcal{L}yn\,\mathcal{X},\ l_1 \succ \dots \succ l_k. \end{cases}$$

One obtains the following[13] factorization of the diagonal series $\mathcal{D}_X$, on $\mathcal{H}_{\sqcup\!\sqcup}(\mathcal{X})$ (see Definition 2.2 and (72)–(73)), which reads [45]

$$(74)\quad \mathcal{D}_{\mathcal{X}} = \sum_{w \in \mathcal{X}^*} S_w \otimes P_w = \prod_{l \in \mathcal{L}yn\,\mathcal{X}}^{\searrow} e^{S_l \otimes P_l}, \text{(decreasing lexcographical ordered product)}.$$

Similarly, $\mathcal{L}yn\,Y$ forms a pure transcendence basis of the A-quasi-shuffle algebra $(A\langle Y \rangle, \uplus, 1_{Y^*})$ (see [27, 29]). In the case when A is a $\mathbb{Q}$-algebra, one also endows $\mathcal{P}_{\uplus}^{Y}$ the linear basis $\{\Pi_w\}_{w \in Y^*}$, expanded by decreasing PBW basis after any basis $\{\Pi_l\}_{l \in \mathcal{L}yn\,Y}$, homogeneous in weight, and its graded dual basis $\{\Sigma_w\}_{w \in Y^*}$ (containing the pure transcendence basis $\{\Sigma_l\}_{l \in \mathcal{L}yn\,Y}$ of the A-quasi-shuffle algebra) [27, 29]. By Theorem 3.2, the bases of homogeneous polynomials $\{\Pi_w\}_{w \in Y^*}$ and $\{\Sigma_w\}_{w \in Y^*}$ of $\mathcal{U}(\mathcal{P}_{\uplus}^{Y})$ are images by φ_{π_1} and by the adjoint mapping of its inverse of $\{P_w\}_{w \in Y^*}$ and $\{S_w\}_{w \in Y^*}$, respectively.
Algorithmically, these can be constructed directly and recursively by [27, 29]

$$(75)\quad \begin{cases} \Pi_{y_s} = \pi_1(y_s), & \text{for } y_s \in Y, \\ \Pi_l = [\Pi_{l_1}, \Pi_{l_2}], & \text{for } l \in \mathcal{L}yn\,Y \setminus Y,\ st(l) = (l_1, l_2), \\ \Pi_w = \Pi_{l_1}^{i_1} \dots \Pi_{l_k}^{i_k}, & \text{for } w = l_1^{i_1} \dots l_k^{i_k},\ \text{with } l_1, \dots, l_k \in \mathcal{L}yn\,Y,\ l_1 \succ \dots \succ l_k \end{cases}$$

and then by duality,

$$(76)\quad \begin{cases} \Sigma_{y_s} = y_s & \text{for } y_s \in Y, \\ \Sigma_l = \displaystyle\sum_{(**)} \frac{y_{s_{k_1} + \dots + s_{k_i}}}{i!} \Sigma_{l_1 \dots l_n}, & \text{for } l \in \mathcal{L}yn\,Y \setminus Y,\ st(l) = (l_1, l_2), \\ \Sigma_w = \dfrac{\Sigma_{l_1}^{\uplus i_1} \uplus \dots \uplus \Sigma_{l_k}^{\uplus i_k}}{i_1! \dots i_k!}, & \text{for } w = l_1^{i_1} \dots l_k^{i_k},\ \text{with } l_1, \dots, l_k \in \mathcal{L}yn\,Y,\ l_1 \succ \dots \succ l_k. \end{cases}$$

In $(**)$, the sum is taken over all $\{k_1, \dots, k_i\} \subset \{1, \dots, k\}$ and $l_1 \succeq \dots \succeq l_n$ such that $(y_{s_1}, \dots, y_{s_k})$ is derived from $(y_{s_{k_1}}, \dots, y_{s_{k_i}}, l_1, \dots, l_n)$ by transitive closure of the relations on standard sequences [4, 45].

[13] MSR is an abbreviation of G. Mélançon, M.P. Schützenberger and C. Reutenauer.

One also has the factorization of the diagonal series $\mathcal{D}_Y$, on $\mathcal{H}_{⧢}(Y)$ (see Definition 2.2 and (75)–(76)), which reads[14] [29]

$$(77)\quad \mathcal{D}_Y = \sum_{w\in Y^*} \Sigma_w \otimes \Pi_w = \prod_{l\in\mathcal{L}yn\,Y}^{\searrow} e^{\Sigma_l\otimes\Pi_l}, \quad \text{(decreasing lexcographical ordered product)}.$$

4. Representative series

By (11), representative (or rational) series are the representative functions on the free monoid. These functions were considered on groups in [7, 9].

Definition 4.1. — Let $S \in A\langle\langle\mathcal{X}\rangle\rangle$ (resp. $A\langle\mathcal{X}\rangle$) and $P \in A\langle\mathcal{X}\rangle$ (resp. $A\langle\langle\mathcal{X}\rangle\rangle$).

1. The *left* and the *right shifts*[15] of S by P, $P \triangleright S$ and $S \triangleleft P$, are defined, for any $w \in \mathcal{X}^*$, by $\langle P \triangleright S \mid w\rangle = \langle S \mid wP\rangle$ and $\langle S \triangleleft P \mid w\rangle = \langle S \mid Pw\rangle$, respectively.

2. Let $A = K$ be a field then one defines also the Sweedler's dual $\mathcal{H}^{\circ}_{⧢}(\mathcal{X})$ (resp. $\mathcal{H}^{\circ}_{⧢}(Y)$) of $\mathcal{H}_{⧢}(\mathcal{X})$ (resp. $\mathcal{H}_{⧢}(Y)$) by

$$S \in \mathcal{H}^{\circ}_{⧢}(\mathcal{X}) \text{ (resp. } \mathcal{H}^{\circ}_{⧢}(Y)) \iff \Delta_{\mathtt{conc}}(S) = \sum_{i\in I_{finite}} G_i \otimes D_i,$$

   where $\{G_i, D_i\}_{i\in I_{finite}}$ are series.

Remark 4.2. — The series $\{G_i, D_i\}_{i\in I_{finite}}$ can be chosen in $\mathcal{H}^{\circ}_{⧢}(\mathcal{X})$ (see [29]).

Theorem 4.3 **([19, 20, 45]).** — *The following assertions are equivalent*

1. *The shifts* $\{S \triangleleft w\}_{w\in\mathcal{X}^*}$ *(resp.* $\{w \triangleright S\}_{w\in\mathcal{X}^*}$*) lie in a finitely generated shift-invariant* A*-module* [32].

2. *The series* S *belongs to the (algebraic) closure of* $\widehat{A.\mathcal{X}}$ *by the rational operations*[16] $\{\mathtt{conc}, +, *\}$ *(within* $A\langle\langle\mathcal{X}\rangle\rangle$*).*

3. *There is an integer* n *and matrices* $\nu \in M_{1,n}(A), \eta \in M_{n,1}(A)$ *and morphism of monoids* $\mu : \mathcal{X}^* \to M_{n,n}(A)$ *such that* $\langle S \mid w\rangle = \nu\mu(w)\eta$*, for* $w \in \mathcal{X}^*$.

The triplet (ν, μ, η) *is called linear representation of* S *of rank* n.

Remark 4.4. —

1. The shifts operators are associative and mutually commute, i.e. $S \triangleleft (P \triangleleft R) = (S \triangleleft P) \triangleleft R$, $P \triangleright (R \triangleright S) = (P \triangleright R) \triangleright S$, $(P \triangleleft S) \triangleright R = P \triangleleft (S \triangleright R)$ and then, for any $x, y \in \mathcal{X}$ and $w \in \mathcal{X}^*$, one has $x \triangleright (wy) = (yw) \triangleleft x = \delta_x^y w$.

[14] Again all tensor products will be taken over A. Note that this factorization holds for any enveloping algebra as announced in [45]. Of course, the diagonal series no longer exists and must be replaced by the identity $\mathrm{Id}_{\mathcal{U}}$.
[15] These are called *residuals* and extend shifts of functions in harmonic analysis [33].
[16] In here, $\widehat{A.\mathcal{X}}$ is understood as the set of all series of the form $\sum_{x\in\mathcal{X}} a_x x$.

2. A series satisfying one of the conditions of Theorem 4.3 is called *rational*. The A-module of these series is denoted by $A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ and it is closed by $\{\mathtt{conc}, +, *\}$ [1]. It is, in fact, a unital A-algebra with respected to one of the products $\{\mathtt{conc}, \shuffle, \stuffle\}$ (see also Proposition 4.5 below).

One has the following constructions of linear representations (only the last one is new and the first ones are already treated in [33], see also [15]). Those of $R_1 \shuffle R_2$ and $R_1 \stuffle R_2$ base on coproducts and tensor product of linear representations:

Proposition 4.5. — *The module $A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ (resp. $A^{\mathrm{rat}}\langle\langle Y\rangle\rangle$) is closed by $\shuffle$ (resp. $\stuffle$). Moreover, for any $i = 1, 2$, let $R_i \in A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ and (ν_i, μ_i, η_i) be its representation of dimension n_i. Then the linear representation of*

$$R_i^* \quad \textit{is} \left(\begin{pmatrix} 0 & 1 \end{pmatrix}, \left\{ \begin{pmatrix} \mu_i(x) + \eta_i\nu_i\mu_i(x) & 0 \\ \nu_i\eta_i & 0 \end{pmatrix} \right\}_{x\in\mathcal{X}}, \begin{pmatrix} \eta_i \\ 1 \end{pmatrix} \right),$$

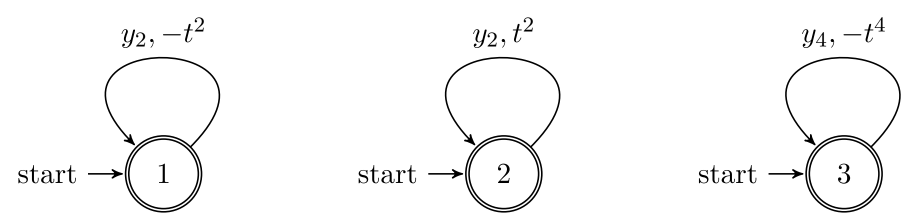
$$\textit{that of } R_1 + R_2 \textit{ is} \left(\begin{pmatrix} \nu_1 & \nu_2 \end{pmatrix}, \left\{ \begin{pmatrix} \mu_1(x) & \mathbf{0} \\ \mathbf{0} & \mu_2(x) \end{pmatrix} \right\}_{x\in\mathcal{X}}, \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \right),$$

$$\textit{that of } R_1R_2 \quad \textit{is} \left(\begin{pmatrix} \nu_1 & 0 \end{pmatrix}, \left\{ \begin{pmatrix} \mu_1(x) & \eta_1\nu_2\mu_2(x) \\ 0 & \mu_2(x) \end{pmatrix} \right\}_{x\in\mathcal{X}}, \begin{pmatrix} \eta_1\mu_2\eta_2 \\ \eta_2 \end{pmatrix} \right),$$

$$\textit{that of } R_1 \shuffle R_2 \textit{ is } (\nu_1 \otimes \nu_2, \{\mu_1(x) \otimes \mathrm{I}_{n_2} + \mathrm{I}_{n_1} \otimes \mu_2(x)\}_{x\in\mathcal{X}}, \eta_1 \otimes \eta_2),$$

$$\textit{that of } R_1 \stuffle R_2 \textit{ is} \Big(\nu_1 \otimes \nu_2, \{\mu_1(y_k) \otimes \mathrm{I}_{n_2} + \mathrm{I}_{n_1} \otimes \mu_2(y_k) + \sum_{i+j=k} \mu_1(y_i) \otimes \mu_2(y_j)\}_{k\geq 1}, \eta_1 \otimes \eta_2 \Big).$$

Example 4.6 **(Identity** $(-t^2y_2)^* \stuffle (t^2y_2)^* = (-4t^4y_4)^*$ [5]**).** —



$(-t^2y_2)^* \leftrightarrow (\nu_2, \mu_2(y_2), \eta_2) = (1, -t^2, 1)$, $\quad (t^2y_2)^* \leftrightarrow (\nu_1, \mu_1(y_2), \eta_1) = (1, t^2, 1)$, $\quad (-t^4y_4)^* \leftrightarrow (\nu, \mu(y_4), \eta) = (1, -t^4, 1)$.

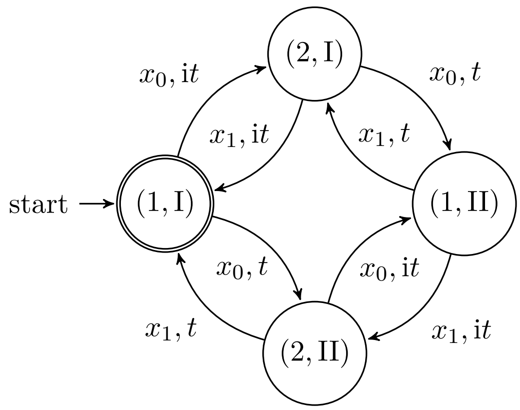
Example 4.7 **(Identity** $(-t^2x_0x_1)^* \shuffle (t^2x_0x_1)^* = (-4t^4x_0^2x_1^2)^*$ [5]**).** —

$(-t^2x_0x_1)^* \leftrightarrow (\nu_2, \{\mu_2(x_0), \mu_2(x_1)\}, \eta_2)$ $\quad (t^2x_0x_1)^* \leftrightarrow (\nu_1, \{\mu_1(x_0), \mu_1(x_1)\}, \eta_1)$

with

$$\nu_1 = \begin{pmatrix}1 & 0\end{pmatrix}, \qquad \mu_1(x_0) = \begin{pmatrix}0 & t\\ 0 & 0\end{pmatrix}, \qquad \mu_1(x_1) = \begin{pmatrix}0 & 0\\ t & 0\end{pmatrix}, \qquad \eta_1 = \begin{pmatrix}1\\ 0\end{pmatrix},$$

$$\nu_2 = \begin{pmatrix}1 & 0\end{pmatrix}, \qquad \mu_2(x_0) = \begin{pmatrix}0 & \mathrm{i}t\\ 0 & 0\end{pmatrix}, \qquad \mu_2(x_1) = \begin{pmatrix}0 & 0\\ \mathrm{i}t & 0\end{pmatrix}, \qquad \eta_2 = \begin{pmatrix}1\\ 0\end{pmatrix}.$$



$$(-t^2x_0x_1)^* \sqcup\!\sqcup (t^2x_0x_1)^* \leftrightarrow (\nu, \{\mu(x_0), \mu(x_1)\}, \eta)$$

with $\nu = \nu_1 \otimes \nu_2 = \begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}$ and $\eta = \eta_1 \otimes \eta_2 = {}^t\begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}$ and

$$\mu(x_0) = \mu_1(x_0) \otimes \mathrm{I}_{n_2} + \mathrm{I}_{n_1} \otimes \mu_2(x_0) = \begin{pmatrix}0&0&t&0\\0&0&0&t\\0&0&0&0\\0&0&0&0\end{pmatrix} + \begin{pmatrix}0&\mathrm{i}t&0&0\\0&0&0&0\\0&0&0&\mathrm{i}t\\0&0&0&0\end{pmatrix} = \begin{pmatrix}0&\mathrm{i}t&t&0\\0&0&0&t\\0&0&0&\mathrm{i}t\\0&0&0&0\end{pmatrix},$$

$$\mu(x_1) = \mu_1(x_1) \otimes \mathrm{I}_{n_2} + \mathrm{I}_{n_1} \otimes \mu_2(x_1) = \begin{pmatrix}0&0&0&0\\0&0&0&0\\t&0&0&0\\0&t&0&0\end{pmatrix} + \begin{pmatrix}0&0&0&0\\\mathrm{i}t&0&0&0\\0&0&0&0\\0&0&\mathrm{i}t&0\end{pmatrix} = \begin{pmatrix}0&0&0&0\\\mathrm{i}t&0&0&0\\t&0&0&0\\0&t&\mathrm{i}t&0\end{pmatrix}.$$

Remark 4.8. —

1. Identities of rational series in Examples 4.6–4.7 are used in [5] to study the rationality of certains ratio of polyzetas, of weight $2k$, over π^{2k}.

2. Since $(-t^2x_0x_1)^* = \pi_X((-t^2y_2)^*)$ and $(t^2x_0x_1)^* = \pi_X((t^2y_2)^*)$ and, on the other hand, $(-t^2y_2)^* = \pi_Y((-t^2x_0x_1)^*)$ and $(t^2y_2)^* = \pi_Y((t^2x_0x_1)^*)$, then π_X and π_Y can be viewed as transducers.

Corollary 4.9 **([29]).** — *With notations in Definitions 2.3–2.4, if A is a field K then there exists a finite double family of series $(G_i, D_i)_{i\in I_{finite}}$ satisfying three assertions of Theorem 4.3 or, equivalently, one of the following assertions holds.*

1. $\langle S \,|\, PQ\rangle = \sum_{i\in I_{finite}} \langle G_i \,|\, P\rangle\langle D_i \,|\, Q\rangle$, *for P and $Q \in K\langle\mathcal{X}\rangle$ (resp. $\in K\langle Y\rangle$).*

2. $\Delta_{\mathsf{conc}}(S) = \sum_{i\in I_{finite}} G_i \otimes D_i$.

Hence, the Sweedler's dual of the bialgebra $\mathcal{H}_{\sqcup\sqcup}(\mathcal{X})$ (resp. $\mathcal{H}_{\uplus}(Y)$) is isomorphic to $(K^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle, \sqcup\sqcup, 1_{\mathcal{X}^*}, \Delta_{\mathtt{conc}})$ (resp. $(K^{\mathrm{rat}}\langle\langle Y\rangle\rangle, \uplus, 1_{Y^*}, \Delta_{\mathtt{conc}})$.

Proof. — Let (β,μ,η) of dimension n be a linear representation of $S \in K^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$. It can be also associated to the following linear representations (β,μ,e_i) of L_i and $({}^te_i,\mu,\eta)$ of $R_i, 1 \le i \le n$, of dimension n, where

$$e_i \in \mathcal{M}_{1,n}(K) \quad \text{and} \quad {}^te_i = (0 \cdots 0 \underset{i}{1} 0 \cdots 0).$$

Hence, for any u and $v \in \mathcal{X}^*$, using the morphism of monoids μ, let us formulate the proof given in [29] as follows

$$\langle S \mid uv\rangle = \beta\mu(u)\mu(v)\eta = \sum_{i=1}^n (\beta\mu(u)e_i)({}^te_i\mu(v)\eta) = \sum_{i=1}^n \langle L_i \mid u\rangle\langle R_i \mid v\rangle, \tag{78}$$

$$\langle \Delta_{\mathtt{conc}}(S) \mid u \otimes v\rangle = \langle S \mid uv\rangle = \sum_{i=1}^n \langle L_i \mid u\rangle\langle R_i \mid v\rangle = \sum_{i=1}^n \langle L_i \otimes R_i \mid u \otimes v\rangle. \tag{79}$$

One deduces then the following criterion yielding the expected results

$$S \in K^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle \iff \Delta_{\mathtt{conc}}(S) = \sum_{i=1}^n L_i \otimes R_i.$$

Finally, according to Defintion 2.4, it follows the final conclusion concerning the Sweedler's dual of bialgebras $\mathcal{H}_{\sqcup\sqcup}(\mathcal{X})$ (resp. $\mathcal{H}_{\uplus}(Y)$). □

Definition 4.10. — Any series $S \in A\langle\langle\mathcal{X}\rangle\rangle$ is called

1. syntactically exchangeable if and only if it is constant on multi-homogeneous classes, i.e. $(\forall\ u,v \in \mathcal{X}^*)([(\forall\ x \in \mathcal{X})(|u|_x = |v|_x)] \Rightarrow \langle S \mid u\rangle = \langle S \mid v\rangle)$. The set of syntactically exchangeable series is denoted by $A^{\mathrm{synt}}_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$.

2. rationally exchangeable if and only if it admits a representation (ν,μ,η) such that the matrices $\{\mu(x)\}_{x\in\mathcal{X}}$ commute and the set of these series, a shuffle subalgebra of $A\langle\langle X\rangle\rangle$, is denoted by $A^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$.

Remark 4.11. — S is syntactically exchangeable if and only if it is of the form[17]

$$S = \sum_{\substack{\alpha\in\mathbb{N}^{(\mathcal{X})},\\ \mathrm{supp}(\alpha)=\{x_1,\ldots,x_k\}}} s_\alpha x_1^{\alpha(x_1)} \sqcup\sqcup \cdots \sqcup\sqcup x_k^{\alpha(x_k)}.$$

When $A = K$ is a field, the rational exchangeable series (Definition 4.10, item 2) are exactly those that admit a representation with commuting matrices (at least the minimal one is such) and it is taken as definition as, even for rings, implying syntactic exchangeability (Definition 4.10, item 1).

[17]Recall that $\mathcal{X}$ could be infinite and the support of the map $\alpha : \mathcal{X} \longrightarrow \mathbb{N}$ is finite.

***Theorem 4.12* (See [17, 29]).** —

1. *In all cases, one has* $A^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle \subset A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle \cap A^{\mathrm{synt}}_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$. *The equality holds when* A *is a field and, letting* $A^{\mathrm{rat}}_{\mathrm{fin}}\langle\langle Y\rangle\rangle = \bigcup_{F\subset_{finite}Y} A^{\mathrm{rat}}\langle\langle F\rangle\rangle$, *the algebra of series over finite subalphabets, one has*
$$A^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle = A^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \sqcup\!\sqcup A^{\mathrm{rat}}\langle\langle x_1\rangle\rangle = \bigsqcup_{x\in X} A^{\mathrm{rat}}\langle\langle x\rangle\rangle$$
and
$$A^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle Y\rangle\rangle \cap A^{\mathrm{rat}}_{\mathrm{fin}}\langle\langle Y\rangle\rangle = \bigcup_{k\geq 0}\bigsqcup_{j=1}^{k} A^{\mathrm{rat}}\langle\langle y_j\rangle\rangle \subsetneq A^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle Y\rangle\rangle.$$

2. *One has* $A^{\mathrm{rat}}\langle\langle x\rangle\rangle = \{P(1-xQ)^{-1}\}_{P,Q\in A[x]}$ *(for* $x\in\mathcal{X}$*) and if* $A=K$ *is an algebraically closed field of characteristic zero then one also has* $K^{\mathrm{rat}}\langle\langle x\rangle\rangle = \mathrm{span}_K\{(ax)^* \sqcup\!\sqcup K\langle x\rangle | a\in K\}$.

3. *Series* $\left(\sum_{x\in\mathcal{X}}\alpha_x x\right)^*$ *are* `conc`*-characters. Any* `conc`*-character is of this form.*

4. A *is supposed without zero divisors. If the family* $(\varphi_i)_{i\in I}$ *is* $\mathbb{Z}$*-linearly independent within* $\widehat{A\mathcal{X}}$ *then the family* $\mathcal{L}yn(\mathcal{X}) \uplus \{\varphi_i^*\}_{i\in I}$ *is* A*-algebraically free within* $(A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle, \sqcup\!\sqcup, 1_{\mathcal{X}^*})$.

5. *In particular, if* A *is a ring without zero divisors then* $\{x^*\}_{x\in\mathcal{X}}$ *(resp.* $\{y^*\}_{y\in Y}$*) are algebraically independent over* $(A\langle\mathcal{X}\rangle, \sqcup\!\sqcup, 1_{\mathcal{X}^*})$ *(resp.* $(A\langle Y\rangle, \uplus, 1_{Y^*})$*) within* $(A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle, \sqcup\!\sqcup, 1_{\mathcal{X}^*})$ *(resp.* $(A^{\mathrm{rat}}\langle\langle Y\rangle\rangle, \uplus, 1_{Y^*})$*).*

Proof. —

1. The inclusion is obvious in view of Definition 4.10. For the equality, it suffices to prove that, when A is a field, every rational and exchangeable series admits a representation with commuting matrices. This is true of any minimal representation as shows the computation of shifts (see [17, 19, 29]).

   Now, if $\mathcal{X}$ is finite, then (all matrices commute)
$$\sum_{w\in\mathcal{X}^*}\mu(w)w = \left(\sum_{x\in\mathcal{X}}\mu(x)x\right)^* = \bigsqcup_{x\in\mathcal{X}}(\mu(x)x)^*$$
   and the result comes from the fact that R is a linear combination of matrix elements. As regards the second equality, inclusion $\supset$ is straightforward. We remark that $\bigcup_{k\geq 1} A^{\mathrm{rat}}\langle\langle y_1\rangle\rangle \sqcup\!\sqcup \cdots \sqcup\!\sqcup A^{\mathrm{rat}}\langle\langle y_k\rangle\rangle$ is directed as these algebras are nested in one another. With this in view, the reverse inclusion comes from the fact that every $S\in A^{\mathrm{rat}}_{\mathrm{fin}}\langle\langle Y\rangle\rangle$ is a series over a finite alphabet and the result follows from the first equality.

2. This is nothing but a theorem by Kronecker (see [53]), rephrased with terms and notations of [1].

3. Let $S = \left(\sum_{x\in\mathcal{X}}\alpha_x\, x\right)^*$. Then $\langle S\,|\,1_{\mathcal{X}^*}\rangle = 1_A$. Furthermore, if $w = xu$ then $\langle S\,|\,xu\rangle = \alpha_x\langle S\,|\,u\rangle$. Thus, by induction on the length, $\langle S\,|\,x_1\ldots x_k\rangle = \prod_{i=1}^k \alpha_{x_i}$ showing that S is a `conc`-character. Conversely, by Schützenberger's reconstruction lemma, we have $S =$

$\langle S \,|\, 1_{\mathcal{X}^*}\rangle.1_A + \sum_{x\in\mathcal{X}} x.x^{-1}S$. But, if S is a conc-character (i.e. $\langle S \,|\, 1_{\mathcal{X}^*}\rangle = 1$ and $x^{-1}S = \langle S \,|\, x\rangle S$) then the previous expression reads $S = 1_A + (\sum_{x\in\mathcal{X}} \langle S \,|\, x\rangle x)S$. The last equality is equivalent to $S = \left(\sum_{x\in\mathcal{X}} \langle S \,|\, x\rangle x\right)^*$ proving the claim.

4. As $(A\langle\mathcal{X}\rangle, ⧢)$ and $(A\langle Y\rangle, \uplus)$ are enveloping algebras, this property is an application of the fact that, on an enveloping $\mathcal{U}$, the characters are linearly independent with respect to the convolution algebra $\mathcal{U}_\infty^*$ (see the general construction and proof in [16]. Here, this convolution algebra $(\mathcal{U}_\infty^*)$ contains the polynomials (is equal in case of finite $\mathcal{X}$). Now, consider a monomial $(\varphi_{i_1}^*)^{⧢\alpha_1} \dots (\varphi_{i_n}^*)^{⧢\alpha_n} = \left(\sum_{k=1}^n \alpha_{i_k}\varphi_{i_k}\right)^*$. The $\mathbb{Z}$-linear independence of the monomials in $(\varphi_i)_{i\in I}$ implies that all these monomials are linearly independent over $A\langle\mathcal{X}\rangle$ which proves algebraic independence of the family $(\varphi_i)_{i\in I}$.

   To end with, the fact that $\mathcal{L}yn(\mathcal{X}) \uplus \{\varphi_i^*\}_{i\in I}$ is algebraically free comes from Radford theorem $(A\langle\mathcal{X}\rangle, ⧢, 1_{\mathcal{X}^*}) \simeq A[\mathcal{L}yn(\mathcal{X})]$ and the transitivity of polynomial algebras (see [2]).

5. Comes directly as an application of the preceding point. □

Remark 4.13. —

1. The last inclusion of Theorem 4.12.1 is strict as shows the example of the following identity, living in $A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle Y\rangle\rangle$ but not in $A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle Y\rangle\rangle \cap A_{\mathrm{fin}}^{\mathrm{rat}}\langle\langle Y\rangle\rangle$

$$\begin{aligned}(ty_1 + t^2y_2 + \dots)^* &= \lim_{k\to+\infty}(ty_1 + \dots + t^k y_k)^* \\ &= \lim_{k\to+\infty}(ty_1)^* ⧢ \dots ⧢ (t^k y_k)^* = \bigsqcup_{k\geq 1}(t^k y_k)^*.\end{aligned}$$

2. Item 2 can be rephrased in terms of stars as $A^{\mathrm{rat}}\langle\langle x\rangle\rangle = \{P(xQ)^*\}_{P,Q\in A[x]}$ holds for every ring and is therefore characteristic free, unlike the shuffle version requiring algebraic closure and denominators.

***Corollary 4.14* (Kleene stars of the plane).** — *Let $R, L \in A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ such that $L^* = R$ ($\langle R \,|\, 1_{\mathcal{X}^*}\rangle = 1_A$ and $\langle L \,|\, 1_{\mathcal{X}^*}\rangle = 0$). The following assertions are equivalent.*

1. *R is a* conc*-character of $(A\langle\mathcal{X}\rangle, \mathtt{conc}, 1_{\mathcal{X}^*})$.*

2. *There is a family of coefficients $(c_x)_{x\in\mathcal{X}}$ such that $R = (\sum_{x\in\mathcal{X}} c_x x)^*$.*

3. *The series R admits a linear representation of dimension one*[18]*.*

4. *L belongs to the plane $A.\mathcal{X}$.*

5. *L is an infinitesimal* conc*-character of $(A\langle\mathcal{X}\rangle, \mathtt{conc}, 1_{X^*})$.*

Proof. —

1 $\iff$ 2. — This corresponds to the point 3 of Theorem 4.12 above.

2 $\iff$ 3. — This is a direct consequence of Theorem 4.3.

[18]The dimension is here (as in [1]) the size of the matrices.

$2 \iff 4$. — This is obvious, by construction (in which L is viewed as the $⧢$-logarithm of R). Indeed, since $(c_x x)^n = (c_x x)^{⧢ n}/n!$, for any $n \in \mathbb{N}$, doing as in Remark 4.13, one has

$$R = \Big(\sum_{x\in\mathcal{X}} c_x x\Big)^* = \bigsqcup\!\!\bigsqcup_{x\in\mathcal{X}} (c_x x)^* = \bigsqcup\!\!\bigsqcup_{x\in\mathcal{X}} \exp_{⧢}(c_x x) = \exp_{⧢}\Big(\sum_{x\in\mathcal{X}} c_x x\Big).$$

$4 \iff 5$. — If L is an infinitesimal character then, by Definitions 2.3–2.4,

$$\forall\, u, v \in \mathcal{X}^*, \langle L \,|\, uv\rangle = \langle L \,|\, u\rangle\langle v \,|\, 1_{\mathcal{X}^*}\rangle + \langle u \,|\, 1_{\mathcal{X}^*}\rangle\langle L \,|\, v\rangle.$$

Hence, for any $w = uv \in \mathcal{X}^{\geq 2}$ with $u, v \in \mathcal{X}^+$, one gets $\langle L \,|\, w\rangle = \langle L \,|\, uv\rangle = 0$. In addition, for $u = v = 1_{\mathcal{X}^*}$, one also gets $\langle L \,|\, 1_{\mathcal{X}^*}\rangle = 0$ and it follows that $L = \sum_{x\in\mathcal{X}} \langle L \,|\, x\rangle x$. Conversely, since $\langle uv \,|\, x\rangle = \langle u \,|\, x\rangle\langle v \,|\, 1_{\mathcal{X}^*}\rangle + \langle u \,|\, 1_{\mathcal{X}^*}\rangle\langle v \,|\, x\rangle = 0$, for $u, v \in \mathcal{X}^+$ and $x \in \mathcal{X}$, then by the pairing in (19), one deduces that

$$\langle L \,|\, uv\rangle = \sum_{x\in\mathcal{X}} \langle L \,|\, x\rangle\langle uv \,|\, x\rangle = 0$$

meaning that L is an infinitesimal $\texttt{conc}$-character. □

Remark 4.15. — In Corollary 4.14, if $A = K$ being a field, point 1 (resp. 5) can be rephrased as "R is a group like element" (resp. "L is a primitive element") of $K^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$, for $\Delta_{\texttt{conc}}$. Indeed, in (53)–(55), if $S \in K\langle\langle Y\rangle\rangle$ (resp. $K\langle\langle\mathcal{X}\rangle\rangle$ is a $⧢\!\!\!\!-$ (resp. $⧢, \texttt{conc}$)-character of $(K\langle Y\rangle, \texttt{conc}, 1_{Y^*})$ (resp. $(K\langle\mathcal{X}\rangle, \texttt{conc}, 1_{\mathcal{X}^*})$ then

1. Since $S \otimes S = \sum_{u,v\in\mathcal{X}^*} \langle S \,|\, u\rangle\langle S \,|\, v\rangle u \otimes v$ then $\Delta_{⧢\!\!\!\!-}(S) = S \otimes S$ (resp. $\Delta_{⧢}(S) = S \otimes S$ and $\Delta_{\texttt{conc}}(S) = S \otimes S$).

2. Since $\Delta_{⧢\!\!\!\!-}$ (resp. $\Delta_{⧢}$ and $\Delta_{\texttt{conc}}$) and the maps $T \longmapsto T \otimes 1_{Y^*}$ and $T \longmapsto 1_{Y^*} \otimes T$ (resp. $T \longmapsto T \otimes 1_{\mathcal{X}^*}$ and $T \longmapsto 1_{\mathcal{X}^*} \otimes T$) are continuous homomorphisms then[19] $\Delta_{⧢\!\!\!\!-}(\log_{\texttt{conc}} S) = \log_{\texttt{conc}} S \otimes 1_{\mathcal{X}^*} + 1_{\mathcal{X}^*} \otimes \log_{\texttt{conc}} S$ (resp. $\Delta_{⧢}(\log_{\texttt{conc}} S) = \log_{\texttt{conc}} S \otimes 1_{\mathcal{X}^*} + 1_{\mathcal{X}^*} \otimes \log_{\texttt{conc}} S$ and $\Delta_{\texttt{conc}}(\log_{\texttt{conc}} S) = \log_{\texttt{conc}} S \otimes 1_{\mathcal{X}^*} + 1_{\mathcal{X}^*} \otimes \log_{\texttt{conc}} S$).

Then S is group like for $\{\Delta_{⧢\!\!\!\!-}, \Delta_{⧢}, \Delta_{\texttt{conc}}\}$, if and only if $\log_{\texttt{conc}} S$ is primitive meaning that the equivalence, between 1 and 5, is an extension of the Ree's theorem which was established for $⧢$ (see [45]) and adapted for $⧢\!\!\!\!-$ (see [29]).

Proposition 4.16. — *Let $\alpha_x, \beta_x, a_s, b_s$, be complex numbers ($x \in \mathcal{X}$ and $s \geq 1$). Then*

$$\Big(\sum_{x\in\mathcal{X}} \alpha_x x\Big)^* ⧢ \Big(\sum_{x\in\mathcal{X}} \beta_x x\Big)^* = \Big(\sum_{x\in\mathcal{X}} (\alpha_x + \beta_x) x\Big)^*,$$

$$\Big(\sum_{s\geq 1} a_s y_s\Big)^* ⧢\!\!\!\!- \Big(\sum_{s\geq 1} b_s y_s\Big)^* = \Big(\sum_{s\geq 1} (a_s + b_s) y_s + \sum_{r,s\geq 1} a_s b_r y_{s+r}\Big)^*.$$

[19]Here, $\log_{\texttt{conc}} S \otimes 1_{Y^*}$ and $1_{Y^*} \otimes \log_{\texttt{conc}} S$ (resp. $\log_{\texttt{conc}} S \otimes 1_{\mathcal{X}^*}$ and $1_{\mathcal{X}^*} \otimes \log_{\texttt{conc}} S$) commute.

Proof. — Let us use $\Delta_{\sqcup\!\sqcup}$ (resp. $\Delta_{⧢}$) defined in (35) and then (54) (resp. (42) and then (55)) and, for any $x_i \in \mathcal{X}, y_t \in Y$, apply (49)

$$\begin{aligned}
&\left\langle \left(\sum_{x\in\mathcal{X}} \alpha_x x\right)^* \sqcup\!\sqcup \left(\sum_{x\in\mathcal{X}} \beta_x x\right)^* \middle| x_i \right\rangle \\
&\qquad = \left\langle \left(\sum_{x\in\mathcal{X}} \alpha_x x\right)^* \otimes \left(\sum_{x\in\mathcal{X}} \beta_x x\right)^* \middle| \Delta_{\sqcup\!\sqcup}(x_i) \right\rangle \\
&\qquad = \left\langle \left(\sum_{x\in\mathcal{X}} \alpha_x x\right)^* \otimes \left(\sum_{x\in\mathcal{X}} \beta_x x\right)^* \middle| x_i \otimes 1_{X^*} + 1_{X^*} \otimes x_i \right\rangle \\
&\qquad = \alpha_i + \beta_i \\
&\qquad = \left\langle \left(\sum_{x\in\mathcal{X}} (\alpha_x + \beta_x) x\right)^* \middle| x_i \right\rangle, \\
&\left\langle \left(\sum_{s\geq 1} a_s y_s\right)^* ⧢ \left(\sum_{s\geq 1} b_s y_s\right)^* \middle| y_t \right\rangle \\
&\qquad = \left\langle \left(\sum_{s\geq 1} a_s y_s\right)^* \otimes \left(\sum_{s\geq 1} b_s y_s\right)^* \middle| \Delta_{⧢}(y_t) \right\rangle \\
&\qquad = \left\langle \left(\sum_{s\geq 1} a_s y_s\right)^* \otimes \left(\sum_{s\geq 1} b_s y_s\right)^* \middle| y_t \otimes 1_{Y^*} + 1_{Y^*} \otimes y_t + \sum_{r,s\geq 1, r+s=t} y_s \otimes y_r \right\rangle \\
&\qquad = a_t + b_t + \sum_{r,s\geq 1, r+s=t} a_s b_r \\
&\qquad = \left\langle \left(\sum_{s\geq 1} (a_s + b_s) y_s + \sum_{r,s\geq 1} a_s b_r y_{s+r}\right)^* \middle| y_t \right\rangle. \qquad \square
\end{aligned}$$

Remark 4.17. — Since $\sum_{r,s\geq 1} a_s b_r y_{s+r} = \sum_{r>s\geq 1} (a_s b_r + a_r b_s) y_{s+r} + \sum_{s\geq 1} a_s b_s y_{2s}$ then $\left(\sum_{s\geq 1} a_s y_s\right)^* ⧢ \left(\sum_{s\geq 1} b_s y_s\right)^* = \left(\sum_{s\geq 1}((a_s + b_s) y_s + a_s b_s y_{2s}) + \sum_{r>s\geq 1}(a_s b_r + a_r b_s) y_{s+r}\right)^*$. Note also that since, for any $x \in \mathcal{X}$ and $n \geq 0$, one has $x^n = x^{\sqcup\!\sqcup n}/n!$ then $\left(\sum_{x\in\mathcal{X}} \alpha_x x\right)^* = \exp_{\sqcup\!\sqcup}(\sum_{x\in\mathcal{X}} \alpha_x x)$ and $\sqcup\!\sqcup$ is commutative then $\exp_{\sqcup\!\sqcup}(A + B) = \exp_{\sqcup\!\sqcup}(A) \exp_{\sqcup\!\sqcup}(B)$. But it is more complicated for $⧢$.

Example 4.18. — For any $y_s, y_r \in Y$ and $a_s, a_r \in \mathbb{C}$, one has (see also Example 4.6)

$$(a_s y_s)^* ⧢ (a_r y_r)^* = (a_s y_s + a_r y_r + a_s a_r y_{s+r})^*, \quad (-a_s y_s)^* ⧢ (a_s y_s)^* = (-a_s^2 y_{2s})^*.$$

Then, for any $c \in \mathbb{C}$ and $y_k \in Y$, one has

1. $(c y_k^*)^{⧢ 2} = (c y_k)^* ⧢ (c y_k)^* = (2 c y_k + c^2 y_{2k})^*$,
2. $(c y_k^*)^{⧢ 3} = (2 c y_k + c^2 y_{2k})^* ⧢ (c y_k)^* = (3 c y_k + 3 c^2 y_{2k} + c^3 y_{3k})^*$.

Corollary 4.19. — 1. *Let $k,n\in\mathbb{N}$, $c\in\mathbb{C}$ and $x\in\mathcal{X}$. Then*

$$\langle (cx)^* \sqcup\!\sqcup (1+cx)^n \mid x^k\rangle = \binom{n+k}{k} c^k.$$

2. *Let $x\in\mathcal{X}$, $y_k\in Y$ and $c\in\mathbb{C}$, $n\in\mathbb{N}_{\geq 1}$. One has*[20]

$$((cx)^*)^{\sqcup\!\sqcup n} = (ncx)^*, \quad ((cx)^*)^n = (cx)^* \sqcup\!\sqcup (1+cx)^{n-1},$$
$$((cy_k)^*)^{\uplus n} = \left(\sum_{i=1}^{n}\binom{n}{i} c^i y_{ik}\right)^* = \biguplus_{i=1}^{n}\left(\binom{n}{i} c^i y_{ik}\right)^*.$$

3. *For any $k,m\in\mathbb{N}$, $x_1,\ldots,x_m\in\mathcal{X}$ and $l_1,\ldots,l_m\in\mathbb{N}$, $l_1+\cdots+l_m=k$, let*

$$P_k := x_1^{l_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup x_m^{l_m} \quad \textit{and} \quad L_k := \mathrm{supp}(P_k).$$

Then, for any $n_1,\ldots,n_m\in\mathbb{N}, c, c_1,\ldots,c_m\in\mathbb{C}\setminus\{0\}$ and $w\in L_k$, one has[21]*,*

$$\left\langle \coprod_{i=1}^{m} ((c_i x_i)^*)^{n_i+1} \,\middle|\, w \right\rangle = \sum_{l_1+\cdots+l_m=k} \binom{n_1+l_1}{l_1}\ldots\binom{n_m+l_m}{l_m} c_1^{l_1}\ldots c_m^{l_m},$$
$$\left\langle \coprod_{i=1}^{m} ((c x_i)^*)^{n_i+1} \,\middle|\, w \right\rangle = \binom{n_1+\cdots+n_m+k}{k} c^k.$$

Proof. —

1. By (49), with $S=(cx)^*, R=(1+cx)^n$ and $(cx)^k \sqcup\!\sqcup (cx)^i = \binom{k+i}{k}(cx)^{k+i}$, one has

$$(cx)^* \sqcup\!\sqcup (1+cx)^n = \sum_{k\geq 0}\sum_{i=0}^{n}\binom{k+i}{k}\binom{n}{i}(cx)^{k+i} = \sum_{k\geq 0}\left(\sum_{i=0}^{n}\binom{k}{k-i}\binom{n}{i}\right)(cx)^k$$

and the Chu–Vandermonde identity [51] yields the expected result.

2. Since $(cx)^* = \exp_{\sqcup\!\sqcup}(cx)$ then $((cx)^*)^{\sqcup\!\sqcup n} = \exp_{\sqcup\!\sqcup}(ncx) = (ncx)^*$. The two last identities are obvious for $n=1$ and supposed to hold, up to rank $n\geq 1$. Next, by $((cx)^*)^{n+1} = (cx)^*((cx)^*)^n$ and $((cy_k)^*)^{\uplus n+1} = (cy_k)^* \uplus ((cy_k)^*)^{\uplus n}$ and then by induction hypothesis,

[20] $((cx)^*)^n = \underbrace{(cx)^*\ldots(cx)^*}_{n \text{ times}}$, $((cx)^*)^{\sqcup\!\sqcup n} = \underbrace{(cx)^* \sqcup\!\sqcup \cdots \sqcup\!\sqcup (cx)^*}_{n-1 \text{ times } \sqcup\!\sqcup}$, $((cy_k)^*)^{\uplus n} = \underbrace{(cx)^* \uplus \cdots \uplus (cx)^*}_{n-1 \text{ times } \uplus}$.

[21] Recall that, for any positive integer k and nonnegative integers $m, n_1, n_2, \ldots, n_m$, the generalized Chu–Vandermonde's identity is expressed as follows [51]

$$\sum_{l_1+l_2+\cdots+l_m=k}\binom{n_1}{l_1}\binom{n_2}{l_2}\ldots\binom{n_m}{l_m} = \binom{n_1+n_2+\cdots+n_m}{k}.$$

one obtains successively

$$\begin{aligned}
((cx)^*)^{n+1} &= (cx)^*((cx)^*)^n \\
&= (cx)^*\Bigg(\sum_{k\geq 0}\binom{n-1+k}{k}(cx)^k\Bigg) \quad \text{(by Item 1)} \\
&= \sum_{k\geq 0}\Bigg(\sum_{l=0}^{k}\binom{n-1+l}{l}\Bigg)(cx)^k \quad \text{(by (48))} \\
&= \sum_{k\geq 0}\binom{n+k}{n}(cx)^k \quad \text{(by the Chu–Vandermonde identity)} \\
&= (cx)^* ⧢ (1+cx)^n \quad \text{(by Item 1)},
\end{aligned}$$

$$\begin{aligned}
(cy_k^*)^{⧢ n+1} &= (cy_k + \sum_{i=1}^{n}\binom{n}{i}c^i y_{ik} + \sum_{i=1}^{n}\binom{n}{i}c^{i+1}y_{(i+1)k})^* \\
&= (cy_k + \binom{n}{1}cy_k + \sum_{i=2}^{n}\binom{n}{i}c^i y_{ik} + \sum_{i=2}^{n+1}\binom{n}{i-1}c^i y_{ik})^* \\
&= \Bigg(\binom{n+1}{1}cy_k + \sum_{i=2}^{n}\Bigg(\binom{n}{i}+\binom{n}{i-1}\Bigg)c^i y_{ik} + \binom{n+1}{n+1}c^{n+1}y_{(n+1)k}\Bigg)^* \\
&= \Bigg(\binom{n+1}{1}cy_k + \sum_{i=2}^{n}\binom{n+1}{i}c^i y_{ik} + \binom{n+1}{n+1}c^{n+1}y_{(n+1)k}\Bigg)^* \\
&= \Bigg(\sum_{i=1}^{n+1}\binom{n+1}{i}c^i y_{ik}\Bigg)^* \\
&= ⧢_{i=1}^{n}\Bigg(\binom{n}{i}c^i y_{ik}\Bigg)^* \quad \text{(by Proposition 4.16)}.
\end{aligned}$$

3. By Items 1–2 and Proposition 4.16, one gets

$$\begin{aligned}
⧢_{i=1}^{m}((c_ix_i)^*)^{n_i+1} &= ⧢_{i=1}^{m}(c_ix_i)^* ⧢ (1+c_ix_i)^{n_i} = ⧢_{i=1}^{m}\sum_{l_i\geq 0}\binom{n_i+l_i}{l_i}(c_ix_i)^{l_i} \\
&= \sum_{k\geq 0}\sum_{l_1+\cdots+l_m=k}\binom{n_1+l_1}{l_1}\ldots\binom{n_m+l_m}{l_m}c_1^{l_1}\ldots c_m^{l_m}P_k.
\end{aligned}$$

It follows then the expected results. □

Definition 4.20 **([50]).** — Let $\mathcal{L}$ be the Lie algebra. Then $\mathcal{L}$ is said to be

1. *nilpotent* if and only if there exists an integer $k \geq 1$ such that the sequence $\{\mathcal{L}^n\}_{n\geq 1}$, defined recursively as follows

$$\mathcal{L}^1 = \mathcal{L}, \quad \mathcal{L}^{n+1} = [\mathcal{L}, \mathcal{L}^n],$$

satisfies $\mathcal{L}^{k+1} = \{0\}$.

2. *solvable* if and only if there exists an integer $k \geq 1$ such that the sequence $\{\mathcal{L}^{(n)}\}_{n\geq 1}$, defined recursively as follows
$$\mathcal{L}^{(1)} = \mathcal{L}, \quad \mathcal{L}^{(n+1)} = [\mathcal{L}^{(n)}, \mathcal{L}^{(n)}],$$
satisfies $\mathcal{L}^{(k+1)} = \{0\}$.

To determine the output y in (3) of the nonlinear dynamical system in (1), the noncommutative generating series σf are approximately computed by using the rational series of the following forms for which their linear representations are examined by Theorems 4.21–4.23 below [29].

$$E_1 x_{i_1} \ldots E_j x_{i_j} E_{j+1}, \quad \text{where } x_{i_1}, \ldots, x_{i_j} \in X,\ E_1, \ldots, E_j \in A^{\mathrm{rat}}\langle\langle x_0 \rangle\rangle, \tag{80}$$
$$E_1 x_{i_1} \ldots E_j x_{i_j} E_{j+1}, \quad \text{where } x_{i_1}, \ldots, x_{i_j} \in X,\ E_1, \ldots, E_j \in A^{\mathrm{rat}}\langle\langle x_1 \rangle\rangle, \tag{81}$$
$$E_1 x_{i_1} \ldots E_j x_{i_j} E_{j+1}, \quad \text{where } x_{i_1}, \ldots, x_{i_j} \in X,\ E_1, \ldots, E_j \in A^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X \rangle\rangle. \tag{82}$$

***Theorem 4.21* (Triangular sub bialgebras of $(A^{\mathrm{rat}}\langle\langle \mathcal{X} \rangle\rangle, ⧢, 1_{X^*}, \Delta_{\mathtt{conc}})$).** — *Let $\rho = (\nu, \mu, \eta)$ be a representation of $R \in A^{\mathrm{rat}}\langle\langle \mathcal{X} \rangle\rangle$ and let $\mathcal{L}$ be the Lie algebra generated by the matices*[22] *$\{\mu(x)\}_{x\in\mathcal{X}}$. Then*

1. *If $\{\mu(x)\}_{x\in\mathcal{X}}$ mutually commute and if the alphabet is finite, any rational exchangeable series decomposes as $R = \sum_{i=1}^{n} ⧢_{x\in\mathcal{X}} R_x^{(i)}$, with $R_x^{(i)} \in A^{\mathrm{rat}}\langle\langle x \rangle\rangle$.*

2. *If $\mathcal{L}$ consists of upper-triangular matrices then $R \in A^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle \mathcal{X} \rangle\rangle ⧢ A\langle \mathcal{X} \rangle$.*

3. *Let $M(x) := \mu(x)x$, for $x \in \mathcal{X}$. Then $M(R) = \sum_{w\in\mathcal{X}^*} \langle R \mid w \rangle \mu(w) w$ and then $R = \nu M(\mathcal{X}^*)\eta$. Moreover,*

   (a) *Since A contains $\mathbb{Q}$ then, by (74), one has*
   $$M(\mathcal{X}^*) = \prod_{l\in\mathcal{L}yn\,\mathcal{X}}^{\searrow} e^{S_l \mu(P_l)}, \textit{(decreasing lexcographical ordered product)}.$$
   By (77), one has in addition,
   $$M(Y^*) = \prod_{l\in\mathcal{L}yn\,Y}^{\searrow} e^{\Sigma_l \mu(\Pi_l)}, \textit{(decreasing lexcographical ordered product)}.$$

   (b) *If $\{\mu(x)\}_{x\in\mathcal{X}}$ are upper-triangular then there exists a diagonal (resp. strictly upper-triangular) letter matrix $D(\mathcal{X})$ (resp. $N(\mathcal{X})$) such that*
   $$M(\mathcal{X}) = D(\mathcal{X}) + N(\mathcal{X})$$
   and then, by Lazard factorization [40], one has
   $$M(\mathcal{X}^*) = ((D(\mathcal{X}^*)N(\mathcal{X}))^* D(\mathcal{X}^*)).$$

[22] $\mathcal{L}$ depends on μ, i.e. R.

(c) *For $X = \{x_0, x_1\}$, similarly (by Lazard factorization again),*

$$M((x_0 + x_1)^*) = (M(x_1^*)M(x_0))^* M(x_1^*) = (M(x_0^*)M(x_1))^* M(x_0^*)$$

and the modules generated by the families series in the forms (80)–(82) *are closed by* conc *and* $\sqcup\!\sqcup$. *Furthermore, it follows that R is a linear combination of series in the form* (80) *(resp.* (81)*) if $M(x_1^*)M(x_0)$ (resp. $M(x_0^*)M(x_1)$) is strictly upper-triangular.*

In all the sequel, $A = K$ is supposed to be an algebraically closed field of characteristic zero. In order to establish Theorem 4.23 below, Lie's theorem [50] (essentially true over algebraically closed fields characteristic zero) and the following Lemma 4.22 are used.

Lemma 4.22. — *Let (ν, τ, η) a representation of S of dimension r such that, for all $x \in \mathcal{X}$, $(\tau(x) - c(x)I_r)$ is strictly upper triangular, then $S \in K^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle \sqcup\!\sqcup K\langle\mathcal{X}\rangle$.*

Proof. — Let $(e_i)_{1 \le i \le r}$ be the canonical basis of $M_{1,r}(K)$. We construct the representations of S_1 and S_2

$$\rho_1 = (\nu, (x \longmapsto \tau(x) - c(x)I_r), \eta) \quad \text{and} \quad \rho_2 = (e_1, (x \longmapsto c(x)I_r), {}^t e_1)$$

and remark that $S_1 \sqcup\!\sqcup S_2$ admits the representation

$$\rho_3 = (\nu \otimes e_1, ((\tau(x) - c(x)I_r) \otimes I_r + I_r \otimes c(x)I_r)_{x \in \mathcal{X}}, \eta \otimes {}^t e_1)$$

as $I_r \otimes c(x)I_r = c(x)I_r \otimes I_r$, ρ_3 is, in fact, $(\nu \otimes e_1, (\tau(x) \otimes I_r)_{x \in \mathcal{X}}, \eta \otimes {}^t e_1)$ which represents S, the result now comes from the fact that $S_1 \in K\langle\mathcal{X}\rangle$ and $S_2 = \left(\sum_{x \in \mathcal{X}} c(x)x\right)^* \in K^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$. □

***Theorem 4.23* (Triangular sub bialgebras of $(K^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle, \sqcup\!\sqcup, 1_{X^*}, \Delta_{\mathtt{conc}})$).** — *With the notations in Definition 4.20, one has*

1. *$\mathcal{L}$ is commutative if and only if $R \in K^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$,*
2. *$\mathcal{L}$ is nilpotent if and only if $R \in K^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle \sqcup\!\sqcup K\langle\mathcal{X}\rangle$,*
3. *$\mathcal{L}$ is solvable if and only if R is a linear combination of series in form* (82).

Proof. — 1. Since $\mathcal{L}$ is commutative then, due to the commutation of matrices, for any $x, y \in \mathcal{X}$ and $p, s \in \mathcal{X}^*$, one has $\langle R \,|\, pxys\rangle = \langle R \,|\, pyxs\rangle$. Conversely, since ρ is minimal then there is P_i and $Q_i \in K\langle\mathcal{X}\rangle$ $(i = 1 \ldots n)$ such that, for any $u \in \mathcal{X}^*$, (see [1, 19])

$$\mu(u) = (\langle P_i \triangleright R \triangleleft Q_i \,|\, u\rangle)_{1 \le i,j \le n} = (\langle R \,|\, Q_i u P_i\rangle)_{1 \le i,j \le n}.$$

Now, for any $x, y \in \mathcal{X}$,

$$\mu(xy) = (\langle R \,|\, Q_i xy P_i\rangle)_{1 \le i,j \le n} \overset{(*)}{=} (\langle R \,|\, Q_i yx P_i\rangle)_{1 \le i,j \le n} = \mu(yx),$$

(equality $\overset{(*)}{=}$ being due to exchangeability).

2. Since $\mathcal{L}$ is nilpotent then let K^n be the space of the representation of $\mathcal{L}$ given by μ and $\bigoplus_{j=1}^m V_j$ be a decomposition of K^n into indecomposable $\mathcal{L}$-modules (see [14] for characteristic 0, or [3] for arbitrary characteristic), we know that V_j is an $\mathcal{L}$-module and

the action of $\mathcal{L}$ is triangularisable with constant diagonals inside each sector V_j. Thus, it is an invertible matrix P in $\mathrm{GL}(n,K)$ such that, for any $x \in \mathcal{X}$,

$$P\mu(x)P^{-1} = \mathrm{blockdiag}(T_1, T_2 \dots, T_k) = \begin{pmatrix} T_1 & 0 & \dots & 0 \\ 0 & T_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & T_k \end{pmatrix},$$

where the T_j's are upper triangular matrices with constant coefficients in the diagonal. It means that $T_j(x) = \lambda(x)I + N(x)$, where $N(x)$ is strictly upper-triangular[23]. Set d_j to be the dimension of T_j (so that $n = \sum_{j=1}^{m} d_j$), partitioning $\nu P^{-1} = \nu'$ (resp. $P\eta = \eta'$) with these dimensions we get blocks so that each (ν'_j, T_j, η'_j) is the representation of a series R_j and $R = \sum_{j=1}^{m} R_j$. By Lemma 4.22, each R_j belongs to $K^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle \sqcup\!\sqcup K\langle\mathcal{X}\rangle$ then so is their sum R.

Conversely, if $\rho_i = (\nu_i, \tau_i, \eta_i), i = 1, 2$, are two representations then

$$\begin{aligned}
&[\tau_1(x) \otimes I_r + I_r \otimes \tau_2(x), \tau_1(y) \otimes I_r + I_r \otimes \tau_2(y)] \\
&\qquad = [\tau_1(x) \otimes I_r, \tau_1(y) \otimes I_r] + [\tau_1(x) \otimes I_r, I_r \otimes \tau_2(y)] \\
&\qquad\qquad + [I_r \otimes \tau_2(x), \tau_1(y) \otimes I_r] + [I_r \otimes \tau_2(x), I_r \otimes \tau_2(y)] \\
&\qquad = [\tau_1(x), \tau_1(y)] \otimes I_r + I_r \otimes [\tau_2(x), \tau_2(y)]
\end{aligned}$$

because

$$\begin{aligned}
[\tau_1(x) \otimes I_r, \tau_1(y) \otimes I_r] &= \tau_1(x)\tau_1(y) \otimes I_r - \tau_1(y)\tau_1(x) \otimes I_r \\
&= [\tau_1(x), \tau_1(y)] \otimes I_r, \\
[\tau_1(x) \otimes I_r, I_r \otimes \tau_2(y)] &= \tau_1(x) \otimes \tau_2(y) - \tau_2(y) \otimes \tau_1(x) = 0, \\
[I_r \otimes \tau_2(x), \tau_1(y) \otimes I_r] &= \tau_1(y) \otimes \tau_2(x) - \tau_1(y) \otimes \tau_2(x) = 0, \\
[I_r \otimes \tau_2(x), I_r \otimes \tau_2(y)] &= I_r \otimes \tau_2(x)\tau_2(y) - I_r \otimes \tau_2(y)\tau_2(x) \\
&= I_r \otimes [\tau_2(x), \tau_2(y)].
\end{aligned}$$

A similar formula holds for m-fold brackets (Dynkin combs), so that if $\mathcal{L}(\tau_i)$'s are nilpotent, the Lie algebra $\mathcal{L}(\tau_1 \otimes I_r + I_r \otimes \tau_2)$ is also nilpotent. The point here comes from the fact that series in $K^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$ as well as in $K\langle\mathcal{X}\rangle$ admit nilpotent representations, so, let (α, τ, β) such a representation and (α', τ', β') its minimal quotient (obtained by minimization, see [1]), then $\mathcal{L}(\tau')$ is nilpotent as a quotient of $\mathcal{L}(\tau)$. Now two minimal representations being isomorphic, $\mathcal{L}(\mu)$ is isomorphic to $\mathcal{L}(\tau)$ and then it is nilpotent.

3. Since the Lie algebra $\mathcal{L}$ (generated by $\{\mu(x)\}_{x\in X}$) is solvable (see Definition 4.20) then, by a Lie's theorem [50], the matrices $\{\mu(x)\}_{x\in X}$ are simultaneously upper triangularisable (as in the above case of nilpotent $\mathcal{L}$). Hence, in the favorable change of bases, one supposes (without loss generality) that the linear representation (ν, μ, η) of R is such that each matrix $\mu(x)$ is upper-triangular, and let $D(\mathcal{X})$ (resp. $N(\mathcal{X})$) be the diagonal (resp. strictly upper-triangular) letter matrix such that $M(\mathcal{X}) = D(\mathcal{X}) + N(\mathcal{X})$. Then

$$R = \nu M(\mathcal{X}^*)\eta = \nu(D(\mathcal{X}^*)N(\mathcal{X}))^* D(\mathcal{X}^*)\eta.$$

[23] Even, as K is infinite, there is a global linear form on $\mathcal{L}$, λ_{lin} such that, for all $g \in \mathcal{L}$, $PgP^{-1} - \lambda_{lin}(g)I$ is strictly upper-triangular.

Since $D(\mathcal{X}^*)N(\mathcal{X})$ is nilpotent of order n then

$$(D(\mathcal{X}^*)N(\mathcal{X}))^* = \sum_{j=0}^{n}(D(\mathcal{X}^*)N(\mathcal{X}))^j.$$

Hence, letting $\mathcal{S}$ be the vector space generated by forms of type (82) which is closed by concatenation, one has

$$D(\mathcal{X}^*)N(\mathcal{X}) \in \mathcal{S}^{n\times n}\text{and then}(D(\mathcal{X}^*)N(\mathcal{X}))^* \in \mathcal{S}^{n\times n}.$$

Finally, $R = \nu M(\mathcal{X}^*)\eta \in \mathcal{S}$ which is the claim.

Conversely, as sums and quotients of solvable representations are solvable, it suffices to show that a single form of type (82) admits a solvable representation and end by quotient and isomorphism as in Item 2. From Proposition 4.5, we get the fact that, if R_i admits solvable representations so does R_1R_2, then the claim follows from the fact that, firstly, single letters admit solvable (even nilpotent) representations and secondly series of $\sqcup\!\sqcup_{x\in\mathcal{X}}\{K^{\mathrm{rat}}\langle\langle x\rangle\rangle\}$ admit solvable representations. Finally, we choose (or construct) a solvable representation of R, call it (α,τ,β) and $(\alpha',\tau'\beta')$ its minimal quotient, then $\mathcal{L}(\tau')$ is solvable as a quotient of $\mathcal{L}(\tau)$. Now two minimal representations being isomorphic, $\mathcal{L}(\mu)$ is isomorphic to $\mathcal{L}(\tau)$, hence solvable. ☐

Remark 4.24. —

1. Denoting by $K^{\mathrm{rat}}_{\mathrm{nil}}\langle\langle\mathcal{X}\rangle\rangle$ (resp. $K^{\mathrm{rat}}_{\mathrm{sol}}\langle\langle\mathcal{X}\rangle\rangle$), the set of rational series such that $\mathcal{L}(\mu)$ is nilpotent (resp. solvable), we get a tower of sub Hopf algebras of the Sweedler's dual, $K^{\mathrm{rat}}_{\mathrm{nil}}\langle\langle\mathcal{X}\rangle\rangle \subset K^{\mathrm{rat}}_{\mathrm{sol}}\langle\langle\mathcal{X}\rangle\rangle \subset \mathcal{H}^{\circ}_{\sqcup\!\sqcup}(\mathcal{X})$.

2. For an example of series S with solvable representation but such that $S \notin K^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle \sqcup\!\sqcup K\langle\mathcal{X}\rangle$, one can take $\mathcal{X} = \{a,b\}$ and $S = a^*b(-a)^*$.

5. Conclusion

In this work, various products $\{\mathtt{conc}, \sqcup\!\sqcup, \uplus\}$ of noncommutative formal series, with coefficients in a commutative ring A containing $\mathbb{Q}$, and their coproducts $\{\Delta_{\mathtt{conc}}, \Delta_{\sqcup\!\sqcup}, \Delta_{\uplus}\}$ are examined, in Section 2. Basing on various pairs of dual bases, contructed in Section 3, the representative (or rational) series, viewed as functions on the monoid $(\mathcal{X}^*, 1_{\mathcal{X}^*})$ (resp. $(Y^*, 1_{Y^*})$) within their associated bialgebra $(A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle, \sqcup\!\sqcup, 1_{\mathcal{X}^*}, \Delta_{\mathtt{conc}})$ (resp. $(A^{\mathrm{rat}}\langle\langle Y\rangle\rangle, \uplus, 1_{Y^*}, \Delta_{\mathtt{conc}})$), were factorized and decomposed, in Section 4. In particular, for A being a field K. This bialgebra coincides with the Sweedler's dual of the bialgebra $(K\langle\mathcal{X}\rangle, \mathtt{conc}, 1_{\mathcal{X}^*}, \Delta_{\sqcup\!\sqcup})$ (resp. $(K\langle Y\rangle, \mathtt{conc}, 1_{Y^*}, \Delta_{\uplus})$).

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