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THE ALGEBRAIC GROUPS LEADING TO SIMULTANEOUS APPROXIMATION OF AN ALGEBRAIC NUMBER AND ITS SQUARE

by

Fujimori Masami

Abstract. — We determine the algebraic groups which have a close relation to simultaneous approximation of an algebraic number and its square.

Résumé. — On détermine les groupes algébriques qui ont une étroite relation avec l'approximation simultanée d'un nombre algébrique et de son carré.

1. Introduction

Denote respectively by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and $\overline{\mathbb{Q}}\,(\hookrightarrow \mathbb{C})$ the ring of rational integers, the field of rational numbers, the field of real numbers, the field of complex numbers, and the algebraic closure of $\mathbb{Q}$ thought in $\mathbb{C}$. Let α be an element of $\overline{\mathbb{Q}} \setminus \mathbb{Q}$; q, r, s three indeterminates; ε an arbitrarily fixed positive constant; and $|\cdot|$ the usual absolute value on $\mathbb{R}$. When α belongs to $\mathbb{R}$ and the degree of α over $\mathbb{Q}$ is at least 3, finiteness of the number of rational integral solutions to the simultaneous approximation inequalities

$$\left|\alpha - \frac{r}{q}\right| < \frac{1}{|q|^{3/2+\varepsilon}}, \quad \left|\alpha^2 - \frac{s}{q}\right| < \frac{1}{|q|^{3/2+\varepsilon}}$$

is deduced from finiteness of the number of rational integral solutions to a parametric system of linear inequalities

$$|q| < Q^{2-\delta}, \quad |-q\alpha + r| < \frac{1}{Q^{1+\delta}}, \quad |-q\alpha^2 + s| < \frac{1}{Q^{1+\delta}} \quad (Q > 1),$$

where Q is a variable real parameter and δ is an appropriate positive number. The latter fact is a consequence of the subspace theorem of Schmidt, which is a generalization of the famous Roth's theorem (cf. e.g. [4, VI Section 3]).

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Key words and phrases. — simultaneous approximation, general Roth system, semi-stable filtration, Hodge filtration, Hodge group.

Let $x_1, \ldots, x_n$ be indeterminates; $l_1, \ldots, l_n$ linearly independent linear forms in $x_1, \ldots, x_n$ with coefficients in $\overline{\mathbb{Q}} \cap \mathbb{R}$; and $c(1), \ldots, c(n)$ real constant numbers with $\sum_{k=1}^{n} c(k) = 0$. The system $\mathcal{S} = (l_1, \ldots, l_n; c(1), \ldots, c(n))$ is called a *general* ROTH *system* if the simultaneous linear inequalities

$$(1) \qquad |l_k| < \frac{1}{Q^{c(k)+\delta}} \quad (Q > 1;\ k = 1, \ldots, n)$$

have only a finite number of rational integral solutions for each arbitrarily fixed positive number δ. The system $\mathcal{S}_\alpha = \left(q, -q\alpha + r, -q\alpha^2 + s; -2, 1, 1\right)$ in the previous paragraph is an example of general ROTH system. The subspace theorem of SCHMIDT tells us in particular that whether a given system is a general ROTH system or not can be known from a certain aspect of a filtered vector space derived from the given system. To describe this phenomenon, we need a few terminology.

Put $V = x_1\mathbb{Q} \oplus \cdots \oplus x_n\mathbb{Q}$. We associate the system $\mathcal{S} = (l_1, \ldots, l_n; c(1), \ldots, c(n))$ with a filtration $F_{\mathcal{S}}^{\cdot} V$ over $\overline{\mathbb{Q}}$ on V given by

$$F_{\mathcal{S}}^{i} V = \sum_{i \le c(k)} l_k \overline{\mathbb{Q}} \quad (i \in \mathbb{R}).$$

The filtration thus obtained is descending, exhaustive, separated, and left-continuous in the sense that we have

$$F_{\mathcal{S}}^{i} V \supset F_{\mathcal{S}}^{j} V \quad (i \le j), \qquad \bigcup_{i \in \mathbb{R}} F_{\mathcal{S}}^{i} V = V \otimes_{\mathbb{Q}} \overline{\mathbb{Q}},$$
$$\bigcap_{i \in \mathbb{R}} F_{\mathcal{S}}^{i} V = 0, \qquad \text{and} \qquad \bigcap_{i < j} F_{\mathcal{S}}^{i} V = F_{\mathcal{S}}^{j} V.$$

Notice that if $l_k \in F_{\mathcal{S}}^{i} V$, then the parametric system of linear inequalities (1) requires any solution to satisfy a linear inequality

$$|l_k| < \frac{1}{Q^{i+\delta}}$$

for some value of the parameter Q which depends on the solution.

Let V be a finite dimensional non-zero vector space over $\mathbb{Q}$ equipped with a filtration $F^{\cdot} V$ ($i \in \mathbb{R}$) over $\overline{\mathbb{Q}}$ as above. Let $F^{w+} V = \cup_{w<j} F^{j} V$ and $\mathrm{gr}^{w}(F^{\cdot} V) = F^{w} V / F^{w+} V$. A real number

$$\mu(V) = \mu(V, F^{\cdot} V) = \frac{1}{\dim_{\mathbb{Q}} V} \sum_{w \in \mathbb{R}} w \dim_{\overline{\mathbb{Q}}} \mathrm{gr}^{w}(F^{\cdot} V)$$

is called the *slope* of the filtered vector space $V = (V, F^{\cdot} V)$. It is an average of indices at which the filtration narrows. A filtered vector space V or its filtration is said to be *semi-stable* if for any non-zero vector subspace W over $\mathbb{Q}$ of V with the induced sub-filtration over $\overline{\mathbb{Q}}$, the inequality $\mu(W) \le \mu(V)$ is valid. We denote by $\mathcal{C}_0^{\mathrm{ss}}(\mathbb{Q}, \overline{\mathbb{Q}})$ the category of finite dimensional vector spaces over $\mathbb{Q}$ equipped with semi-stable filtration over $\overline{\mathbb{Q}}$ of slope zero. The morphisms in $\mathcal{C}_0^{\mathrm{ss}}(\mathbb{Q}, \overline{\mathbb{Q}})$ are the linear maps over $\mathbb{Q}$ between their underlying vector spaces which respect filtrations when linearly extended over $\overline{\mathbb{Q}}$.

A distinction between the general ROTH systems and the others is drawn as follows:

***Theorem 1.1* (Schmidt, cf. e.g.** [4, VI Theorem 2B]**).** — *The filtration $F_{\mathcal{S}}^{\cdot} V$ derived from a general* ROTH *system $\mathcal{S}$ is semi-stable of slope zero. Conversely, every object of $\mathcal{C}_0^{\mathrm{ss}}(\mathbb{Q}, \overline{\mathbb{Q}})$ whose filtration is defined over $\overline{\mathbb{Q}} \cap \mathbb{R}$ comes from a general* ROTH *system.*

For objects $V = (V, F^{\cdot}V)$ and $W = (W, F^{\cdot}W)$ in $\mathcal{C}_0^{\mathrm{ss}}(\mathbb{Q}, \overline{\mathbb{Q}})$, their tensor product $V \otimes W$ is the vector space $V \otimes_{\mathbb{Q}} W$ equipped with the filtration

$$F^i(V \otimes_{\mathbb{Q}} W) = \sum_{i=j+k} F^jV \otimes_{\overline{\mathbb{Q}}} F^kW \quad (i \in \mathbb{R}).$$

The tensor product $V \otimes W$ is again semi-stable of slope zero ([1, 5]), which implies the following:

***Theorem 1.2* (Faltings, Totaro).** — *Let $\omega_0^{\mathrm{ss}}(\mathbb{Q}, \overline{\mathbb{Q}})$ be the forgetful tensor functor of $\mathcal{C}_0^{\mathrm{ss}}(\mathbb{Q}, \overline{\mathbb{Q}})$ to the tensor category $\mathrm{Vec}_{\mathbb{Q}}$ of finite dimensional vector spaces over $\mathbb{Q}$. The tensor category $\mathcal{C}_0^{\mathrm{ss}}(\mathbb{Q}, \overline{\mathbb{Q}})$ is equivalent to the tensor category $\mathrm{Rep}_{\mathbb{Q}} \mathrm{Aut}\, \omega_0^{\mathrm{ss}}(\mathbb{Q}, \overline{\mathbb{Q}})$ of finite dimensional representations over $\mathbb{Q}$ of the affine group scheme $\mathrm{Aut}\, \omega_0^{\mathrm{ss}}(\mathbb{Q}, \overline{\mathbb{Q}})$ of natural equivalences of the functor $\omega_0^{\mathrm{ss}}(\mathbb{Q}, \overline{\mathbb{Q}})$.*

A very interesting byproduct of these two theorems is the fact that a general Roth system is always obtained from a representation of some algebraic group defined over $\mathbb{Q}$ and vice versa. Let $\check{V}$ be the $\mathbb{Q}$-vector space of linear forms in the indeterminates q, r, s. What we are concerned about in our present paper is the filtration $F_\alpha^{\cdot}$ on $\check{V}$ defined over $\overline{\mathbb{Q}}$ given by

$$F_\alpha^i \check{V} = \begin{cases} \check{V} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} & (i \leq -2) \\ (-q\alpha + r)\overline{\mathbb{Q}} \oplus (-q\alpha^2 + s)\overline{\mathbb{Q}} & (-2 < i \leq 1) \\ 0 & (i > 1), \end{cases}$$

with which a general Roth system $\mathcal{S}_\alpha = \left(q, -q\alpha + r, -q\alpha^2 + s; -2, 1, 1\right)$ at the beginning of this section is associated. Before stating the result of our present paper, we review what is known about the filtration derived from the Roth inequality

$$\left|\alpha - \frac{r}{q}\right| < \frac{1}{|q|^{2+\varepsilon}},$$

or parametric simultaneous linear inequalities

$$|q| < Q^{1-\delta}, \quad |-q\alpha + r| < \frac{1}{Q^{1+\delta}} \quad (Q > 1).$$

Let $\check{W} = q\mathbb{Q} \oplus r\mathbb{Q}$. We define a filtration $F_\alpha^{\cdot}\check{W}$ on $\check{W} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ as

$$F_\alpha^i \check{W} = \begin{cases} \check{W} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} & (i \leq -1) \\ (-q\alpha + r)\overline{\mathbb{Q}} & (-1 < i \leq 1) \\ 0 & (i > 1). \end{cases}$$

The filtered vector space $\check{W} = (\check{W}, F_\alpha^{\cdot}\check{W})$ is readily seen to be an object of $\mathcal{C}_0^{\mathrm{ss}}(\mathbb{Q}, \overline{\mathbb{Q}})$. When $\alpha \in \overline{\mathbb{Q}} \cap \mathbb{R}$, the (classical) Roth system $(q, -q\alpha + r; -1, 1)$ is associated with it. For any $\alpha \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$, we have proved in [3] that $\check{W}$ is in the image of a fully faithful tensor functor ι from the tensor category of finite dimensional representations over $\mathbb{Q}$ of a 1-dimensional anisotropic torus over $\mathbb{Q}$ (which varies with α) or the special linear group SL_2 of degree 2 according as the number α is quadratic over $\mathbb{Q}$ or not. The functor ι is compatible with the respective forgetful functors to the tensor category of finite dimensional vector spaces over $\mathbb{Q}$. This means the filtered vector space $\check{W}$ may be regarded as a representation of one of the algebraic groups determined by α. We would like to recall the definition of ι in some detail when $\alpha \notin \mathbb{R}$.

Let $\mathbb{G}_{\mathrm{m}}$ be the standard 1-dimensional multiplicative group, $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ the complex conjugation restricted to $\overline{\mathbb{Q}}$,

$$\beta = \sigma(\alpha), \quad \text{and} \quad P = \begin{pmatrix} 1 & 1 \\ \alpha & \beta \end{pmatrix}.$$

An embedding e defined over $\overline{\mathbb{Q}}$ of $\mathbb{G}_{\mathrm{m}}$ into SL_2 is given as, using the usual identification $\mathbb{G}_{\mathrm{m}}(R) \simeq R^{\times}$ (the multiplicative group of invertible elements in R) for a $\overline{\mathbb{Q}}$-algebra R,

$$\begin{aligned} e(c) &= {}^tP^{-1} \begin{pmatrix} c^{-1} & 0 \\ 0 & c \end{pmatrix} {}^tP \qquad (c \in R^{\times}) \\ &= \frac{1}{\beta - \alpha} \begin{pmatrix} \beta & -\alpha \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c^{-1} & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 1 & \beta \end{pmatrix}. \end{aligned}$$

Denote by T_α the image of e which is a subtorus over $\overline{\mathbb{Q}}$ of SL_2. The smallest subgroup $\breve{H}$ defined over $\mathbb{Q}$ of SL_2 that includes T_α when the base field is extended over $\overline{\mathbb{Q}}$ is T_α itself or the whole group SL_2 [3, Section 4].

Put $\psi(1) = -q\beta + r$ and $\psi(2) = -q\alpha + r$. The torus T_α is naturally identified with the 1-dimensional multiplicative group $T = \mathrm{Spec}(\overline{\mathbb{Q}}[q,r]/(1 - \psi(1)\psi(2)))$ whose generators of its character group are $\psi(1)$ and $\psi(2)$. The identification is given by the map

$$T \ni (q, r) \mapsto \begin{pmatrix} r - q(\alpha+\beta) & -q\alpha\beta \\ q & r \end{pmatrix} \in T_\alpha \subset \mathrm{SL}_2$$

[3, Lemma 2.5 & Remark 2.6]. The tori T and T_α are generally defined over $\overline{\mathbb{Q}} \cap \mathbb{R}$. If α is quadratic over $\mathbb{Q}$, then they are defined over $\mathbb{Q}$.

To the triple of the group $\breve{H}$, the inclusion map $\kappa = \mathrm{incl}\colon T_\alpha \hookrightarrow \breve{H} \times_{\mathbb{Q}} \overline{\mathbb{Q}}$, and the cocharacter $e\colon \mathbb{G}_{\mathrm{m}} \times_{\mathbb{Q}} \overline{\mathbb{Q}} \to T_\alpha$, apply the method of construction of a tensor functor $\iota_{\breve{H},\kappa,e}\colon \mathrm{Rep}_{\mathbb{Q}}(\breve{H}) \to \mathcal{C}(\mathbb{Q}, \overline{\mathbb{Q}})$ in our former paper [3, Section 1]. Remember that for a finite dimensional representation space V over $\mathbb{Q}$ of $\breve{H}$, we have defined the filtration $V^{\cdot} = V^{\cdot}_{\kappa,e}$ over $\overline{\mathbb{Q}}$ of $\iota_{\breve{H},\kappa,e}(V)$ as

$$V^i = \bigoplus_{i \le \langle \phi, e \rangle} V_\phi \quad (i \in \mathbb{R}).$$

Here $\langle\,\cdot\,,\cdot\,\rangle$ is the canonical $\mathbb{Z}$-valued pairing between the characters and the cocharacters of T_α and V_ϕ is the subspace over $\overline{\mathbb{Q}}$ of $V \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ on which T_α acts by multiplication of a character ϕ via the map $\kappa = \mathrm{incl}$. The functor $\iota = \iota_{\breve{H},\kappa,e}$ is fully faithful [3, Theorem 3.8]. The standard representation of SL_2 on $\breve{W} = q\mathbb{Q} \oplus r\mathbb{Q}$ restricted to $\breve{H}$ defines a representation of $\breve{H}$. We have a direct sum decomposition

$$\breve{W} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} = \breve{W}_{\psi(1)} \oplus \breve{W}_{\psi(2)} = (-q\beta + r)\overline{\mathbb{Q}} \oplus (-q\alpha + r)\overline{\mathbb{Q}}.$$

Since $\langle \psi(1), e \rangle = -1$ and $\langle \psi(2), e \rangle = 1$, the filtration on $\breve{W}$ attached by the functor ι coincides with the filtration $F^{\cdot}_\alpha \breve{W}$ defined earlier. Thus the filtered vector space $(\breve{W}, F^{\cdot}_\alpha \breve{W})$ is in the image of a fully faithful tensor functor ι from the tensor category of finite dimensional representations over $\mathbb{Q}$ of an algebraic group $\breve{H}$ defined over $\mathbb{Q}$.

Let $\mathbb{A}$ be the 1-dimensional affine space, which is a ring scheme. Put $N = \psi(1)\psi(2) = (-q\beta + r)(-q\alpha + r) \in \mathbb{R}[q, r]$. By means of the basis $-\alpha, 1$ of $\mathbb{C}$ as an $\mathbb{R}$-vector space, the

Weil restriction $\operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{A}$ from $\mathbb{C}$ to $\mathbb{R}$ of $\mathbb{A}$ is coordinated as $\operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{A} \simeq \operatorname{Spec}(\mathbb{R}[q,r])$. The correspondence

$$\left(\operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{A}\right)(\mathbb{R}) = \mathbb{C} \simeq \operatorname{Spec}(\mathbb{R}[q,r])(\mathbb{R})$$

is such that for $a, b \in \mathbb{R}$

$$\mathbb{C} \ni a(-\alpha) + b \ \longmapsto\ (q \mapsto a,\ r \mapsto b) \in \operatorname{Spec}(\mathbb{R}[q,r])(\mathbb{R}).$$

The function N is the complex norm map $\operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{A} \to \mathbb{A}$. The Deligne torus $\mathbb{S} = \operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{\mathrm{m}}$ is an open subscheme of $\operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{A}$ and in our coordinate corresponds to $\operatorname{Spec}(\mathbb{R}[q,r]_N)$, where $\mathbb{R}[q,r]_N$ is the localization ring of $\mathbb{R}[q,r]$ by the multiplicative system of non-negative powers of N. The functions $\psi(1)$ and $\psi(2)$ are a pair of generators of the character group of a 2-dimensional torus $\mathbb{S}$. Our torus $T \times_{\overline{\mathbb{Q}} \cap \mathbb{R}} \mathbb{R} = \operatorname{Spec}(\mathbb{R}[q,r]/(1-N))$ is the kernel of the norm $N \colon \mathbb{S} \to \mathbb{G}_{\mathrm{m}}$.
We denote by $\breve{W}$ the $\mathbb{Q}$-vector space $q\mathbb{Q} \oplus r\mathbb{Q}$ again. The $\mathbb{R}$-vector space $\breve{W} \otimes_{\mathbb{Q}} \mathbb{R}$ can be regarded as (the set of $\mathbb{R}$-valued points of) the dual vector space to $\operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{A} \simeq \operatorname{Spec}(\mathbb{R}[q,r])$. The multiplication on a ring scheme $\mathbb{A}$ induces a canonical action of the Deligne torus $\mathbb{S} = \operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{\mathrm{m}}$ on a 2-dimensional vector space $\operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{A}$, hence on $\breve{W} \otimes_{\mathbb{Q}} \mathbb{R}$. Using the above coordinates of $\operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{A}$ and of $\mathbb{S}$, the action of $(q_0, r_0) \in \mathbb{S}$ on $\breve{W} \otimes_{\mathbb{Q}} \mathbb{R}$ is expressed in matrix form as

$$(q,r) \longmapsto (q,r) \begin{pmatrix} r_0 - q_0(\alpha+\beta) & -q_0\alpha\beta \\ q_0 & r_0 \end{pmatrix}.$$

The Hodge decomposition of $\breve{W} \otimes_{\mathbb{Q}} \mathbb{C}$ reads as

$$\breve{W} \otimes_{\mathbb{Q}} \mathbb{C} = \breve{W}^{0,1} \oplus \breve{W}^{1,0},$$

where

$$\breve{W}^{0,1} = \breve{W}_{\psi(1)} \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \quad \text{and} \quad \breve{W}^{1,0} = \breve{W}_{\psi(2)} \otimes_{\overline{\mathbb{Q}}} \mathbb{C}.$$

We understand that the Hodge filtration on $\breve{W} \otimes_{\mathbb{Q}} \mathbb{C}$ is (essentially) the same as the base field extension from $\overline{\mathbb{Q}}$ to $\mathbb{C}$ of the filtration $F_\alpha^{\cdot}\breve{W}$. Our algebraic group $\breve{H}$ is nothing but the Hodge group (the special Mumford-Tate group) of the $\mathbb{Q}$-Hodge structure $\breve{W}$.
In this way, we see our category $\mathcal{C}(\mathbb{Q}, \overline{\mathbb{Q}})$ contains $\mathbb{Q}$-Hodge structures defined over $\overline{\mathbb{Q}}$. For any choice of an object $V \in \mathcal{C}(\mathbb{Q}, \overline{\mathbb{Q}})$, to determine the algebraic group G defined over $\mathbb{Q}$ such that V ought to be considered a representation of G is a generalization of the problem of finding out the Hodge group of a prescribed Hodge structure. In particular, an extension of our result [3] for the 2-dimensional objects to higher dimensional objects is not a simple matter.
Now we explain our result in the present paper.
When α is (rational or) quadratic over $\mathbb{Q}$, the filtration $F_\alpha^{\cdot}\breve{V}$ is not semi-stable (cf. Appendix), hence we assume the degree of α over $\mathbb{Q}$ is at least 3.

Theorem 1.3. — *If the algebraic number α is cubic over $\mathbb{Q}$, then there exists a fully faithful tensor functor ι of the category $\operatorname{Rep}_{\mathbb{Q}} T_\alpha$ of finite dimensional representations over $\mathbb{Q}$ of a two-dimensional anisotropic torus T_α over $\mathbb{Q}$ into the tensor category $\mathcal{C}_0^{\mathrm{ss}}(\mathbb{Q}, \overline{\mathbb{Q}})$ such that the group $T_\alpha(\mathbb{Q})$ of $\mathbb{Q}$-valued points of the torus T_α is isomorphic to the kernel of the norm map of the cubic number field $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$, such that the functor ι commutes with the forgetful functors to the tensor category $\mathrm{Vec}_{\mathbb{Q}}$ of finite dimensional vector spaces over $\mathbb{Q}$, and such that its image contains the filtered vector space $(\breve{V}, F_\alpha^{\cdot}\breve{V})$.*

If the algebraic number α is not *cubic over $\mathbb{Q}$, then there exists a fully faithful tensor functor ι of the category $\mathrm{Rep}_{\mathbb{Q}}\,\mathrm{SL}_3$ of finite dimensional representations over $\mathbb{Q}$ of the special linear group SL_3 of degree 3 into $\mathcal{C}_0^{\mathrm{ss}}(\mathbb{Q},\overline{\mathbb{Q}})$ such that the functor ι is compatible with the forgetful tensor functors to $\mathrm{Vec}_{\mathbb{Q}}$ and such that the image of ι contains the filtered vector space $(\check{V}, F_\alpha^{\cdot}\check{V})$.*

The method of proof is similar to that in [3].
Statements and proofs are given over arbitrary fields in the body of the paper. The conclusion becomes a little weak when the base field has a positive characteristic.
The plan of the paper is as follows: In the first two sections, we take up the case when the Galois closure of the field generated by a given α over the base field is abelian of type $(2,2,\ldots)$. The next two sections treat the remaining cases. In the last section, we make clear what are the Hodge-like groups in the situation of our present paper when the characteristic of the base field is zero. The results in Section 2 and Section 3 are newly obtained. The results in the other sections have been announced at a meeting [2] but their proofs are not yet published.
As we have said above, although the results and proofs of our former paper [3] and those of the present one are alike, determination of Hodge-like groups for arbitrarily given filtered vector spaces is not easy in general. Its confirmation in our present case is already a bit complicated. So, we believe the contribution of our present work to the literature would be helpful especially for the people who feel an interest in our former paper [3] and who want to know results in simultaneous approximation cases. This is why the author has written this paper.

2. Commutative case of type $(2,2,\ldots)$

In this section, we define several things that we need in Section 3. We see some properties of them.
Let K be an arbitrary field. We denote respectively by $\mathbb{G}_{\mathrm{m}}$ and by SL_3 the standard 1-dimensional split multiplicative group and the special linear group of degree 3 whose base fields are both viewed as K. Let K^{sep} be a separable algebraic closure of K and $\alpha \in K^{\mathrm{sep}}$ such that $\omega^2(\alpha)=\alpha$ for all $\omega \in \mathrm{Gal}\,(K^{\mathrm{sep}}/K)$. In this situation, the Galois closure of the field generated by α over K is a finite abelian extension of K (cf., e.g., [3, Lemma 4.3]). Assume that the extension degree of $K(\alpha)$ over K is at least 4. Fix a (finite or infinite) Galois extension field L of K containing α and also fix $\sigma,\tau \in \mathrm{Gal}\,(K^{\mathrm{sep}}/K)$ with $\sigma(\alpha)\neq\alpha$, $\tau(\alpha)\neq\alpha$, and $\sigma(\alpha)\neq\tau(\alpha)$. Note that we have $\sigma\tau(\alpha)=\tau\sigma(\alpha)\neq\alpha$. Elements $\beta,\gamma,\delta \in L$ and an element $P \in \mathrm{GL}_3(L)$ are respectively defined as

$$\beta=\sigma^{-1}(\alpha)=\sigma(\alpha),\ \gamma=\tau(\alpha),\ \delta=\sigma\tau(\alpha),\ \text{and}\ P=\begin{pmatrix}1&1&1\\ \alpha&\beta&\gamma\\ \alpha^2&\beta^2&\gamma^2\end{pmatrix}.$$

Let e_1, e_2, and e_3 be the embeddings defined over L of $\mathbb{G}_{\mathrm{m}}$ into SL_3 given respectively by, using the usual identification $\mathbb{G}_{\mathrm{m}}(R)\simeq R^{\times}$ (the group of invertible elements in R) for an

L-algebra R,

$$e_1(c) = {}^tP^{-1}\begin{pmatrix} c^{-2} & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix}{}^tP, \quad e_2(c) = {}^tP^{-1}\begin{pmatrix} c & 0 & 0 \\ 0 & c^{-2} & 0 \\ 0 & 0 & c \end{pmatrix}{}^tP,$$

and $e_3(c) = (e_1(c)e_2(c))^{-1}$ $(c \in R^\times)$. We denote respectively by T_1, T_2, T_3 the subgroups over L of SL_3 which are the images of e_1, e_2, e_3.

We note that

$$P^{-1} = D \cdot \begin{pmatrix} \beta\gamma & -\beta-\gamma & 1 \\ \gamma\alpha & -\gamma-\alpha & 1 \\ \alpha\beta & -\alpha-\beta & 1 \end{pmatrix},$$

where

$$D = \mathrm{diag}\left(\frac{1}{(\beta-\alpha)(\gamma-\alpha)}, \frac{1}{(\gamma-\beta)(\alpha-\beta)}, \frac{1}{(\alpha-\gamma)(\beta-\gamma)}\right).$$

Put

$$\varepsilon_1 = \frac{(\beta-\delta)(\gamma-\delta)}{(\beta-\alpha)(\gamma-\alpha)}, \quad \varepsilon_2 = \frac{(\gamma-\delta)(\alpha-\delta)}{(\gamma-\beta)(\alpha-\beta)},$$

$$\varepsilon_3 = \frac{(\alpha-\delta)(\beta-\delta)}{(\alpha-\gamma)(\beta-\gamma)}, \quad \text{and} \quad E_{i\,j} = \frac{\varepsilon_i}{\varepsilon_j}\ (i,j = 1,2,3).$$

We have

$$P^{-1}\sigma(P) = \begin{pmatrix} 0 & 1 & \varepsilon_1 \\ 1 & 0 & \varepsilon_2 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & E_{1\,3} \\ 0 & 1 & E_{2\,3} \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$P^{-1}\tau(P) = \begin{pmatrix} 0 & \varepsilon_1 & 1 \\ 0 & \varepsilon_2 & 0 \\ 1 & \varepsilon_3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & E_{1\,2} & 0 \\ 0 & 1 & 0 \\ 0 & E_{3\,2} & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and

$$P^{-1}\sigma\tau(P) = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ \varepsilon_2 & 0 & 1 \\ \varepsilon_3 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ E_{2\,1} & 1 & 0 \\ E_{3\,1} & 0 & 1 \end{pmatrix}\begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Let A_1, A_2, and A_3 be the L-valued points of SL_3 defined respectively as

$$A_1 = {}^tP^{-1}\begin{pmatrix} 1 & -E_{2\,1} & -E_{3\,1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}{}^tP, \; A_2 = {}^tP^{-1}\begin{pmatrix} 1 & 0 & 0 \\ -E_{1\,2} & 1 & -E_{3\,2} \\ 0 & 0 & 1 \end{pmatrix}{}^tP,$$

and

$$A_3 = {}^tP^{-1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -E_{1\,3} & -E_{2\,3} & 1 \end{pmatrix}{}^tP.$$

We see for an L-algebra R and $a, b, c \in R^\times$ that

$$\begin{aligned}\sigma({}^tP^{-1})\operatorname{diag}(a,b,c)\,\sigma({}^tP) &= {}^tP^{-1}A_3\operatorname{diag}(b,a,c)\,A_3^{-1}\,{}^tP\\ &= {}^tP^{-1}\begin{pmatrix} b & 0 & 0\\ 0 & a & 0\\ E_{1\,3}(c-b) & E_{2\,3}(c-a) & c\end{pmatrix}{}^tP,\\ \tau({}^tP^{-1})\operatorname{diag}(a,b,c)\,\tau({}^tP) &= {}^tP^{-1}A_2\operatorname{diag}(c,b,a)\,A_2^{-1}\,{}^tP\\ &= {}^tP^{-1}\begin{pmatrix} c & 0 & 0\\ E_{1\,2}(b-c) & b & E_{3\,2}(b-a)\\ 0 & 0 & a\end{pmatrix}{}^tP,\end{aligned}$$

and

$$\begin{aligned}\sigma\tau({}^tP^{-1})\operatorname{diag}(a,b,c)\,\sigma\tau({}^tP) &= {}^tP^{-1}A_1\operatorname{diag}(a,c,b)\,A_1^{-1}\,{}^tP\\ &= {}^tP^{-1}\begin{pmatrix} a & E_{2\,1}(a-c) & E_{3\,1}(a-b)\\ 0 & c & 0\\ 0 & 0 & b\end{pmatrix}{}^tP.\end{aligned}$$

Denote by $\operatorname{Int}(A)$ the conjugation left action on SL_3 of $A \in \mathrm{SL}_3(L)$. Making use of the above equations, we understand that the relations

$$e_2 = \operatorname{Int}(\sigma(A_3))\circ\sigma(e_1) = \operatorname{Int}(\tau(A_2))\circ\tau(e_2) = \operatorname{Int}(\sigma\tau(A_1))\circ\sigma\tau(e_3) \tag{2}$$

and

$$e_3 = \operatorname{Int}(\sigma(A_3))\circ\sigma(e_3) = \operatorname{Int}(\tau(A_2))\circ\tau(e_1) = \operatorname{Int}(\sigma\tau(A_1))\circ\sigma\tau(e_2) \tag{3}$$

hold. At the same time, we obtain the next proposition:

Proposition 2.1. — *Let $T_{\alpha;\sigma,\tau}$ be the maximal torus $T_2\,T_3$ of $\mathrm{SL}_3\times_K L$. The smallest subgroup $\breve{G}$ defined over K of SL_3 which includes the torus $T_{\alpha;\sigma,\tau}$ when the base field is extended to L is the whole group SL_3.*

3. Representation in the exceptional case

In this section, we prove our filtered vector space should be regarded as a representation of the special linear group of degree 3 if the Galois closure of the field generated by a primarily given number is abelian of type $(2, 2, \ldots)$.

The symbol $U_{i\,j}$ $(i, j = 1, 2, 3;\ i \neq j)$ designates the 1-dimensional unipotent subgroup over L of SL_3 whose conjugate ${}^tPU_{i\,j}{}^tP^{-1}$ is the standard 1-dimensional unipotent subgroup with its non-diagonals all zero except the (i, j)-coefficient. As an example, for an L-algebra R, the additive group of R-valued points of $U_{1\,2}$ is given by

$$U_{1\,2}(R) = {}^tP^{-1}\begin{pmatrix} 1 & R & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}{}^tP.$$

Let $T_{\alpha;\sigma,\tau}$ be the maximal torus in Section 2 of $\mathrm{SL}_3\times_K L$. We denote by $\chi(2)$ and $\chi(3)$ the characters on $T_{\alpha;\sigma,\tau}$ dual to the cocharacters e_2 and e_3 in Section 2. We see for an L-algebra

R and $b, c \in R^\times$; $u_{ij} \in R$ ($i, j = 1, 2, 3$; $i \neq j$) that

$$\begin{pmatrix} bc & 0 & 0 \\ 0 & b^{-2}c & 0 \\ 0 & 0 & bc^{-2} \end{pmatrix} \begin{pmatrix} 1 & u_{12} & u_{13} \\ u_{21} & 1 & u_{23} \\ u_{31} & u_{32} & 1 \end{pmatrix} \begin{pmatrix} b^{-1}c^{-1} & 0 & 0 \\ 0 & b^2c^{-1} & 0 \\ 0 & 0 & b^{-1}c^2 \end{pmatrix} = \begin{pmatrix} 1 & b^3u_{12} & c^3u_{13} \\ b^{-3}u_{21} & 1 & b^{-3}c^3u_{23} \\ c^{-3}u_{31} & b^3c^{-3}u_{32} & 1 \end{pmatrix}.$$

This equality means the group, say U_{32}, is the unipotent subgroup over L of SL_3 on which the maximal torus $T_{\alpha;\sigma,\tau}$ acts (by the inner automorphism from the left) via the character $3\chi(2) - 3\chi(3)$ (additive notation).

To the triple of the group $\breve{G} = \mathrm{GL}_3$, the inclusion map $\kappa = \mathrm{incl}\colon T_{\alpha;\sigma,\tau} \hookrightarrow \breve{G} \times_K L$, and the cocharacter $e = e_1\colon \mathbb{G}_\mathrm{m} \times_K L \to T_{\alpha;\sigma,\tau}$, apply the method of construction of a tensor functor $\iota_{\breve{G},\kappa,e}\colon \mathrm{Rep}_K(\breve{G}) \to \mathcal{C}(K, L)$ in our former paper [3, Section 1]. Recall that for a finite dimensional representation space V over K of $\breve{G}$, we have defined a filtration over L of $\iota_{\breve{G},\kappa,e}(V)$ as

$$V^i = V^i_{\kappa,e} = \bigoplus_{i \leq \langle \phi, e\rangle} V_\phi \quad (i \in \mathbb{R}),$$

where V_ϕ is the subspace over L of $V \otimes_K L$ on which $T_{\alpha;\sigma,\tau}$ acts by multiplication of a character ϕ via the map $\kappa = \mathrm{incl}$.

Example 3.1 **(representation corresponding to a simultaneous approximation).** — Let s_{ij} be an indeterminate considered a function on SL_3 defined as the matrix coefficient in the i-th row and the j-th column for each indices i and j. Put $q = s_{31}$, $r = s_{32}$, and $s = s_{33}$. Let $\breve{V}$ be the vector space over K spanned by q, r, and s in the ring of functions over K on SL_3. By the translation to the right on SL_3, the vector space $\breve{V}$ becomes a representation space of $\breve{G} = \mathrm{SL}_3$. Since the action of the torus $T_{\alpha;\sigma,\tau}$ is defined as, for an arbitrary L-algebra R, $b \in R^\times \simeq T_2(R)$, and $c \in R^\times \simeq T_3(R)$,

$$\begin{pmatrix} q & r & s \end{pmatrix} \longmapsto \begin{pmatrix} q & r & s \end{pmatrix} {}^tP^{-1} \begin{pmatrix} bc & 0 & 0 \\ 0 & b^{-2}c & 0 \\ 0 & 0 & bc^{-2} \end{pmatrix} {}^tP,$$

where

$${}^tP^{-1} = \begin{pmatrix} \beta\gamma & \gamma\alpha & \alpha\beta \\ -\beta-\gamma & -\gamma-\alpha & -\alpha-\beta \\ 1 & 1 & 1 \end{pmatrix} \cdot D \qquad (D\text{: a diagonal matrix})$$

is the same as the transposed inverse of the matrix P in Section 2, we see that

$$(q\beta\gamma - r(\beta+\gamma) + s)\, L = \breve{V}_{\chi(2)+\chi(3)},$$
$$(q\gamma\alpha - r(\gamma+\alpha) + s)\, L = \breve{V}_{-2\chi(2)+\chi(3)},$$

and

$$(q\alpha\beta - r(\alpha+\beta) + s)\, L = \breve{V}_{\chi(2)-2\chi(3)}.$$

The relation $e_1+e_2+e_3=0$ taken into account, the filtration of $\iota_{\breve{G},\kappa,e}(\breve{V})$ is given by

$$F_\alpha^i\breve{V}=\begin{cases}\breve{V}\otimes_K L & \text{for } i\leq -2\\ \breve{V}_{-2\chi(2)+\chi(3)}\oplus\breve{V}_{\chi(2)-2\chi(3)} & \text{for } -2<i\leq 1\\ 0 & \text{for } i>1.\end{cases}$$

On the other hand, we have

$$(q\gamma\alpha-r(\gamma+\alpha)+s)-(q\alpha\beta-r(\alpha+\beta)+s)=(-q\alpha+r)(\beta-\gamma)$$

and

$$(q\gamma\alpha-r(\gamma+\alpha)+s)\,(\alpha+\beta)-(q\alpha\beta-r(\alpha+\beta)+s)\,(\gamma+\alpha)=(-q\alpha^2+s)(\beta-\gamma).$$

Thus the filtration $F_\alpha^{\cdot}\breve{V}$ can be written as

$$F_\alpha^i\breve{V}=\begin{cases}\breve{V}\otimes_K L & \text{for } i\leq -2\\ (-q\alpha+r)L\oplus(-q\alpha^2+s)L & \text{for } -2<i\leq 1\\ 0 & \text{for } i>1.\end{cases}$$

Note that the filtration $F_\alpha^{\cdot}\breve{V}$ does not depend on the choice of the element σ nor $\tau\in\mathrm{Gal}(L/K)$.

Remember the definition of a quantity m [3, Defintion 3.2] for an element x of a filtered vector space $V\otimes_K L$:

$$m(x)=\sup\{i\mid V^i\ni x\}\tag{4}$$

Proposition 3.2. — *Suppose we are given a representation space V over K of $\breve{G}=\mathrm{SL}_3$. For a character $\phi=a\cdot\chi(2)+b\cdot\chi(3)$ $(a,b\in\mathbb{Z})$ of the torus $T_{\alpha;\sigma,\tau}$, let*

$$\phi^\circ=-(a+b)\cdot\chi(2)+b\cdot\chi(3)\quad\textit{and}\quad\phi^\dagger=a\cdot\chi(2)-(a+b)\cdot\chi(3).$$

If $x\in V_\phi\setminus\{0\}$, then there exist elements $y\in V_{\phi^\circ}\setminus\{0\}$ and $z\in V_{\phi^\dagger}\setminus\{0\}$ such that

$$\sigma(x)-y\in\bigoplus_{k,l\geq 0;\,(k,l)\neq(0,0)}V_{\phi^\circ+3k(\chi(2)-\chi(3))-3l\chi(3)}$$

and

$$\tau(x)-z\in\bigoplus_{k,l\geq 0;\,(k,l)\neq(0,0)}V_{\phi^\dagger+3k(\chi(3)-\chi(2))-3l\chi(2)}.$$

In particular, we have

$$m(\sigma(x))=\langle\phi^\circ,e\rangle=\langle\phi,e_2\rangle\quad\textit{and}\quad m(\tau(x))=\langle\phi^\dagger,e\rangle=\langle\phi,e_3\rangle.$$

Proof. — Note first that $A_3\in U_{3\,1}U_{3\,2}$. We have remarked at the beginning of this section that the unipotent subgroups $U_{3\,1}$ and $U_{3\,2}$ correspond respectively to the characters $-3\chi(3)$ and $3\chi(2)-3\chi(3)$ of the maximal torus $T_{\alpha;\sigma,\tau}$. As is well-known, the group $U_{3\,1}U_{3\,2}$ sends an element x of V_ϕ to an affine space

$$x+\bigoplus_{k,l\geq 0;\,(k,l)\neq(0,0)}V_{\phi-3k\chi(3)+3l(\chi(2)-\chi(3))}.$$

We get an expression

$$A_3^{-1}x = x + \sum_{k,l\geq 0;\,(k,l)\neq(0,0)} x_{k,l}, \qquad x_{k,l} \in V_{\phi-3k\chi(3)+3l(\chi(2)-\chi(3))}.$$

Applying $\sigma \in \mathrm{Gal}(L/K)$, we have

$$\sigma(A_3)^{-1}\sigma(x) = \sigma(x) + \sum_{k,l\geq 0;\,(k,l)\neq(0,0)} \sigma(x_{k,l}),$$

namely,

$$\sigma(x) = \sigma(A_3x) + \sum_{k,l\geq 0;\,(k,l)\neq(0,0)} \sigma(A_3x_{k,l}).$$

For an L-algebra R and $c \in R^\times$, we know by the relations (2) and (3)

$$\begin{aligned} e_2(c)\,\sigma(A_3x) &= \sigma(A_3)\,\sigma(e_1)(c)\,\sigma(A_3)^{-1}\cdot\sigma(A_3)\,\sigma(x) \\ &= \sigma(A_3)\,c^{-a-b}\sigma(x) = c^{-a-b}\sigma(A_3x) \end{aligned}$$

and

$$\begin{aligned} e_3(c)\,\sigma(A_3x) &= \sigma(A_3)\,\sigma(e_3)(c)\,\sigma(A_3)^{-1}\cdot\sigma(A_3)\,\sigma(x) \\ &= \sigma(A_3)\,c^{b}\sigma(x) = c^{b}\sigma(A_3x), \end{aligned}$$

in other words $\sigma(A_3x) \in V_{\phi^\circ}$. We obtain similarly

$$e_2(c)\,\sigma(A_3x_{k,l}) = c^{-a-b+3k}\sigma(A_3x_{k,l})$$

and

$$e_3(c)\,\sigma(A_3x_{k,l}) = c^{b-3k-3l}\sigma(A_3x_{k,l}),$$

i.e., $\sigma(A_3x_{k,l}) \in V_{\phi^\circ+3k(\chi(2)-\chi(3))-3l\chi(3)}$. We see that $y = \sigma(A_3x)$ meets the requirement, for $\sigma(A_3x) \neq 0$ if $x \neq 0$.
The equality

$$\langle \phi^\circ + 3k(\chi(2)-\chi(3)) - 3l\chi(3), e\rangle = a + 3l$$

implies that

$$\sigma(x) \in V^a \quad \text{and} \quad \sigma(x) \notin V^i \ (i > a),$$

hence $m(\sigma(x)) = a = \langle \phi^\circ, e\rangle = \langle \phi, e_2\rangle$.
The assertion concerning $\tau(x)$ is derived in the same manner. □

Lemma 3.3. — *For any non-zero finite dimensional representation space V over K of $\check{G} = \mathrm{SL}_3$, we have $\mu(\iota_{\check{G},\kappa,e}(V)) = 0$, where μ is the slope function of filtered vector spaces (cf. e.g.* [3, Definition 1.12]*).*

Proof. — Completely the same as the proof of Lemma 3.5 in [3]. □

Lemma 3.4. — *To an arbitrary 1-dimensional vector subspace W over K of an SL_3-representation V over K, attach the sub-filtration over L of $\iota_{\mathrm{SL}_3,\kappa,e}(V)$. We have $\mu(W) \leq 0$.*

Proof. — In the coefficient extension $V \otimes_K L$, a non-zero vector $w \in W$ is written

$$w = w_1 + \cdots + w_r, \quad w_i \in V_{\psi(i)} \setminus \{0\},$$

where $V_{\psi(i)}$ is the subspace over L of $V \otimes_K L$ on which the torus $T_{\alpha;\sigma,\tau}$ acts via a character $\psi(i)$. We may assume that the characters $\psi(i)$ are pairwise distinct. By the definitions of the sub-filtration and the quantity $m(\,\cdot\,)$ recalled in (4), we have

$$\mu(W) = \min_{1 \leq i \leq r} \langle \psi(i), e \rangle = m(w).$$

Let

$$a = \min_{1 \leq i \leq r} \langle \psi(i), e_2 \rangle, \quad m' = \min \{ \langle \psi(i), e \rangle \mid \langle \psi(i), e_2 \rangle = a \},$$

and $\phi = a\chi(2) - (a + m')\chi(3)$. There exists a unique number j such that $\psi(j) = \phi$. Applying $\sigma \in \mathrm{Gal}(L/K)$ to w, we have

$$w = \sigma(w) = \sigma(w_1) + \cdots + \sigma(w_r).$$

From Proposition 3.2, we see that

$$w = \sigma(w) \in \sum_{i=1}^{r} \bigoplus_{k,l \geq 0} V_{\psi^\circ(i) + 3k(\chi(2) - \chi(3)) - 3l\chi(3)}$$

and that there exists an element $y \in V_{\phi^\circ} \setminus \{0\}$ with

$$\sigma(w_j) - y \in \bigoplus_{k,l \geq 0;\, (k,l) \neq (0,0)} V_{\psi^\circ(j) + 3k(\chi(2) - \chi(3)) - 3l\chi(3)}.$$

Since for all i

$$\langle \psi^\circ(i) + 3k(\chi(2) - \chi(3)) - 3l\chi(3), e \rangle = \langle \psi(i), e_2 \rangle + 3l \geq a$$

and for i such that $\langle \psi(i), e_2 \rangle = a$

$$\langle \psi^\circ(i) + 3k(\chi(2) - \chi(3)) - 3l\chi(3), e_2 \rangle = \langle \psi(i), e \rangle + 3k \geq m',$$

the eigenvector y with respect to the action of the torus $T_{\alpha;\sigma,\tau}$ does not cancel out in the sum $w = \sigma(w_1) + \cdots + \sigma(w_r)$. We get the equality $m(w) = m(\sigma(w)) = a$.
Put this time

$$b = \min_{1 \leq i \leq r} \langle \psi(i), e_3 \rangle, \quad m'' = \min \{ \langle \psi(i), e \rangle \mid \langle \psi(i), e_3 \rangle = b \},$$

and $\varphi = -(m'' + b)\chi(2) + b\chi(3)$. There is a unique index h with $\psi(h) = \varphi$. In the same way as in the previous paragraph, we know the simultaneous equalities

$$\langle \psi^\dagger(i) + 3k(\chi(3) - \chi(2)) - 3l\chi(2), e \rangle = b$$

and

$$\langle \psi^\dagger(i) + 3k(\chi(3) - \chi(2)) - 3l\chi(2), e_3 \rangle = m''$$

are possible only when $i = h$ and $k = l = 0$. We see $m(w) = m(\tau(w)) = b$.
Each character $\psi(i)$ is expressed as $\psi(i) = a_i\chi(2) + b_i\chi(3)$ $(a_i, b_i \in \mathbb{Z})$. By the definitions of $m(\,\cdot\,), a, b$, we obtain

$$m(w) \leq \langle \psi(i), e \rangle = -a_i - b_i \leq -a - b = -2m(w),$$

i.e., $\mu(W) = m(w) \leq 0$. □

Proposition 3.5. — *For any non-zero finite dimensional representation space V over K of $\breve{G} = \mathrm{SL}_3$, the filtered vector space $\iota_{\breve{G},\kappa,e}(V)$ is semi-stable of slope zero, hence the functor $\iota_{\breve{G},\kappa,e}$ factors through $\mathcal{C}_0^{\mathrm{ss}}(K,L)$.*

Proof. — The same proof as the one of Proposition 3.7 in [3] is valid. □

Let V be the underlying vector space over K of an object in $\mathcal{C}(K,L)$. Remember that a linear map f over K of V to another underlying space is filtered if and only if

$$m(x) \leq m\left(f(x)\right)$$

for all $x \in V \otimes_K L$.

Theorem 3.6. — *The functor $\iota_{\mathrm{SL}_3,\kappa,e}\colon \mathrm{Rep}_K(\mathrm{SL}_3) \to \mathcal{C}_0^{\mathrm{ss}}(K,L)$ is fully faithful.*

Proof. — Let W be an arbitrary finite dimensional representation space over K of SL_3. For any $y \in W \otimes_K L$, we have a unique expression

$$y = \sum_{\psi \in X} y_\psi, \quad y_\psi \in W_\psi,$$

where X is the character group of the torus $T_{\alpha;\sigma,\tau}$ and W_ψ is the subspace over L of $W \otimes_K L$ on which $T_{\alpha;\sigma,\tau}$ acts by multiplication of a character ψ. We define a set $X(y)$ of characters as

$$X(y) = \{\psi \in X \mid y_\psi \neq 0\}.$$

Assume $y \neq 0$ and let

$$a(y) = \min_{\psi \in X(y)} \langle \psi, e_2 \rangle \quad \text{and} \quad b(y) = \min_{\psi \in X(y)} \langle \psi, e_3 \rangle.$$

By the same and a similar reasoning to the one in the proof of Lemma 3.4, we see that

$$m(\sigma(y)) = a(y) \quad \text{and} \quad m(\tau(y)) = b(y).$$

Hence, for a linear map $f\colon V \to W$ over K between the underlying vector spaces of finite dimensional representations and $x \in V_\phi$ such that $f(x) \neq 0$, we have

$$m(\sigma(x)) = a(x) = \langle \phi, e_2 \rangle, \quad m(\sigma(f(x))) = a(f(x)),$$
$$m(\tau(x)) = b(x) = \langle \phi, e_3 \rangle, \quad \text{and} \quad m(\tau(f(x))) = b(f(x)).$$

On the assumption that f is filtered, we get

$$\langle \phi, e \rangle = m(x) \leq m(f(x)) = \min_{\psi \in X(f(x))} \langle \psi, e \rangle,$$
$$\langle \phi, e_2 \rangle = m(\sigma(x)) \leq m(f(\sigma(x))) = m(\sigma(f(x))) = a(f(x)),$$

and

$$\langle \phi, e_3 \rangle = m(\tau(x)) \leq m(f(\tau(x))) = m(\tau(f(x))) = b(f(x)).$$

Since $e = e_1 = -e_2 - e_3$, the first inequality is equivalent to

$$\max_{\psi \in X(f(x))} \{\langle \psi, e_2 \rangle + \langle \psi, e_3 \rangle\} \leq \langle \phi, e_2 \rangle + \langle \phi, e_3 \rangle.$$

By the definitions of $a(\,\cdot\,)$ and $b(\,\cdot\,)$, these inequalities must be all equalities. We find

$$\langle \psi, e_2 \rangle = a(f(x)) = \langle \phi, e_2 \rangle \quad \text{and} \quad \langle \psi, e_3 \rangle = b(f(x)) = \langle \phi, e_3 \rangle$$

for all $\psi \in X(f(x))$. Thus we obtain

$$X\left(f(x)\right) = \{\phi\} \quad \text{if} \quad f(x) \neq 0.$$

This means

$$f(x) = y_\phi \ \in \ W_\phi \quad \text{for all} \quad x \in V_\phi,$$

that is, that the map f commutes with the action of $T_{\alpha;\sigma,\tau}$. Since f is defined over K, the map commutes with all Galois conjugates $\omega(T_{\alpha;\sigma,\tau})$ $(\omega \in \mathrm{Gal}(L/K))$ and so with $\breve{G} = \mathrm{SL}_3$. □

4. The remaining cases

In this section, we collect what are required in Section 5.

The symbols $K, \mathbb{G}_\mathrm{m}, \mathrm{SL}_3$, and K^{sep} being as in Section 2, let α be an element of K^{sep} such that there exists an element $\sigma \in \mathrm{Gal}\left(K^{\mathrm{sep}}/K\right)$ with $\sigma^2(\alpha) \neq \alpha$. Fix a (finite or infinite) Galois extension field L of K containing α. Fix $\sigma \in \mathrm{Gal}\left(K^{\mathrm{sep}}/K\right)$ such that $\sigma^2(\alpha) \neq \alpha$. Elements $\beta, \gamma \in L$ and an element $P \in \mathrm{GL}_3(L)$ are respectively defined as

$$\beta = \sigma^{-1}(\alpha), \quad \gamma = \sigma^{-2}(\alpha), \quad P = \begin{pmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{pmatrix}.$$

With these β, γ, and P, we define embeddings e_1, e_2, e_3 over L of $\mathbb{G}_\mathrm{m}$ into SL_3 by the same expressions as in Section 2. Their images are respectively written T_1, T_2, T_3 as before.

Put

$$\begin{aligned}
\lambda_1 &= \frac{\left(\beta - \sigma^{-1}(\gamma)\right)\left(\gamma - \sigma^{-1}(\gamma)\right)}{(\beta - \alpha)(\gamma - \alpha)}, \quad & \lambda_2 &= \frac{\left(\gamma - \sigma^{-1}(\gamma)\right)\left(\alpha - \sigma^{-1}(\gamma)\right)}{(\gamma - \beta)(\alpha - \beta)}, \\
\lambda_3 &= \frac{\left(\alpha - \sigma^{-1}(\gamma)\right)\left(\beta - \sigma^{-1}(\gamma)\right)}{(\alpha - \gamma)(\beta - \gamma)}, \quad & \Lambda_2 &= \frac{\lambda_2}{\lambda_1}, \quad \Lambda_3 = \frac{\lambda_3}{\lambda_1}, \\
\nu_1 &= \frac{\left(\beta - \sigma(\alpha)\right)\left(\gamma - \sigma(\alpha)\right)}{(\beta - \alpha)(\gamma - \alpha)}, & \nu_2 &= \frac{\left(\gamma - \sigma(\alpha)\right)\left(\alpha - \sigma(\alpha)\right)}{(\gamma - \beta)(\alpha - \beta)}, \\
\nu_3 &= \frac{\left(\alpha - \sigma(\alpha)\right)\left(\beta - \sigma(\alpha)\right)}{(\alpha - \gamma)(\beta - \gamma)}, & N_1 &= \frac{\nu_1}{\nu_3}, \quad \text{and} \quad N_2 = \frac{\nu_2}{\nu_3}.
\end{aligned}$$

Note that when $\sigma^3(\alpha) = \alpha$, we have $\lambda_2 = \lambda_3 = \nu_1 = \nu_2 = 0$. We know

$$P^{-1}\,\sigma^{-1}(P) = \begin{pmatrix} 0 & 0 & \lambda_1 \\ 1 & 0 & \lambda_2 \\ 0 & 1 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \Lambda_2 & 1 & 0 \\ \Lambda_3 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$P^{-1}\,\sigma(P) = \begin{pmatrix} \nu_1 & 1 & 0 \\ \nu_2 & 0 & 1 \\ \nu_3 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & N_1 \\ 0 & 1 & N_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu_3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let A and B be the L-valued points of SL_3 defined respectively as

$$A = {}^tP^{-1} \begin{pmatrix} 1 & -\Lambda_2 & -\Lambda_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} {}^tP \qquad \text{and} \qquad B = {}^tP^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -N_1 & -N_2 & 1 \end{pmatrix} {}^tP.$$

We see for an L-algebra R and $a, b, c \in R^\times$; $u \in R$ that

$$\sigma^{-1}({}^tP^{-1})\,\mathrm{diag}(a,b,c)\,\sigma^{-1}({}^tP) = {}^tP^{-1}A\,\mathrm{diag}(c,a,b)\,A^{-1}\,{}^tP = {}^tP^{-1}\begin{pmatrix} c & \Lambda_2(c-a) & \Lambda_3(c-b) \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}{}^tP,$$

$$\sigma({}^tP^{-1})\,\mathrm{diag}(a,b,c)\,\sigma({}^tP) = {}^tP^{-1}B\,\mathrm{diag}(b,c,a)\,B^{-1}\,{}^tP = {}^tP^{-1}\begin{pmatrix} b & 0 & 0 \\ 0 & c & 0 \\ N_1(a-b) & N_2(a-c) & a \end{pmatrix}{}^tP,$$

$$\sigma^{-1}({}^tP^{-1})\begin{pmatrix} 1 & u & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\sigma^{-1}({}^tP) = {}^tP^{-1}\begin{pmatrix} 1 & 0 & -\Lambda_2 u \\ 0 & 1 & u \\ 0 & 0 & 1 \end{pmatrix}{}^tP,$$

and

$$\sigma({}^tP^{-1})\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & u & 1 \end{pmatrix}\sigma({}^tP) = {}^tP^{-1}\begin{pmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ -N_2 u & 0 & 1 \end{pmatrix}{}^tP.$$

In particular, we have

$$e_2 = \mathrm{Int}(\sigma(A)) \circ \sigma(e_3) = \mathrm{Int}(\sigma^{-1}(B)) \circ \sigma^{-1}(e_1) \tag{5}$$

and

$$e_3 = \mathrm{Int}(\sigma(A)) \circ \sigma(e_1) = \mathrm{Int}(\sigma^{-1}(B)) \circ \sigma^{-1}(e_2), \tag{6}$$

where the symbol Int implies the conjugation left action on SL_3 as before. We find the following:

Proposition 4.1. — *Let $T_{\alpha;\sigma}$ be the maximal torus $T_2\,T_3$ of $\mathrm{SL}_3 \times_K L$ and $\breve{G}$ the smallest subgroup defined over K of SL_3 which includes the torus $T_{\alpha;\sigma}$ when the base field is extended to L. If $\sigma^3(\alpha) \neq \alpha$, then $\breve{G} = \mathrm{SL}_3$. If the extension field $K(\alpha)$ is cubic over K, then the torus $T_{\alpha;\sigma}$ is defined over K, hence $\breve{G} = T_{\alpha;\sigma}$.*

Remark 4.2. — Let q, r, s be three indeterminates. When α is cubic over K, we observe that the torus $T_{\alpha;\sigma}$ is naturally isomorphic to a 2-dimensional anisotropic torus over K (cf. [3, Lemma 2.5])

$$\mathrm{Spec}\left(K[q,r,s]\Big/\left(1 - \prod_{i=0}^{2}\sigma^{-i}(q\beta\gamma - r(\beta+\gamma) + s)\right)\right),$$

the functions $q\gamma\alpha - r(\gamma+\alpha) + s$ and $q\alpha\beta - r(\alpha+\beta) + s$ being considered generators of its character group. Hence, the group $T_{\alpha;\sigma}(K)$ of K-valued points on the torus $T_{\alpha;\sigma}$ is identified with the multiplicative subgroup of $K(\alpha)$ composed of the elements of norm 1.

5. Representation in the remaining cases

In this section, we show our filtered vector space should be considered a representation of the algebraic group defined in Section 4 if the Galois closure of the field generated by a primarily given number is non-abelian or is abelian of type other than $(2, 2, \ldots)$.
Let $U_{i\,j}$ $(i, j = 1, 2, 3;\ i \neq j)$ be the 1-dimensional unipotent subgroups over L of SL_3 similar to those in Section 3, $T_{\alpha;\sigma}$ the maximal torus over L of SL_3 in Section 4, and $\chi(2), \chi(3)$ the characters on $T_{\alpha;\sigma}$ dual to the cocharacters e_2, e_3 in Section 4. The expression in terms of $\chi(2), \chi(3)$ of the character by multiplication of which the torus $T_{\alpha;\sigma}$ acts on $U_{i\,j}$ is the same as in Section 3.
To the triple of the group $\breve{G}$ in Section 4, the inclusion map $\kappa = \mathrm{incl}\colon T_{\alpha;\sigma} \hookrightarrow \breve{G} \times_K L$, and the cocharacter $e = e_1\colon \mathbb{G}_{\mathrm{m}} \times_K L \to T_{\alpha;\sigma}$ in Section 4, apply the method of construction of a tensor functor $\iota_{\breve{G},\kappa,e}\colon \mathrm{Rep}_K(\breve{G}) \to \mathcal{C}(K, L)$ in our former paper [3, Section 1]. The vector space $\breve{V}$ over K spanned by three indeterminates q, r, s equipped with the filtration over L

$$F^i_\alpha \breve{V} = \begin{cases} \breve{V} \otimes_K L & \text{for } i \leq -2 \\ (-q\alpha + r)L \oplus (-q\alpha^2 + s)L & \text{for } -2 < i \leq 1 \\ 0 & \text{for } i > 1 \end{cases}$$

is an object in the image of the functor $\iota_{\breve{G},\kappa,e}$ as in the exceptional case of Section 3.

Proposition 5.1. — *Given a character $\phi = a \cdot \chi(2) + b \cdot \chi(3)$ $(a, b \in \mathbb{Z})$ of the torus $T_{\alpha;\sigma}$, let $\phi^\circ = b \cdot \chi(2) - (a + b) \cdot \chi(3)$. For a representation space V over K of $\breve{G}$, if $x \in V_\phi \setminus \{0\}$, then there exists an element $y \in V_{\phi^\circ} \setminus \{0\}$ such that*

$$\sigma(x) - y \in \bigoplus_{k,l \geq 0;\, (k,l) \neq (0,0)} V_{\phi^\circ - 3k\chi(3) + 3l(\chi(2) - \chi(3))}.$$

In particular, we have $m(\sigma(x)) = \langle \phi^\circ, e \rangle = \langle \phi, e_2 \rangle$, where m is the function described in (4).

Proof. — First, we assume that $\sigma^3(\alpha) \neq \alpha$. We know $\breve{G} = \mathrm{SL}_3$. The assertion is confirmed in the same fashion as in the proof of Proposition 3.2.
Next, assume that $\sigma^3(\alpha) = \alpha$. In this case, by the definitions of Λ_2 and Λ_3, the L-valued point A equals the identity, hence we know from the relations (5) and (6) that

$$e_2 = \sigma(e_3) \qquad \text{and} \qquad e_3 = \sigma(e_1).$$

For an L-algebra R and $c \in R^\times$, we obtain

$$e_2(c)\,\sigma(x) = \sigma(e_3)(c)\,\sigma(x) = c^b \sigma(x)$$

and

$$e_3(c)\,\sigma(x) = \sigma(e_1)(c)\,\sigma(x) = c^{-(a+b)} \sigma(x).$$

Thus $y = \sigma(x) \in V_{\phi^\circ} \setminus \{0\}$ suffices. □

Lemma 5.2. — *Any 1-dimensional representation defined over K of $\breve{G}$ is trivial.*

Proof. — Let V be a 1-dimensional representation space over K of $\breve{G}$. The torus $T_{\alpha;\sigma}$ acts on $V \otimes_K L$ via a character ϕ, in other words, $V \otimes_K L = V_\phi$. We see from Proposition 5.1 that $\phi^\circ = \phi$, which forces $\phi = 0$. The rest of proof is the same as the latter part of the proof of Lemma 3.4 in [3]. □

Using this lemma, we get the next:

Lemma 5.3. — *For any non-zero finite dimensional representation space V over K of $\breve{G}$, we have $\mu(\iota_{\breve{G},\kappa,e}(V)) = 0$, where μ is the slope function of filtered vector spaces (cf. e.g.* [3, Definition 1.12]*).*

Lemma 5.4. — *To an arbitrary 1-dimensional vector subspace W over K of a $\breve{G}$-representation V over K, attach the sub-filtration over L of $\iota_{\breve{G},\kappa,e}(V)$. We have $\mu(W) \leq 0$.*

Proof. — In the coefficient extension $V \otimes_K L$, a non-zero vector $w \in W$ is written

$$w = w_1 + \cdots + w_r, \quad w_i \in V_{\psi(i)} \setminus \{0\}$$

as in the proof of Lemma 3.4.
We have

$$\mu(W) = \min_{1 \leq i \leq r} \langle \psi(i), e \rangle = m(w).$$

Let

$$a = \min_{1 \leq i \leq r} \langle \psi(i), e_2 \rangle, \quad a' = \min \{ \langle \psi(i), e_3 \rangle \mid \langle \psi(i), e_2 \rangle = a \},$$

and $\phi = a\chi(2) + a'\chi(3)$. There exists a unique number j with $\psi(j) = \phi$. Applying $\sigma \in \mathrm{Gal}(L/K)$ to w, we have

$$w = \sigma(w) = \sigma(w_1) + \cdots + \sigma(w_r).$$

From Proposition 5.1, we see that

$$w = \sigma(w) \in \sum_{i=1}^{r} \bigoplus_{k,l \geq 0} V_{\psi^\circ(i) - 3k\chi(3) + 3l(\chi(2) - \chi(3))}$$

and that there exists an element $y \in V_{\phi^\circ} \setminus \{0\}$ such that

$$\sigma(w_j) - y \in \bigoplus_{k,l \geq 0;\, (k,l) \neq (0,0)} V_{\psi^\circ(j) - 3k\chi(3) + 3l(\chi(2) - \chi(3))}.$$

Since for all i

$$\langle \psi^\circ(i) - 3k\chi(3) + 3l(\chi(2) - \chi(3)), e \rangle = \langle \psi(i), e_2 \rangle + 3k \geq a$$

and for i such that $\langle \psi(i), e_2 \rangle = a$

$$\langle \psi^\circ(i) - 3k\chi(3) + 3l(\chi(2) - \chi(3)), e_2 \rangle = \langle \psi(i), e_3 \rangle + 3l \geq a',$$

the eigenvector y with respect to the action of the torus $T_{\alpha;\sigma}$ does not cancel out in the sum $w = \sigma(w_1) + \cdots + \sigma(w_r)$. We get the equality $m(w) = m(\sigma(w)) = a$ and a number h with $\psi(h) = \phi^\circ$.
By the definition of a, we obtain

$$a \leq \langle \psi(h), e_2 \rangle = \langle \phi^\circ, e_2 \rangle = \langle \phi, e_3 \rangle = a',$$

hence

$$a = m(w) \leq \langle \psi(j), e \rangle = \langle \phi, e \rangle = -a - a' \leq -2a.$$

This is possible only when $0 \geq a = m(w) = \mu(W)$. □

Proposition 5.5. — *For any finite dimensional representation space V over K of $\breve{G}$, the filtered vector space $\iota_{\breve{G},\kappa,e}(V)$ is semi-stable of slope zero, hence the functor $\iota_{\breve{G},\kappa,e}$ factors through $\mathcal{C}_0^{\mathrm{ss}}(K,L)$.*

Proof. — The same proof as the one of Proposition 3.7 in [3] is valid. □

Theorem 5.6. — *The functor $\iota_{\breve{G},\kappa,e}\colon \mathrm{Rep}_K(\breve{G}) \to \mathcal{C}_0^{\mathrm{ss}}(K,L)$ is fully faithful.*

Proof. — Let W be an arbitrary finite dimensional representation space over K of $\breve{G}$. For any $y \in W \otimes_K L$, we have a unique expression

$$y = \sum_{\psi \in X} y_\psi, \quad y_\psi \in W_\psi$$

as in the proof of Theorem 3.6.
We define a set $X(y)$ of characters as

$$X(y) = \{\psi \in X \mid y_\psi \neq 0\}.$$

Assume $y \neq 0$ and let

$$a(y) = \min_{\psi \in X(y)} \langle \psi, e_2 \rangle, \qquad b(y) = \min_{\psi \in X(y)} \langle \psi, e_3 \rangle,$$
$$b'(y) = \min_{\psi \in X(y)} \{\langle \psi, e_2 \rangle \mid \langle \psi, e_3 \rangle = b(y)\},$$

and $\varphi(y) = b'(y)\chi(2) + b(y)\chi(3)$. By a similar reasoning to the one in the proof of Lemma 5.4, we see that

$$m(\sigma(y)) = a(y), \qquad \varphi^\circ(y) \in X(\sigma(y)),$$

and

$$a(\sigma(y)) \overset{\mathrm{def}}{=} \min_{\psi \in X(\sigma(y))} \langle \psi, e_2 \rangle = \langle \varphi^\circ(y), e_2 \rangle = b(y).$$

Hence, for a linear map $f\colon V \to W$ over K between the underlying vector spaces of finite dimensional representations and $x \in V_\phi$ such that $f(x) \neq 0$, we have

$$m(\sigma(x)) = a(x) = \langle \phi, e_2 \rangle, \qquad m(\sigma(f(x))) = a(f(x)),$$
$$m\Big(\sigma^2(x)\Big) = a(\sigma(x)) = b(x) = \langle \phi, e_3 \rangle,$$

and

$$m\Big(\sigma^2(f(x))\Big) = a(\sigma(f(x))) = b(f(x)).$$

On the assumption that f is filtered, we get

$$\langle \phi, e \rangle = m(x) \leq m(f(x)) = \min_{\psi \in X(f(x))} \langle \psi, e \rangle,$$
$$\langle \phi, e_2 \rangle = m(\sigma(x)) \leq m(f(\sigma(x))) = m(\sigma(f(x))) = a(f(x)),$$

and

$$\langle \phi, e_3 \rangle = m(\sigma^2(x)) \leq m\Big(f(\sigma^2(x))\Big) = m\Big(\sigma^2(f(x))\Big) = b(f(x)).$$

Since $e = e_1 = -e_2 - e_3$, the first inequality is equivalent to

$$\max_{\psi \in X(f(x))} \{\langle \psi, e_2 \rangle + \langle \psi, e_3 \rangle\} \leq \langle \phi, e_2 \rangle + \langle \phi, e_3 \rangle.$$

By the definitions of $a(\,\cdot\,)$ and $b(\,\cdot\,)$, these inequalities must be all equalities. We obtain

$$\langle \psi, e_2 \rangle = a(f(x)) = \langle \phi, e_2 \rangle \quad \text{and} \quad \langle \psi, e_3 \rangle = b(f(x)) = \langle \phi, e_3 \rangle$$

for all $\psi \in X(f(x))$. Thus the rest of proof goes through the same path as in the proof of Theorem 3.6. $\square$

6. The group in characteristic zero

We shall make explicit the group defined in Section 4 when the characteristic of the base field is zero, using the one-to-one correspondence between Lie algebras and connected Lie groups. We calculate the Lie algebra $\breve{\mathfrak{g}}$ of the group $\breve{G}$ defined in Section 4 as a subalgebra of the Lie algebra $\mathfrak{sl}_3$ of the special linear group SL_3 of degree 3. We identify the Lie algebra $\mathfrak{gl}_3$ of the general linear group GL_3 of degree 3 with the Lie algebra of all matrices of degree 3 and regard $\mathfrak{sl}_3$ as the subalgebra of trace zero. For each $i, j = 1, 2, 3$, the element of $\mathfrak{gl}_3$ which is identified with the matrix whose (i, j)-coefficient is 1 and whose other coefficients are 0 is denoted by $E_{i\,j}$.

To ease notation, we examine in fact the Lie algebra ${}^tP\,\breve{\mathfrak{g}}\,{}^tP^{-1}$ of the conjugate group ${}^tP\,\breve{G}\,{}^tP^{-1}$, where P is the matrix used to define $\breve{G}$ in Section 4 and ${}^tP$ is its transpose.

For $i = 1, 2, 3$, set

$$Z_i = 1 - 3E_{i\,i} \in \mathfrak{sl}_3(\mathbb{Z}).$$

Lemma 6.1. — $Z_1,\ Z_2,\ Z_3 \in {}^tP\,\breve{\mathfrak{g}}\,{}^tP^{-1}$

Proof. — Recall that the tori T_i $(i = 1, 2, 3)$ are respectively the images of the embeddings e_i $(i = 1, 2, 3)$ over L of $\mathbb{G}_\mathrm{m}$ into SL_3. The morphism e_1, for example, is written in matrix form as

$$e_1(c) = {}^tP^{-1}\begin{pmatrix} c^{-2} & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix}{}^tP.$$

Differentiating both sides with respect to c, we see

$$\frac{d}{dc}e_1(c) = {}^tP^{-1}\begin{pmatrix} -2c^{-3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}{}^tP.$$

The torus T_1 is included in $\breve{G}$ by definition. We get

$${}^tP^{-1}\,Z_1\,{}^tP = {}^tP^{-1}\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}{}^tP \in \breve{\mathfrak{g}}.$$

In the same way, we obtain ${}^tP^{-1}\,Z_2\,{}^tP,\ \ {}^tP^{-1}\,Z_3\,{}^tP\ \in \breve{\mathfrak{g}}$. $\square$

Fix temporarily an arbitrary element τ of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$. Denote by $\mathbf{c}_i$ $(i = 1, 2, 3)$ the i-th column vector of the matrix ${}^tP\,\tau({}^tP^{-1})$. We denote similarly by $\mathbf{r}_i$ $(i = 1, 2, 3)$ the i-th row vector of the matrix $\tau({}^tP)\,{}^tP^{-1}$. Put $A = \mathbf{c}_1\mathbf{r}_1$, $B = \mathbf{c}_2\mathbf{r}_2$, and $C = \mathbf{c}_3\mathbf{r}_3$. They are matrices of degree 3. We call respectively $a_{i\,j}$, $b_{i\,j}$, and $c_{i\,j}$ the coefficients in the i-th row and the j-th column of the matrices A, B, and C: $A = (a_{i\,j}),\ B = (b_{i\,j}),\ C = (c_{i\,j})$. Set

$$W_A = 1 - 3A,\ W_B = 1 - 3B,\ \text{and}\ W_C = 1 - 3C.$$

Lemma 6.2. — $W_A,\ W_B,\ W_C \in {}^tP\,\breve{\mathfrak{g}}\,{}^tP^{-1}$

Proof. — We have

$$ {}^tP\,\tau(e_1)(c)\,{}^tP^{-1} = {}^tP\,\tau({}^tP^{-1}) \begin{pmatrix} c^{-2} & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix} \tau({}^tP)\,{}^tP^{-1}. $$

Hence

$$ \begin{aligned} {}^tP\,\breve{\mathfrak{g}}\,{}^tP^{-1} \ni {}^tP\,\tau({}^tP^{-1})Z_1\tau({}^tP)\,{}^tP^{-1} &= 1 - 3\,{}^tP\,\tau({}^tP^{-1})E_{1\,1}\tau({}^tP)\,{}^tP^{-1} \\ &= 1 - 3(\mathbf{c}_1, \mathbf{0}, \mathbf{0}) \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{pmatrix} \\ &= 1 - 3\mathbf{c}_1\mathbf{r}_1. \end{aligned} $$

We see in a similar fashion for $i = 2, 3$

$$ 1 - 3\mathbf{c}_i\mathbf{r}_i = {}^tP\,\tau({}^tP^{-1})Z_i\tau({}^tP)\,{}^tP^{-1} \in {}^tP\,\breve{\mathfrak{g}}\,{}^tP^{-1}\,. $$

□

Lemma 6.3. — $\frac{1}{9}[Z_i, W_A] = E_{i\,i}A - AE_{i\,i}$ *(*$i = 1, 2, 3$*)*

Proof. — By definition, we find instantly

$$ \frac{1}{9}[Z_i, W_A] = [E_{i\,i}, A] = E_{i\,i}A - AE_{i\,i}. $$

□

Lemma 6.4. — *For* $i \neq j$*, we have* $\frac{1}{27}\,[Z_i, [Z_j, W_A]] = a_{i\,j}E_{i\,j} + a_{j\,i}E_{j\,i}$*.*

Proof. — From Lemma 6.3, we see for $i \neq j$

$$ \begin{aligned} \frac{1}{27}\,[Z_i, [Z_j, W_A]] &= -[E_{i\,i}, E_{j\,j}A - AE_{j\,j}] \\ &= E_{i\,i}AE_{j\,j} + E_{j\,j}AE_{i\,i} \\ &= a_{i\,j}E_{i\,j} + a_{j\,i}E_{j\,i}. \end{aligned} $$

□

Lemma 6.5. — *For* $i \neq j$*, we have* $\frac{1}{81}\,[Z_i, [Z_i, [Z_j, W_A]]] = a_{j\,i}E_{j\,i} - a_{i\,j}E_{i\,j}$*.*

Proof. — Immediate from Lemma 6.4. □

Proposition 6.6. — *For each distinct indices* i *and* j *(*$i, j = 1, 2, 3$*), we have* $a_{i\,j}E_{i\,j} \in {}^tP\,\breve{\mathfrak{g}}\,{}^tP^{-1}$*. Explicitly:*

$$ a_{i\,j}E_{i\,j} = \frac{1}{54}\,[Z_i, [Z_j, W_A]] - \frac{1}{162}\,[Z_i, [Z_i, [Z_j, W_A]]] $$

Proof. — Combination of Lemma 6.4 with Lemma 6.5. □

Corollary 6.7. — *For each distinct indices* i *and* j *(*$i, j = 1, 2, 3$*), we also have* $b_{i\,j}E_{i\,j} \in {}^tP\,\breve{\mathfrak{g}}\,{}^tP^{-1}$ *and* $c_{i\,j}E_{i\,j} \in {}^tP\,\breve{\mathfrak{g}}\,{}^tP^{-1}$*.*

Proof. — Replace A with B or with C in Lemmas 6.3–6.5 and Proposition 6.6. □

Lemma 6.8. — *Suppose α is* not *cubic, where $\alpha \in K^{\mathrm{sep}}$ is the element used to define the group $\breve{G}$. For $i \neq j$, there exists $\tau \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$, which may depend on the pair (i, j), such that at least one of $a_{i\,j}$, $b_{i\,j}$, or $c_{i\,j}$ becomes non-zero.*

Proof. — By definition

$$ {}^tP = \begin{pmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{pmatrix}. $$

As we saw in Section 2, we have

$$ {}^tP^{-1} = \begin{pmatrix} \beta\gamma & \gamma\alpha & \alpha\beta \\ -\beta-\gamma & -\gamma-\alpha & -\alpha-\beta \\ 1 & 1 & 1 \end{pmatrix} \cdot D. $$

Here D is a diagonal matrix and $\det D \neq 0$. Thus we know for any $\tau \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$

$$ \begin{aligned} &{}^tP\,\tau({}^tP^{-1}) \\ &= \begin{pmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{pmatrix} \begin{pmatrix} \tau(\beta)\tau(\gamma) & \tau(\gamma)\tau(\alpha) & \tau(\alpha)\tau(\beta) \\ -\tau(\beta)-\tau(\gamma) & -\tau(\gamma)-\tau(\alpha) & -\tau(\alpha)-\tau(\beta) \\ 1 & 1 & 1 \end{pmatrix} \cdot \tau(D) \\ &= \begin{pmatrix} (\alpha-\tau(\beta))(\alpha-\tau(\gamma)) & (\alpha-\tau(\gamma))(\alpha-\tau(\alpha)) & (\alpha-\tau(\alpha))(\alpha-\tau(\beta)) \\ (\beta-\tau(\beta))(\beta-\tau(\gamma)) & (\beta-\tau(\gamma))(\beta-\tau(\alpha)) & (\beta-\tau(\alpha))(\beta-\tau(\beta)) \\ (\gamma-\tau(\beta))(\gamma-\tau(\gamma)) & (\gamma-\tau(\gamma))(\gamma-\tau(\alpha)) & (\gamma-\tau(\alpha))(\gamma-\tau(\beta)) \end{pmatrix} \cdot \tau(D) \end{aligned} $$

and

$$ \tau({}^tP)\,{}^tP^{-1} = \begin{pmatrix} (\tau(\alpha)-\beta)(\tau(\alpha)-\gamma) & (\tau(\alpha)-\gamma)(\tau(\alpha)-\alpha) & (\tau(\alpha)-\alpha)(\tau(\alpha)-\beta) \\ (\tau(\beta)-\beta)(\tau(\beta)-\gamma) & (\tau(\beta)-\gamma)(\tau(\beta)-\alpha) & (\tau(\beta)-\alpha)(\tau(\beta)-\beta) \\ (\tau(\gamma)-\beta)(\tau(\gamma)-\gamma) & (\tau(\gamma)-\gamma)(\tau(\gamma)-\alpha) & (\tau(\gamma)-\alpha)(\tau(\gamma)-\beta) \end{pmatrix} \cdot D. $$

We show that there exists $\tau \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$ such that at least one of $a_{1\,2}$, $b_{1\,2}$, or $c_{1\,2}$ does not vanish. Since α is not cubic, we can choose $\tau \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$ so that α does not equal any of $\tau(\alpha)$, $\tau(\beta)$, and $\tau(\gamma)$. With such a choice of τ, by the definitions of $a_{1\,2}$, $b_{1\,2}$, and $c_{1\,2}$, they are respectively equal to $\tau(\alpha)-\gamma$, $\tau(\beta)-\gamma$, and $\tau(\gamma)-\gamma$ modulo multiplication of non-zero elements of K^{sep}. We see in this way at least two of them are non-zero indeed. With the same choice of $\tau \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$, the elements $a_{1\,3}$, $b_{1\,3}$, and $c_{1\,3}$ are respectively $\tau(\alpha)-\beta$, $\tau(\beta)-\beta$, and $\tau(\gamma)-\beta$ up to multiplication of non-zero elements of K^{sep}. Hence at most one of them disappears.

Changing the role of α with that of β or γ in the above discussion, we obtain what we want. ☐

Corollary 6.9. — *On the assumption that α is* not *cubic, we have $\breve{G} = \mathrm{SL}_3$.*

Proof. — Directly follows from Proposition 6.6, Corollary 6.7, and Lemma 6.8. ☐

Theorem 6.10. — *If the base field K is of characteristic zero (and if there exists an element $\sigma \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$ with $\sigma^2(\alpha) \neq \alpha$), then the filtered vector space $(\breve{V}, F^{\cdot}_{\alpha}\breve{V})$ in Section 5 is in the image of a fully faithful tensor functor of the category of finite dimensional representation spaces over K of a 2-dimensional anisotropic torus over K or SL_3 according as the coefficient α is cubic over K or not, the functor being compatible with the forgetful functors to Vec_K.*

Proof. — When the element α is cubic over K, we have already seen in Section 5 that the conclusion is true. When α is not cubic over K, we have confirmed in Corollary 6.9 that the group $\breve{G}$ is identical to SL_3, hence we are done. □

Appendix

Let q, r, s be indeterminates; K an arbitrary field; L an extension field of K; α, β elements of L; and c a positive constant number. We denote by V the vector space over K generated by the indeterminates q, r, s and put

$$l_0 = q, \quad l_1 = -q\alpha + r, \quad l_2 = -q\beta + s.$$

The linear forms l_0, l_1, l_2 in q, r, s constitute a basis of the vector space $V \otimes_K L$ over L. We attach to the vector space V over K a filtration $F^{\cdot}V$ over L defined as

$$F^i V = \begin{cases} V \otimes_K L & (i \leq -2c) \\ l_1\, L \oplus l_2\, L & (-2c < i \leq c) \\ 0 & (i > c). \end{cases}$$

Lemma A.1. — *The filtered vector space $(V, F^{\cdot}V)$ of slope zero is semi-stable if and only if the elements $1, \alpha, \beta$ of L are linearly independent over K.*

Proof. — Suppose V is not semi-stable. There exists a non-zero subspace W over K of V such that its slope $\mu(W)$ is positive (cf. e.g. [3, Definition 1.13]). By the definition of (induced) sub-filtration (cf. e.g. [3, Definition 1.4]), we must have

$$W \otimes_K L \subset l_1\, L \oplus l_2\, L.$$

In particular, for a non-zero $w \in W$ there exist $a, b \in L$ such that

$$w = l_1 a + l_2 b = -q(a\alpha + b\beta) + ra + sb.$$

We observe that $(a, b) \neq (0, 0)$ and that the elements a, b, and $a\alpha + b\beta$ belong in fact to K. Thus $1, \alpha, \beta$ are linearly dependent over K.
Conversely, if $1, \alpha, \beta$ are linearly dependent over K, then there exist $a, b \in K$ such that $(a, b) \neq (0, 0)$ and $a\alpha + b\beta \in K$. Set $w = l_1 a + l_2 b$. We see

$$w = -q(a\alpha + b\beta) + ra + sb \in V \cap (l_1\, L \oplus l_2\, L) \setminus \{0\}.$$

We get $0 \neq wK \subset V$ and $\mu(wK) = c > 0 = \mu(V)$, which indicates V is not semi-stable. □

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