



PUBLICATIONS MATHÉMATIQUES DE BESANÇON

Algèbre et Théorie des nombres

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2026, p. 21-52.

<https://doi.org/10.5802/pmb.68>

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PRESSES UNIVERSITAIRES DE FRANCHE-COMTÉ



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<http://www.centre-mersenne.org/>

e-ISSN : 2592-6616

ON DEFINABLE GROUPS IN REAL CLOSED FIELDS WITH A GENERIC DERIVATION, AND RELATED STRUCTURES

by

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Abstract. — We study finite-dimensional groups definable in models of the theory RCF_∂ of real closed fields with a generic derivation (also known as *CODF*, the theory of closed ordered differential fields [28]). We prove that any such group Γ definably embeds in a “canonical” semialgebraic group G . We explain how our methods work in the general context of strongly model complete theories T of large “geometric” fields with a generic derivation, which includes the cases where T is the theory of pseudofinite fields and $T = \text{Th}(\mathbb{Q}_p)$. We also give a general theorem on recovering a definable group from generic data in the context of geometric theories.

Finally we extend the methods to o -minimal theories with a generic derivation, due to Fornasiero and Kaplan, [10], and open theories of topological fields with a generic derivation, due to Cubides-Kovacsics and the third author, [7].

Résumé. — Nous étudions les groupes définissables, de dimension finie dans les modèles de la théorie RCF_∂ des corps réels-clos munis d’une dérivée générique (aussi connue sous l’acronyme *CODF*, la théorie des corps ordonnés différentiellement clos [28]). Nous montrons qu’un tel groupe Γ se plonge dans un groupe semi-algébrique G « canonique ».

Nous expliquons comment nos méthodes se généralisent aux théories fortement modèles-complètes de corps « géométriques », larges, munis d’une dérivée générique. Ceci comprend les cas de la théorie des corps pseudo-finis et de la théorie du corps des nombres p -adiques. Nous montrons également, dans le contexte des théories géométriques, comment récupérer un groupe définissable à partir de données génériques.

Enfin nous étendons nos méthodes aux théories o -minimales avec une dérivée générique, traitées par Fornasiero et Kaplan, [10], et aux théories « ouvertes » de corps topologiques avec une dérivée générique, étudiées par Cubides-Kovacsics et le troisième auteur, [7].

1. Introduction

Let us begin by saying that this paper has a substantial expository aspect, as will be explained later.

Acknowledgements. — YP is partially supported by ISF grant 290/19. AP is partially supported by NSF grants DMS-1760212, DMS-2054271 and DMS-2502292. FP is partially supported by Fonds de la Recherche Scientifique-FNRS-FRS.

The aim of the paper is to describe finite-dimensional definable groups in models of the theory RCF_∂ , also known as closed ordered differential fields. In particular any such group Γ definably embeds in a group G which is semialgebraic, namely definable in the restriction to the field language. Moreover G will be canonical in a sense we make precise. (We will discuss later the case of infinite-dimensional groups.)

The proof goes through various stages. First, in Section 3, from a finite-dimensional group Γ we obtain a “generically defined” group in RCF , which we may call a semialgebraic pre-group. Then, in Section 4, we construct, from the generically defined group, a definable group G in RCF . Finally we show that Γ definably embeds in G .

Although the results and proofs are first presented for groups definable in models of RCF_∂ , they go through in various ways to more general or related contexts, as mentioned in the abstract. Details are given below and in subsequent sections.

One reason for starting with the theory RCF_∂ is that it is fairly well-known and of interest to several researchers. Another is pedagogical; going through first the proof in a concrete case, and then noting what works in more general contexts.

In [20] we adapt the theory of algebraic D -groups to some of the settings considered in this paper, obtaining in particular the notion of a Nash D -group (a Nash group over a real closed field with a derivation, equipped with a “connection” respecting the group structure). We make heavy use of the constructions in the current paper to equip G with the structure of a D -group (G, s) such that Γ is precisely the set of solutions of the corresponding differential equation on G . This depends on the “canonicity” of G .

A first version of this paper was written and circulated in late 2022. We were then informed that various aspects of the proof already appear in one way or another in the literature (although the full statements of our results do not). Precise details are given towards the end of this introduction. Nevertheless we decided to basically stick with our original presentation of the material. The paper is self-contained, has, in our opinion, some expository value, and also the precise statements of various lemmas are needed (and quoted) in [20].

The theory $CODF$ of closed ordered differential fields was introduced by Michael Singer [28] as an expansion of the theory of real closed ordered fields obtained by adjoining a derivation ∂ and with axioms analogous to L. Blum’s one variable axioms for DCF_0 , bearing in mind the ordering. Singer worked in what he called the *language of ordered fields*, and his axioms for $CODF$ consisted of the axioms for real closed (ordered) fields, the axioms for ordered differential fields (i.e. fields equipped with an ordering and a derivation), as well as the following:

Let $f, g_1, \dots, g_m$ be differential polynomials in one variable y , such that f has order n which is $\geq$ the order of each g_i . Suppose that, viewing f and the g_i as ordinary polynomials of at most $n+1$ variables, there is $(c_0, \dots, c_n)$ such that $f(c_0, \dots, c_n) = 0$ and $g_i(c_0, \dots, c_n) > 0$, for $i = 1, \dots, m$ and $\partial f / \partial y^{(n)}(c_0, \dots, c_n) \neq 0$. Then, viewing f, g_i back as differential polynomials in one indeterminate, there is z such that $f(z) = 0$ and $g_i(z) > 0$ for $i = 1, \dots, m$.

Singer proved in [28] that $CODF$ is the model completion in the language of ordered fields of the theory ODF of ordered differential fields, namely that $CODF$ is the model companion of ODF and that whenever K is a model of $CODF$ and $K_0 \subseteq K$ is a substructure (so model of ODF) then $CODF$ together with the diagram of K_0 is complete. This precisely means that $CODF$ has quantifier elimination in the language of ordered differential fields. Let us stress here that by an ordered differential field, we mean simply a field equipped with both an

ordering and a derivation where no additional relation between the ordering and derivation is assumed.

It is more usual in the model theory of fields to work in the language of rings rather than fields (so in the language of ordered rings rather than ordered fields, when studying ordered fields). So we will work in the language of ordered rings, and remark that it easily follows from Singer's results that *CODF* also has quantifier elimination in this language.

Hence we can redefine *CODF* as follows:

Fact 1.1. —

- (i) *The (universal) theory of ordered differential integral domains, in the language of ordered differential rings, has a model companion, which we call CODF.*
- (ii) *CODF is complete, with quantifier elimination (in the language of ordered differential rings).*

We could work just in the differential ring language, from which point of view *CODF* is the model companion of the theory of formally real differential fields. This is placed in a more general context by Tressl in his work on “uniform model companions” [29] and is relevant to some of our subsequent generalizations, so we give a few details. The reader can see [29] for more references.

From now on L will be the language of rings and L_∂ the language of differential rings. We work throughout in characteristic 0. A field K is said to be *large* [26] if any irreducible curve over K with a smooth K -point has infinitely many K -points. The class of large fields is axiomatizable, and Tressl points out that if T is a model complete theory of large fields in the language L , then the theory $T \cup \{\partial \text{ is a derivation}\}$ has a model companion, which we will call T_∂ . Actually Tressl works with several commuting derivations $\partial_1, \dots, \partial_m$ but we are here just concerned with the case $m = 1$. An important aspect of Tressl's work is that axioms can be given for T_∂ which are uniform across T . These can be formulated as so-called “geometric axioms” (see Fact 2.8 and Remark 2.9(2) of [18]) and will be discussed later. But the connection with *CODF*, is that real closed fields are large, and that the theory *RCF* of real closed fields in the ring language is model-complete, whereby Tressl's theory gives us the model companion RCF_∂ of $RCF \cup \{\partial \text{ is a derivation}\}$. (8.2)(ii) of [29], says:

Fact 1.2. — *CODF, when formulated in the language of differential rings coincides with RCF_∂ .*

From now on we will use the expression RCF_∂ , instead of *CODF*, as it will be easier notation-wise to adjust to the variants T_∂ . Even though one works in the differential ring language, there is of course a unique ordering which is definable, so it makes sense to talk about semialgebraic sets. We may often say “by quantifier elimination” when we mean quantifier elimination in the ordered differential ring language or ordered ring language, depending on the context. We hope there is no confusion for the reader.

Our main result, proved in Section 4 is:

Theorem 1.3. — *Let Γ be a finite-dimensional group definable in a model (K, ∂) of RCF_∂ . Then there is a group G definable in the real closed field K , and a definable (in (K, ∂)) embedding of Γ in G with additional properties given in Remark 4.8.*

The corresponding result for *connected* groups of finite Morley rank definable in differentially closed fields is Lemma 1.1 of [24]. The methods there are stability-theoretic, and a main point of the current paper is to adapt the ideas to unstable situations where there are *many* “types of maximal dimension”.

The result for arbitrary (possibly infinite Morley rank) connected groups definable in differentially closed fields, namely that they are embeddable in algebraic groups, appears in [23], and directly adapting the methods there to the unstable contexts is more problematic, but will be discussed later.

We now want to state formally the generalizations which we will also prove. We will mention the contexts, state the results, then say something about the notation. There are three overlapping contexts.

- In context (A), discussed in Section 5.2, T is a theory of large geometric fields in the language L of rings which is strongly model complete. T_∂ is the model companion of $T \cup \{\partial \text{ is a derivation}\}$, in the language $L_\partial = L \cup \{\partial\}$.
- In context (B), discussed in Section 6.1, T is a complete, model-complete expansion of the theory of real closed fields, in a language L which *expands* the language of ordered rings. T_∂ is the model companion of $T \cup \{\partial \text{ is a compatible derivation}\}$ in the language $L_\partial = L \cup \{\partial\}$.
- In context (C), discussed in Section 6.2, T is an “open theory of large topological fields” of characteristic 0 in a language L expanding the language of rings by constant and relation symbols. T_∂ is the model companion of $T \cup \{\partial \text{ is a derivation}\}$ in the language $L_\partial = L \cup \{\partial\}$.

So note that the language L (so also L_∂) may be different in these three contexts, and the existence of the relevant model companion T_∂ is part of the literature.

Theorem 1.4. — *In each of the contexts (A), (B), (C), let (K, ∂) be a model of T_∂ (and K a model of T in the language L). Let Γ be a group definable in the structure (K, ∂) . Then there is a definable (in (K, ∂)) embedding of Γ in a group G which is definable in the L -structure K (and again with close connections between Γ and G as in Theorem 1.3 above).*

For context (A), geometric fields were introduced in [15], and strongly model complete is also known as “almost QE” and appears in [4]. Anyway, see Theorem 5.11. As mentioned in the abstract, context (B) appears in [10] and details will be given in Section 6.1. See Theorem 6.8. Likewise context (C) appears in [7] and more details appear in Section 6.2. See Theorem 6.12. Each of these contexts subsumes the special case of RCF_∂ and more examples will also be given below.

Finally in this introduction, we mention another generalization of one aspect of the proof of Theorem 1.3 and which plays a role in the proofs of Theorem 1.4. This is the construction of a definable group from a generically definable group, in the context of an arbitrary geometric structure, and extends the “group chunk” methods from [30]. Geometric structures were again introduced in [15].

Theorem 1.5. — *Let M be a geometric structure and let (X, F, g) be a generically defined group on the definable set X (all in the sense of M). Then there is a definable map h of X into a definable group G , with various properties, including that the images under h of the generically defined multiplication F and inversion g on X are multiplication and inversion on G . Also $h(X)$ is “large” in G .*

See Section 5.1 for the precise definitions and Theorem 5.4 there.

Our model-theoretic and differential-algebraic notation is standard. But one can see the volume [3] for a lot of background on these topics. In particular the article by the second author on the model theory of algebraically closed fields gives an account of algebraic varieties which is relevant to the current paper.

We will be working in one-sorted structures (except in Section 6.2), mainly fields equipped with a derivation. In this context $\bar{a}, \bar{b}, \dots$ will denote finite tuples from the “home sort” and likewise for finite tuples of variables. But nevertheless when there is no ambiguity, we may let $a, b, \dots$ denote finite tuples from the universe, and $x, y, \dots$ finite tuples of variables.

Let us say a few words about the possible interest or relevance of work in this paper. There may be a sense that when working with ordered or valued fields one should really posit some additional compatibilities between the derivation and ordering (or valuation), as for example is done in asymptotic differential algebra. Nevertheless there is also interest and work in “real differential algebra” and even “real differential Galois theory” where no such additional compatibilities are assumed. Likewise in the p -adic case. See for example [6]. So establishing “universal domains” for differential algebra over real, p -adic (and even characteristic 0 pseudofinite) fields seems a worthwhile enterprise and this is what is accomplished in a fairly general context by Tressl’s “uniform model companion”. In the background may be also new kinds of semialgebraic *ODE*’s such as logarithmic differential equations on Nash groups, with their own Galois theory, but this remains to be seen.

There is also quite a bit of work on definable groups in various kinds of dense pairs, such as [1]. As for example RCF_∂ is an expansion of the theory of dense pairs of real closed fields, we expect both the results and methods of the current paper to impact this kind of work.

Let us finally discuss connections with other recent work and some history. After a first version of the paper was written and circulated, some connections to and overlaps with their own works were then pointed out by Pantelis Eleftheriou and Silvain Rideau-Kikuchi, whom we thank, and which we will discuss now.

Let us focus on the RCF_∂ case. Our first step, carried out in Section 3, is to construct from a finite-dimensional definable group Γ in RCF_∂ a canonical “generically defined group” (see Definition 5.2) on a definable set in RCF . See Proposition 3.5. The second step is to produce from the generically defined group in RCF a “group chunk” on a definable set in RCF . We do not give here (and do not need or want to give) a definition of a “group chunk” but it is precisely the definable set Y_0 in Lemma 4.2. The third step is to produce from the “group chunk” an actual definable group G in RCF (as a group of “germs” of definable functions on Y_0). See Lemma 4.5. The fourth step is to definably (in RCF_∂) embed Γ in G . See Lemma 4.7. The second and third steps are put in the general context of geometric structures in our Section 5.

In [9], a group chunk theorem appears in a certain axiomatic framework (of a finite-valued definable dimension on definable sets), yielding from a group chunk an interpretable group. This is Theorem 2.2 of [9]. The axiomatic framework includes “geometric structures”, so also

RCF , $\mathcal{o}$ -minimal structures, pseudofinite fields, p -adically closed fields, ... So this coincides with our third step, except that we produce a definable group, regardless of elimination of imaginaries. As our proof of the fourth step depends on the constructions and notations from our steps one, two and three, we stick with our proof of the third step.

Rideau-Kikuchi pointed out his interesting paper [16], joint with Hrushovski. After a bit of translation, Section 3.4 of that paper introduces the notion of a generically defined group on a “definable partial type” and Proposition 3.15 there recovers a type-definable group. (Actually done at the level of generality of $\ast$ -definable or pro-definable groups.) This represents a combination of our second and third steps (the content of Section 5), except for producing a type-definable rather than definable group.

So although we would not describe it as “folklore” this construction of a (type-)definable group from “generic data” in a somewhat general context is represented in various forms in recent literature.

In hindsight a main point of these papers (including ours) is to generalize various constructions, originating with Weil’s theorem on reconstructing a connected algebraic group from generic data, to contexts of definable partial types rather than definable complete types. (For example, given a geometric structure M and definable set X in M , the set of complete types over M concentrating on X of maximal dimension can be viewed as a definable partial type). The second author of the current paper was acquainted with an early version of [16] by Hrushovski alone, but Section 3 in that version on groups with definable generics referred only to definable *complete* generic types.

In addition to Proposition 3.15 of [16], Proposition 3.4 in the same paper shows that a $\ast$ -definable group (i.e. whose domain consists of possibly infinite tuples) with a “definable partial generic type” is a projective limit of definable groups. These results, with some additional work, yield a solution to the case of infinite dimensional groups Γ definable in say RCF_∂ : Γ definable embeds in a semialgebraic group G (although G is no longer canonical in any sense and Γ does not arise from any D -group structure on G). This confirms a suggestion of Rideau-Kikuchi and appears in [25]. But let us note that in the more general contexts (say of our Section 5.2) where T need not have elimination of imaginaries, [25] only yields that G is interpretable, whereas when Γ is finite-dimensional we obtain *definability* of G .

There are two additional technical updates to our current paper. Firstly, owing to a recent paper of Fornasiero and Terzo [11], we can drop the assumption in Section 5.2 and Theorem 5.11 that T is a theory of *large* fields. Also from [25] we can also assume in Section 5.2 and Theorem 5.11 that T is just model complete, rather than almost QE. (The almost QE assumption was used to control definable closure in models of T_∂ , which now is in Corollary 1.42 of [25] only using model completeness.) Let us also mention that the axiomatic context of [25] includes both contexts A and B above, but not context (B).

Acknowledgements. — In addition to the grant support mentioned above, all authors would like to thank the Fields Institute, Toronto, for their hospitality in Fall 2021 when this work began. In particular the second author would like to thank the Simons Foundation and the Fields Institute for a Simons Distinguished Visitor position during this period. The second author would also like to thank Imperial College for its hospitality in the summer of 2022 with a Nelder Visiting Fellowship.

2. Preliminaries and geometric axioms

In this section we work with the complete theory RCF_∂ .

We take L to be the language of rings $\{+, \times, -, 0, 1\}$ and L_∂ the language of differential rings $\{+, \times, -, 0, 1, \partial\}$.

We will write a model M of RCF_∂ as (K, ∂) where K is a field, necessarily real closed, and ∂ is a derivation. A saturated model of RCF_∂ (or universal domain) will be written as $(\mathcal{U}, \partial)$. So note that $\mathcal{U}$ is a saturated real closed field.

For A a subset of $\mathcal{U}$, by an $(L_\partial)_A$ -definable set we mean a set definable over A in the structure $(\mathcal{U}, \partial)$, so defined by an L_∂ -formula with parameters from A . By an L_A -definable set over A we mean the same thing but in the reduct of $\mathcal{U}$ to L , namely defined in the structure $\mathcal{U}$ by an L -formula over A . So the latter is the same as a semialgebraic set over A . As usual we denote by $k(\bar{a})$ the field generated by k and $\bar{a}$ (where k is a subfield of $\mathcal{U}$ and $\bar{a}$ a tuple from $\mathcal{U}$), and we let $k\langle\bar{a}\rangle$ denote the differential field generated by k and $\bar{a}$ where $\bar{a}$ is as before and typically k is a differential subfield of $(\mathcal{U}, \partial)$. Note that $k\langle\bar{a}\rangle$ is precisely $k(\bar{a}, \partial(\bar{a}), \dots, \partial^{(m)}(\bar{a}), \dots)$, where $\partial^{(m)} = \partial \circ \dots \circ \partial$ (m times). If $\bar{a} = (a_1, \dots, a_n)$, $\partial^{(m)}(\bar{a})$ denotes $(\partial^{(m)}(a_1), \dots, \partial^{(m)}(a_n))$. We also sometimes let $a^{(m)}$ denote $\partial^{(m)}(a)$, and likewise for $\bar{a}^{(m)}$.

Fact 2.1. —

- (i) Any $(L_\partial)_A$ -formula $\phi(\bar{x})$ is equivalent, modulo RCF_∂ , to a formula of the form $\theta(\bar{x}, \partial(\bar{x}), \dots, \partial^m(\bar{x}))$ for some m and L_A -formula $\theta(\bar{x}_0, \dots, \bar{x}_m)$.
- (ii) For any $A \subset \mathcal{U}$ the model theoretic algebraic closure of A in the structure $(\mathcal{U}, \partial)$ is the same as the field-theoretic relative algebraic closure of the differential subfield of $\mathcal{U}$ generated by A . Hence, by virtue of the ordering, the model-theoretic definable closure of A in $(\mathcal{U}, \partial)$ is the same as the definable closure in the real closed field $\mathcal{U}$ of the differential field generated by A .
- (iii) If X, Y are $(L_\partial)_A$ -definable sets in $(\mathcal{U}, \partial)$ and $f : X \rightarrow Y$ is $(L_\partial)_A$ -definable then we can partition X into finitely many $(L_\partial)_A$ -definable sets X_i such that for each i , $f|X_i$ is of the form $g(\bar{x}, \partial(\bar{x}), \dots, \partial^{(m)}(\bar{x}))$ for some m and L_A -definable function g .

Explanation. — (i) is precisely quantifier elimination. (ii) follows from the Singer axioms for $CODF$, namely if k is a (small) differential subfield of $\mathcal{U}$ and $a \in \mathcal{U}$ is not algebraic over k in the field-theoretic sense then we conclude from the axioms and compactness/saturation that there are infinitely many elements in $\mathcal{U}$ with the same L_∂ -type as a over k . (iii) follows from quantifier elimination, (ii), and compactness. $\square$

Typically $k, K, \dots$ will denote small differential subfields of $(\mathcal{U}, \partial)$. The following are exactly as in DCF_0 :

Definition 2.2. —

- (i) Let $\bar{a}$ be a finite tuple, and k a differential subfield of $(\mathcal{U}, \partial)$. By $\text{order}(\bar{a}/k)$ we mean the transcendence degree over k of $k\langle\bar{a}\rangle$ if this is finite, and ∞ otherwise. We will also write $\dim_\partial(\bar{a}/k)$ for $\text{order}(\bar{a}/k)$ when the latter is finite.

- (ii) Let X be definable over k . Then $\text{order}(X) = \max\{\text{order}(\bar{a}/k) : \bar{a} \in X\}$ if this is finite and ∞ otherwise. We say that X is “finite-dimensional” if $\text{order}(X)$ is finite. In this case we also write $\dim_{\partial}(X)$ for $\text{order}(X)$.
- (iii) For X definable over k and finite-dimensional, and $\bar{a} \in X$ we will call $\bar{a}$ ∂ -generic in X over k if $\dim_{\partial}(\bar{a}/k) = \dim_{\partial}(X)$.
- (iv) Let $\bar{a}$ and $\bar{b}$ be tuples (even infinite) from $\mathcal{U}$. We will say that $\bar{a}$ is ∂ -independent from $\bar{b}$ over k , if $k\langle\bar{a}\rangle$ is independent from $k\langle\bar{b}\rangle$ over k in the sense of fields.

We can and will distinguish $\dim_{\partial}$, ∂ -genericity, and ∂ -independence from dimension, genericity, independence in RCF by referring to the latter as $\dim_L$, L -genericity, L -independence, when we want to avoid ambiguity. More will be said later, especially about the connection to the corresponding notions in algebraic geometry. When we want to distinguish types in the sense of RCF and types in the sense of RCF_{∂} , we will write $tp_L(\cdot)$ and $tp_{L_{\partial}}(\cdot)$ (or $tp_{\partial}(\cdot)$ for brevity).

We will be talking about algebraic varieties V over $\mathcal{U}$ or over (possibly differential) subfields k of $\mathcal{U}$. First by a k -variety V (or variety over k), we will usually mean an *affine* variety defined by polynomial equations over k and which is k -irreducible. We will explain below why we will take in addition V to be absolutely irreducible (also called geometrically irreducible). But a warning is that we will attach to such V another object $T_{\partial}(V)$ which is a possibly reducible affine variety over k . Secondly our notational conventions are that, in contrast to [2] (where varieties are subsets of K^n , K a real closed field), we can identify a variety V over k with the “functor” taking fields F containing k to the set $V(F)$ of solutions of the set of equations over k defining V . So from this point of view V can be identified with its set of F points for F an algebraically closed field containing k . On the other hand we will be concerned with sets of points $V(K)$ of V in real closed fields $K < \mathcal{U}$ containing k or in $\mathcal{U}$ itself. $V(\mathcal{U})$ will of course be a set definable in $\mathcal{U}$ over k .

We will assume that the base field $k < \mathcal{U}$ over which V is defined is real closed. (In the RCF case this is a mild assumption as it just means that k is definably closed in $\mathcal{U}$.) The varieties we consider will typically arise by choosing a tuple $\bar{a}$ from $\mathcal{U}$ and defining V to be the affine variety over k defined by all the polynomial equations over k vanishing at $\bar{a}$. But then $V(k)$ will be Zariski dense in V (as $k < \mathcal{U}$) and so in fact V will be absolutely irreducible, not only k -irreducible. Let us mention in passing that from [26], the largeness of a field k (such as a real closed field) is equivalent to any k -variety V with a smooth k point having a Zariski-dense set of k -points.

Given a k -variety V , the *algebro-geometric dimension* of V is precisely the maximum of $\text{trdeg}(k(a)/k)$ where a is a point of V in some *algebraically closed* field F containing k . We call such a (witnessing the algebro-geometric dimension of V) a Zariski-generic point of V over k . To say that $V(\mathcal{U})$ is Zariski-dense in V means precisely that we can find a Zariski-generic point of V over k in $V(\mathcal{U})$. So to summarize:

Remark 2.3. — Assume V is a k -variety such that $V(\mathcal{U})$ is Zariski-dense. Then the algebro-geometric dimension of V coincides with the o -minimal dimension of $V(\mathcal{U})$. Hence for points $a \in V(\mathcal{U})$ being Zariski-generic over k and o -minimal generic over k coincide.

Now suppose V to be a k -variety. From differential algebra we have the notion of the prolongation of V , also called the “shifted tangent bundle of V ”, which will be a possibly reducible

variety defined by polynomial equations over k . We denoted this by $T_\partial(V)$, which comes equipped with a projection $\pi : T_\partial(V) \rightarrow V$. $T_\partial(V)$ is also called $\tau(V)$ in the literature (such as [19]).

Here are the details. Assume that V is a k -variety in affine n -space, and let $I_k(V)$ be the ideal in $k[x_1, \dots, x_n]$ consisting of polynomials vanishing on V . For each polynomial $P(x_1, \dots, x_n) \in I_k(V)$, consider the equation over k in variables $x_1, \dots, x_n, u_1, \dots, u_n$,

$$\sum_{i=1, \dots, n} (\partial P / \partial x_i)(\bar{x}) u_i + P^\partial(\bar{x}) = 0$$

Here P^∂ is the polynomial over k obtained from P by applying the derivation ∂ to the coefficients of P .

Then $T_\partial(V)$ is the possibly reducible subvariety of affine $2n$ -space defined by all these displayed equations, as well as $P(\bar{x}) = 0$, as P ranges over $I_k(V)$. And π is the projection to the first n -coordinates.

Definition 2.4. — By a rational D -variety (V, s) defined over a field k , we mean a k -variety V equipped with a rational section, defined over k from V to $T_\partial(V)$. This means that s is given by a system of quotients P/Q of polynomial functions over k in n indeterminates, such that working in an ambient algebraically closed field, s is defined on a nonempty Zariski open (over k) subset U of V and for $a \in U$, $s(a) \in T_\partial(V)$ and $\pi(s(a)) = a$.

When s is a regular map, namely everywhere defined on V , we have the notion of an algebraic D -variety, as defined explicitly by Buium [5] in the general differential algebraic context, but with geometric origins in the notion of an Ehresmann connection. In the literature V is often assumed to be smooth. So we state a fact, just for the record, although it will not be needed for the rest of the paper:

Remark 2.5. —

- (i) When V is smooth, then T_∂ is irreducible (so also a k -variety) as well as smooth, with $\dim(T_\partial(V)) = 2 \dim(V)$.
- (ii) In general, $T_\partial(V)$ may be reducible, with dimension $< 2 \dim(V)$. But nevertheless it is smooth and irreducible over the smooth locus V_{sm} of V . Namely (again working in an ambient algebraically closed field), $\pi^{-1}(V_{sm})$ is a smooth irreducible quasi-affine variety over k .

Proof. —

(i). — It is (1.5) of [19].

(ii). — The proof of (1.4) in [19] gives smoothness of the quasi-affine variety $\pi^{-1}(V_{sm})$, and the proof of (1.5) of [19] gives irreducibility. $\square$

Now $\mathcal{U}$ embeds in a differentially closed field which can be taken to be the algebraic closure $(\bar{\mathcal{U}}, \partial)$ of $\mathcal{U}$ [27] and the geometric axioms for DCF_0 imply that one can find a generic point a of $V(\bar{\mathcal{U}})$ over k , such that $s(a) = (a, \partial(a))$.

Lemma 1.6 of [4] says that the analogous statement holds for RCF_∂ :

Fact 2.6. — *Under the assumptions that k is a real-closed differential subfield of $\mathcal{U} \models RCF_\partial$, (V, s) a rational D -variety over k and $V(\mathcal{U})$ is Zariski-dense, there is $a \in V(\mathcal{U})$, a generic point of V over k in the algebro-geometric sense, such that $s(a) = (a, \partial(a))$.*

In fact, the conclusion characterizes (saturated) models of RCF_∂ . Let us note that if $\mathcal{U}$ was replaced by a not necessarily saturated model (K, ∂) of RCF_∂ containing (k, ∂) , then the conclusion of Fact 2.6 would be that for every Zariski open subset U of V , defined over k there is $a \in U(K)$ with $s(a)$ defined and equal to $(a, \partial(a))$.

The main thing we want to do now is to strengthen Fact 2.6. We will need dimension in the $\mathcal{o}$ -minimal or semialgebraic sense as well as various compatibilities with dimension in the algebro-geometric sense (as in Remark 2.3).

We now return to our (saturated) model $(\mathcal{U}, \partial)$ of RCF_∂ with k a (small) real-closed differential subfield of $\mathcal{U}$. Given a semialgebraic set X defined over k , let a be an L -generic point of X over k , and let $p(x) = tp_L(a/k)$. We call p an L -generic type of X over k . Here is the strengthening of Fact 2.6, which can again be considered as giving *stronger* “geometric axioms” for RCF_∂ .

Lemma 2.7. — *Under the same assumptions as Fact 2.6, let $p(x)$ be an L -generic type of $V(\mathcal{U})$ over k . Then there is a realization a of p such that $s(a) = (a, \partial(a))$.*

Proof. — Let us fix a $|k|^+$ -saturated model K of RCF extending $(k, +, \times, <)$. So $p(x)$ is a complete type over k in the sense of K , so is realized by some $a \in V(K)$. By Remark 2.3, a is Zariski-generic in V over k . Let $a = (a_1, \dots, a_n)$. Let $s(a) = (a_1, \dots, a_n, b_1, \dots, b_n)$ so by definition we have that $\sum_{i=1, \dots, n} ((\partial P / \partial x_i)(a_1, \dots, a_n)) b_i + P^\partial(a_1, \dots, a_n) = 0$ for all $P \in I_k(V)$. By the extension theorem for derivations (Theorem 1.1 of [21]) we can extend the derivation ∂ on k to a derivation ∂^* on the field $k(a_1, \dots, a_n)$ by defining $\partial^*(a_i) = b_i$ for $i = 1, \dots, n$. Now extend ∂^* to a derivation ∂^{**} of K . So (K, ∂^{**}) is a model of “ $RCF + \partial$ is a derivation”, hence embeds in a model of RCF_∂ , so without loss of generality into $(\mathcal{U}, \partial)$. As RCF_∂ has quantifier elimination in the ordered differential ring language (and (k, ∂) is a differential field with k real closed), (K, ∂^{**}) embeds in the ordered differential field $\mathcal{U}$ over k . The image of a under this embedding works. $\square$

3. Definable groups of finite dimension

We continue to use notation and conventions from Section 2.

Definition 3.1. —

- (i) If a is a finite tuple from $\mathcal{U}$, and $N \geq 1$ then by $\nabla^{(N)}(a)$ we mean the tuple $(a, \partial(a), \dots, \partial^{(N)}(a))$. We write $\nabla(a)$ for $\nabla^{(1)}(a) = (a, \partial(a))$.
- (ii) For $X \subseteq \mathcal{U}^n$ a L_∂ -definable set, by $\nabla^{(N)}(X)$ we mean $\{\nabla^{(N)}(a) : a \in X\}$. It is also a definable set, defined over the same parameters as X .

For X as in Definition 3.1 (ii), the map $\nabla^{(N)}$ clearly gives a bijection between X and $\nabla^{(N)}(X)$. When $X = \Gamma$ is a definable group over k , then $\nabla^{(N)}$ induces a group structure on $\nabla^{(N)}(\Gamma)$ also defined over k .

Fact 3.2. — Let $\bar{a} \in \mathcal{U}$ and suppose that $\partial^{(n)}(\bar{a})$ is in the field-theoretic algebraic closure of $k(\bar{a}, \partial(\bar{a}), \dots, \partial^{(n-1)}(\bar{a}))$. Then $\partial^{(n+1)}(\bar{a}) \in k(\bar{a}, \partial(\bar{a}), \dots, \partial^{(n-1)}(\bar{a}), \partial^{(n)}(\bar{a}))$. So also $\partial^{(m)}(\bar{a}) \in k(\bar{a}, \partial(\bar{a}), \dots, \partial^{(n-1)}(\bar{a}), \partial^{(n)}(\bar{a}))$ for all $m > n$ (and trivially, for all $m \leq n$).

Proof. — Let a be a coordinate of the tuple $\bar{a}$ and we show that $a^{(n+1)} \in k(\bar{a}, \partial(\bar{a}), \dots, \partial^{(n-1)}(\bar{a}), \partial^{(n)}(\bar{a}))$. Let $P(x)$ be the minimal (monic) polynomial of $a^{(n)}$ over $k(\bar{a}, \dots, \partial^{(n-1)}(\bar{a}))$. Suppose P has degree m . We have that $P(a^{(n)}) = 0$. So $\partial(P(a^{(n)})) = 0$. We then compute that $a^{(n+1)}(\partial P/\partial x)(a^{(n)}) \in k(\bar{a}, \dots, \bar{a}^{(n-1)}, \bar{a}^{(n)})$. But $\partial P/\partial x$ is a polynomial over $k(\bar{a}, \dots, \bar{a}^{(n-1)})$ of degree $< m$, whereby $(\partial P/\partial x)(a^{(n)}) \neq 0$ so we can divide by it to get $a^{(n+1)} \in k(\bar{a}, \dots, \bar{a}^{(n)})$ as required. $\square$

The first aim is to find a reasonably canonical description of finite dimensional sets X and groups Γ definable in $(\mathcal{U}, \partial)$ over k , up to definable bijection (isomorphism) over k . The definable bijection will simply replace Γ by $\nabla^{(N)}(\Gamma)$ for suitable N . This is an adaptation of well-known constructions in DCF_0 , but with a few subtleties, and is one of our main steps.

Proposition 3.3. —

- (I) Let $X \subseteq \mathcal{U}^n$ be a finite-dimensional definable set in $(\mathcal{U}, \partial)$, defined over k . Then, after replacing X by $\nabla^{(N)}(X)$ for suitable N (and replacing n by $n(N+1)$), there is a semialgebraic subset Y of $\mathcal{U}^n$ defined over k and subvarieties $V_1, \dots, V_r$ of affine n -space, defined over k , with Zariski-dense sets of k -points (or $\mathcal{U}$ -points), and for each $i = 1, \dots, r$, a rational section $s_i : V_i \rightarrow T_{\partial}(V_i)$ defined over k , and Zariski open U_i of V_i defined over k on which s_i is defined, such that
 - (a) Y is covered by the $U_i(\mathcal{U})$,
 - (b) For each i , $\dim(V_i) = \dim(Y \cap U_i(\mathcal{U}))$, and
 - (c) $X = \bigcup_{i=1, \dots, r} \{a \in Y \cap U_i(\mathcal{U}) : \nabla(a) = s_i(a)\}$.
- (II) Suppose moreover that $X = \Gamma$ is a finite-dimensional definable group (defined again over k). Then, again replacing Γ by the definably over k isomorphic group $\nabla^{(N)}(\Gamma)$ for suitable N , we have, (with Γ in place of X):
 - (d) There is a partial semialgebraic (over k) function $*$: $Y \times Y \rightarrow Y$, whose restriction to $\Gamma \times \Gamma$ is precisely the group operation on Γ .
 - (e) There is a partial semialgebraic (over k) function $\text{inv} : Y \rightarrow Y$ whose restriction to Γ is precisely group inversion.

Proof. —

(I). — By quantifier elimination, X can be defined by a formula $\theta(\bar{x}, \partial(\bar{x}), \dots, \partial^{(m)}(\bar{x}))$ where θ is an L -formula over k . Note that at this point we can see that $\nabla^{(m)}(X)$ is defined as

$$\{(\bar{x}_0, \bar{x}_1, \dots, \bar{x}_m) : \theta(\bar{x}_0, \dots, \bar{x}_m), \partial(\bar{x}_0, \dots, \bar{x}_{m-1}) = (\bar{x}_1, \dots, \bar{x}_m)\},$$

which does not make use of the finite-dimensionality of X . We will, using finite-dimensionality, refine this description, so as to give the proposition.

By finite-dimensionality and compactness we may find $N \geq m$ such that $\partial^{(N)}(\bar{a})$ is in the field-theoretic algebraic closure of $k(\bar{a}, \dots, \partial^{(N-1)}(\bar{a}))$ for all $\bar{a} \in X$. By Fact 3.2, $\partial^{(r)}(\bar{a}) \in k(\bar{a}, \dots, \partial^{(N-1)}(\bar{a}), \partial^{(N)}(\bar{a}))$ for all $\bar{a} \in X$ and r (in particular for $r = N+1$).

Let us first fix $\bar{a} \in X$, and let $V_{\bar{a}}$ be the affine variety over k with generic (over k) point $\nabla^{(N)}(\bar{a}) (= (\bar{a}, \partial(\bar{a}), \dots, \partial^{(N)}(\bar{a}))$ remember). We have just seen that $\partial^{(N+1)}(\bar{a}) \in k(\nabla^{(N)}(\bar{a}))$, so equals $f(\nabla^{(N)}(\bar{a}))$ for some k -rational function f , defined at $\nabla^{(N)}(\bar{a})$ so defined on some Zariski open set U of $V_{\bar{a}}$ over k . We know that $(\nabla^{(N)}(\bar{a}), \partial(\nabla^{(N)}(\bar{a})))$ is a point on $T_{\partial}(V_{\bar{a}})$, so it follows that $(\nabla^{(N)}(\bar{a}), \partial(\nabla^{(N)}(\bar{a}))) = s_{\bar{a}}(\nabla^{(N)}(\bar{a}))$ for some rational over k section $s_{\bar{a}}$ of $T_{\partial}(V_{\bar{a}}) \rightarrow V_{\bar{a}}$.

Note that this has been accomplished for every $\bar{a}$ in X , but the data in conclusions $V_{\bar{a}}$, $s_{\bar{a}}$ of course depend only on $tp_{\partial}(\bar{a}/k)$.

At this point it is convenient to *replace* X by $\nabla^{(N)}(X)$ via the k -definable bijection $\nabla^{(N)}$. So the “new” X is

$$\{(\bar{x}_0, \bar{x}_1, \dots, \bar{x}_N) : \models \theta(\bar{x}_0, \dots, \bar{x}_m), \bar{x}_{i+1} = \partial(\bar{x}_i), i = 0, \dots, N-1\}.$$

We will add dummy variables so as to consider the formula θ as having free variables $\bar{x}_0, \dots, \bar{x}_N$. We let Y be the set of realizations of θ , an L_k -definable set.

We write a point of the “new” X as $\bar{b} = (\bar{b}_0, \dots, \bar{b}_N)$. So to summarize the situation with this new notation:

- (*) For each $\bar{b} \in X$ we have an irreducible affine variety $V_{\bar{b}}$ (in the appropriate affine space) over k and rational (over k) section $s_{\bar{b}} : V_{\bar{b}} \rightarrow T_{\partial}(V_{\bar{b}})$ such that $\bar{b}$ is a (o -minimal or Zariski) generic point of $V_{\bar{b}}$ over k and $\nabla(\bar{b}) = s_{\bar{b}}(\bar{b})$. Let $U_{\bar{b}}$ be a Zariski-open over k subset of $V_{\bar{b}}$ on which s is defined.

As $V_{\bar{b}}$, $U_{\bar{b}}$ and $s_{\bar{b}}$ depend only on $tp_{\partial}(\bar{b}/k)$, we will write them as V_p , U_p and s_p , where $p = tp_{\partial}(\bar{b}/k)$. Let P be the collection of all such types.

Note that:

- (**) for any $p \in P$, if $\bar{b} \in Y \cap U_p$ and $\nabla(\bar{b}) = s_p(\bar{b})$ then $\bar{b} \in X$.

This is because $\nabla(\bar{b}) = s_p(\bar{b})$ implies that $\partial(\bar{b}_i) = \bar{b}_{i+1}$ for $i = 0, \dots, N-1$ which together with $\bar{b} \in Y$ is equivalent to $\bar{b} \in X$.

On the other hand if $\bar{x} \in X$ then $\bigvee_{p \in P} (\bar{x} \in U_p \wedge \nabla(\bar{x}) = s_p(\bar{x}))$. So by compactness there are (distinct) $p_1, \dots, p_r \in P$ such that X is covered by $\bigcup_{i=1, \dots, r} \{\bar{b} \in U_{p_i} : \nabla(\bar{b}) = s_{p_i}(\bar{b})\}$.

So together with (**) we obtain (a) and (c) of (I) of the Proposition with V_i, U_i for V_{p_i}, U_{p_i} and s_i for s_{p_i} . For (b) notice that for each i , $Y \cap U_i$ is an L_k -definable subset of V_i defined over k which contains an L -generic over k point of V_i so has to have the same L -dimension as V_i . So we have obtained (I).

(II). — Part (I) applies to $X = \Gamma$ as a definable set. The k -definable bijection $\nabla^{(N)}$ between Γ and $\nabla^{(N)}(\Gamma)$ gives a k -definable group structure on $\nabla^{(N)}(\Gamma)$. So we immediately assume that our definable group is $\nabla^{(N)}(\Gamma)$, which we still call Γ and we have (I)(a), (b), (c). Applying Fact 2.1 (iii) to multiplication $m : \Gamma \times \Gamma \rightarrow \Gamma$ and using the fact that for $a \in \Gamma$, $k(a, \partial(a), \dots, \partial^{(m)}(a), \dots) = k(a)$, there is a partition of $\Gamma \times \Gamma$ into finitely many L_{∂} -definable (over k) sets Z_i such that on Z_i , m is given by an L_k -definable function h_i say. Again Z_i is the restriction to $\Gamma \times \Gamma$ of some L_k -definable set $W_i \subseteq Y \times Y$, on which we may assume h_i to be defined. Let W be the union of the W_i (an L_k -definable set). Then by cut and paste we can piece together coherently the h_i ’s to find a L_k -definable function $h : W \rightarrow Y$ such that for each i the restriction of h to Z_i is precisely h_i . Hence the restriction of h to $\Gamma \times \Gamma$

is multiplication. Extend h to a partial L_k -definable function $*$: $Y \times Y \rightarrow Y$, to give (d). Obtaining (e) is similar. $\square$

From our definable (finite-dimensional) group Γ we will construct a semialgebraic analogue of Weil's pre-group [30, Section 1, including Proposition 1], which we could call a semialgebraic pre-group.

We let $\Gamma, Y, V_i, U_i, s_i, *, \text{inv}$ be as in the conclusion of the proposition above. After reordering we may assume that $V_1, \dots, V_s$ are of maximal dimension over k (in the algebraic or semialgebraic sense) among the V_i . So $\dim_L(Y)$ coincides with the algebro-geometric dimension of V_i ($= \dim_L(V_i(\mathcal{U}))$) for each $i = 1, \dots, s$. Note that any L -generic point of Y over k is in U_i . In the next lemma we give the compatibility of the various notions of dimension. See Definition 2.2 and the comments following it.

Lemma 3.4. — *Let Γ be our finite-dimensional L_∂ -definable group (defined again over k). Then with the above notation (including having replaced Γ by suitable $\nabla^{(N)}(\Gamma)$), we have:*

- (i) $\dim_\partial(\Gamma)$ is equal to $\dim_L(Y)$.
- (ii) Let $g \in \Gamma$. Then g is ∂ -generic in Γ over k iff g is L -generic in Y over k iff for some $i = 1, \dots, s$, $g \in V_i$ and is generic in V_i over k in the algebro-geometric sense.
- (iii) Let $g_1, \dots, g_k \in \Gamma$. Then the g_i are ∂ -independent over k iff they are L -independent over k .
- (iv) For any d , let $x_1, \dots, x_d$ be L -generic L -independent (over k) elements of Y . Then there are $g_1, \dots, g_d \in \Gamma$ which are ∂ -generic, ∂ -independent over k , such that $tp_L(x_1, \dots, x_d/k) = tp_L(g_1, \dots, g_d/k)$.
- (v) If g_1, g_2 are ∂ -generic in Γ over k and ∂ -independent over k , then $g_1 g_2$ (product in Γ) is also ∂ -generic in Γ over k and $\{g_1, g_2, g_1 g_2\}$ is pairwise ∂ -independent over k .

Proof. —

(i). — We first note that for all $g \in \Gamma$, $k\langle g \rangle = k(g)$, whereby $\dim_\partial(g/k) = \text{order}(g/k) = \text{trdeg}(k(g)/k)$ which $= \dim_L(g/k)$. Hence $\dim_\partial(\Gamma) \leq \dim(Y)$.

On the other hand, fix $i = 1, \dots, s$. By Fact 2.6 there is $a \in V_i$ which is L -generic over k and with $\nabla(a) = s_i(a)$. But then $a \in \Gamma$. So $\dim_\partial(\Gamma) \geq \dim(V_i) = \dim_L(Y)$.

(ii). — It has the same proof.

(iii). — It is because $k\langle g_i \rangle = k(g_i)$.

(iv). — For each $i = 1, \dots, d$ let $j_i \in \{1, \dots, s\}$ be such that $x_i \in V_{j_i}$. Then $(x_1, \dots, x_d)$ is L -generic in $V_{j_1} \times \dots \times V_{j_d}$ over k . But $(V_{j_1} \times \dots \times V_{j_d}, s_{j_1} \times \dots \times s_{j_d})$ is a rational D -variety over k (satisfying the additional conditions in the hypotheses of Lemma 2.7, namely irreducibility with a Zariski-dense set of $\mathcal{U}$ -points). By Lemma 2.7, there is $(g_1, \dots, g_d)$ satisfying $tp_L(x_1, \dots, x_d/k)$ such that $\nabla(g_i) = s_{j_i}(g_i)$ for each i . But then, because of the construction each $g_i \in \Gamma$, and clearly the g_i are ∂ -generic and ∂ -independent, over k .

(v). — It follows by properties of L -dimension $\square$

Using the notation of Proposition 3.3, we have the following which is exactly the structure of a semialgebraic pregroup on Y .

Proposition 3.5. —

- (i) For L -generic L -independent (over k) x, y in Y , $x * y$ is defined and is also L -generic in Y over k .
- (ii) For x, y, z L -generic L -independent over k , $(x * y)$ is L -independent of z over k and x is L -independent of $(y * z)$ over k and $((x * y) * z) = (x * (y * z))$.
- (iii) For L -generic (over k) $x \in Y$, $\text{inv}(x)$ is defined and L -generic in Y over k , and $\text{inv}(\text{inv}(x)) = x$. Moreover if y is also L -generic and L -independent of x over k , we have that $y = \text{inv}(x) * (x * y)$, $x = (x * y) * \text{inv}(y)$.

Proof. —

(i). — By Lemma 3.4 above, let $g, h \in \Gamma$ such that $tp_L(g, h/k) = tp_L(x, y/k)$. Then $tp_L(g, h, gh/k) = tp_L(x, y, x * y/k)$. By Lemma 3.4(ii) and (v), gh is L -generic in Y over k , so $x * y$ is too.

(ii). — Again by Lemma 3.4(iv) we may assume that x, y, z belong to Γ , and by Lemma 3.4 the statement reduces to its truth for ∂ -independence, which is given by Lemma 3.4(v).

(iii). — Same proof as above. □

4. Producing an RCF -definable group

The main in this section is to use Proposition 3.5 and an adaptation of various group construction results to prove:

Theorem 4.1. — *Let Γ be a finite-dimensional group definable in $(\mathcal{U}, \partial)$. Then there is a group G definable in the real closed field $\mathcal{U}$ and an L_∂ -definable embedding of Γ into G , satisfying the additional conditions in Remark 4.8.*

See also Remark 4.8 at the end of this section for more on the relation between Γ and G . Starting from the data $Y, *, \text{inv}$ in Proposition 3.5 the construction of G is related to Weil's theorem about recovering a (connected) algebraic group from birational data, namely from a generically given group operation on an irreducible variety V [30]. In the Weil context, the generically defined operation is on a single stationary type in ACF_0 (the unique generic type of V). In our context there is an unbounded set of L -generic types of our semialgebraic set Y , which introduces some complications. Nevertheless we produce a suitable “group-chunk” $Y_0 \subseteq Y$ in Lemma 4.2 below, from which we will construct our semialgebraic group from a collection of germs of definable, invertible functions on Y_0 . Similar things appear in a stability-theoretic context in Hrushovski's work on invertible germs of definable functions on a stationary type [13]. The point of view taken here is also influenced by van den Dries' paper [8]. We put this aspect of our work into the general context of geometric structures, in Section 5.1. See the introduction for more references to relevant recent work.

There is another aspect to Weil's theorem which concerns constructing not just a definable (in ACF) group, but an *algebraic* group. The RCF -analogue of this aspect is about constructing a *Nash group* which is done in [22]. This will be relevant to the follow-up paper [20].

We will be working for now in the structure $\mathcal{U}$ (i.e. a saturated real closed field). In any case for now definable (and generic, independent etc.) means in $\mathcal{U}$, namely semialgebraic or L -definable. At the end of the section (Lemma 4.7) we will come back to definability in $(\mathcal{U}, \partial)$. We will go straight in to the proof of Theorem 4.1. We use notation as in Proposition 3.5, namely $(Y, *, \text{inv})$. By a *large* subset Y_0 of Y defined over k we mean a semialgebraic subset Y_0 of Y with $\dim(Y \setminus Y_0) < \dim(Y)$. In particular $\dim(Y_0) = \dim(Y)$. This is equivalent to saying that all generic over k points of Y are in Y_0 . Note that the intersection of finitely many large subsets of Y is large.

Note that if a formula $\phi(x)$ over k holds for all k -generic $a \in Y$, then $\phi(x)$ defines a large subset of Y . Likewise for subsets of $Y \times Y$, $Y \times Y \times Y$. We will also use freely definability of dimension in RCF .

We now let F denote the partial map $*$: $Y \times Y \rightarrow Y$, and I the partial map inv . The properties of Y_0 , F and I in Lemma 4.2 are precisely the “group chunk” axioms.

Lemma 4.2. — *Let Y_0 be the set of $x \in Y$ such that*

- (i) *for a large set of (y, z) in Y^2 , $F(x, y)$, $F(y, z)$, $F(x, F(y, z))$, $F(F(x, y), z)$ are defined and $F(x, F(y, z)) = F(F(x, y), z)$,*
- (ii) *$I(x)$ and $I(I(x))$ are defined and $I(I(x)) = x$,*
- (iii) *for a large set of $y \in Y$, $I(y)$, $F(x, y)$, and $F(I(x), F(x, y))$ are defined, $F(I(x), F(x, y)) = y$, $F(F(x, y), I(y))$ is defined and equals x , and $I(F(x, y)) = F(I(y), I(x))$.*

Then Y_0 is definable over k and is a large subset of Y .

Proof. — This is immediate, using Proposition 3.5, once one sees that Y_0 is definable over k and contains all generic over k points of Y . $\square$

We now consider germs of partial definable functions from Y_0 to Y_0 . Note first that if $f(-)$ is a partial function definable with parameters (such as $F(x, c)$ for some c), then the following are equivalent (see [22])

- (i) *for some tuple of parameters d such that f is definable over k, d , whenever $y \in Y_0$ is generic over k, d then $f(y)$ is defined and in Y_0 ,*
- (ii) *there is a large definable (with parameters) subset U of Y_0 such that f is defined on U , with values in Y_0 .*

If f_1, f_2 satisfy (i) (and (ii)) above, we will say that f_1 and f_2 have the same *germ* on Y_0 , if whenever $y \in Y_0$ is generic over the relevant parameters, then $f_1(y) = f_2(y)$, equivalently, by the above, $f_1 = f_2$ on some large definable subset of Y_0 .

We let $\sim$ be the equivalence relation of having the same germ. Then $\sim$ is definable over k for uniformly definable over k families of functions. Namely if $f(x, z)$ is a partial function definable over k such that for all c , $f(-, c)$ satisfies (i) (and (ii)) above then the equivalence relation $E(z_1, z_2)$: $f(-, z_1) \sim f(-, z_2)$ is k -definable, so the collection of germs of the $f(-, c)$ as c varies is a k -definable set in $\mathcal{U}^{eq}$ (which is the same as $\mathcal{U}$, as RCF eliminates imaginaries). We will consider (partial) definable functions of the form $F_a(-) = F(a, -)$ for $a \in Y_0$. We show that they are “generically invertible”, the set of germs of definable functions $Y_0 \rightarrow Y_0$

generated by the set of F_a is a group which is generated in two steps (and even better). This will be the sought after definable group.

We will work *over* k .

Lemma 4.3. —

- (i) I is a bijection from Y_0 to itself, and $I \circ I$ is the identity on Y_0 .
- (ii) For every $a \in Y_0$, there is a large subset U_a of Y_0 such that F_a is a bijection between U_a and its image $F(U_a)$ in Y_0 which is also large in Y_0 . So we can compose any F_a and F_b to get another function with the same properties.
- (iii) For every $a \in Y_0$, a is determined by the germ of F_a .
- (iv) For every generic $a \in Y_0$ and any $x, y \in Y_0$, if $F_a \circ F_x$ and $F_a \circ F_y$ have the same germ then $x = y$.
- (v) If $x \in Y_0$ and a is generic over x , then $F_x \circ F_a$ has the same germ as F_b for some (generic) $b \in \text{dcl}(x, a) \cap Y_0$. Likewise $F_a \circ F_x$ has the same germ as F_b for some (generic) $b \in \text{dcl}(x, a) \cap Y_0$.
- (vi) The set of germs of $F_a \circ F_b$, for $a, b \in Y_0$, is a group, under composition of germs, where moreover the inverse of the germ of $F_a \circ F_b$ is the germ of $F_{I(b)} \circ F_{I(a)}$.

Proof. —

(i). — It is immediate from Lemma 4.2(ii).

(ii). — Fix $a \in Y_0$. Let U_a be $\{y \in Y_0 : F_a(y) \text{ is defined and in } Y_0, \text{ and } F_{I(a)}(F_a(y)) \text{ is defined and equals } y\}$. Then U_a is definable (over a). Suppose $y \in Y_0$ is generic over a , so by Lemma 4.2(ii), $F_a(y)$ is defined and $F_{I(a)}(F_a(y)) = y$ and in particular $F_a(y)$ is also generic over a so in Y_0 . Hence if $y \in Y_0$ is generic over a , then $y \in U_a$. It follows that U_a is large in Y_0 .

Note that it follows that $F_a|_{U_a}$ is a bijection between U_a and its image, which by definition is also a subset of Y_0 .

It remains to show that the image $F_a(U_a)$ is large. But as in the first part of the proof of (ii) (and the fact that $I(a) \in Y_0$) $U_{I(a)}$ is large. But $U_{I(a)}$ is contained in $F_a(U_a)$ so the latter is large too.

So as stated in the lemma, given $a, b \in Y_0$, the composition $F_a \circ F_b$ will give a bijection between two large subsets of Y_0 .

(iii). — Let $a, b \in Y_0$ and suppose that $F_a \sim F_b$. Choose $c \in Y_0$ generic over a, b . Then $F_a(c) = F_b(c) = d$ say. By Lemma 4.2(iii), $F_d(I(c))$ is defined and equals a , and also equals b , so $a = b$.

(iv). — Suppose that $F_a \circ F_x \sim F_a \circ F_y$. Let z (in Y_0 of course) be generic over a, x, y . So $F_a(F(x, z)) = F_a(F(y, z))$. By part (ii) $F(x, z) = F(y, z)$. By Lemma 4.2(iii), $F(F(x, z), I(z)) = x$ and $F(F(y, z), I(z)) = y$, so $x = y$.

(v). — First consider $F_x \circ F_a$. Let y be generic over x, a , then $(F_x \circ F_a)(y) = F(x, F(a, y)) = F(F(x, a), y)$ by Lemma 4.2(i) and properties of dimension. But $F(x, a) = b$ is generic (over x too) and clearly y is generic over x, a, b so $F_x \circ F_a$ has the same germ as F_b .

Now consider $F_a \circ F_x$. By the first part of our proof of (v), $F_{I(x)} \circ F_{I(a)}$ has the same germ as F_c for some generic c . It suffices to prove:

Claim. — $F_a \circ F_x$ has the same germ as $F_{I(c)}$.

Choose y generic over x, a, c . Let $F(x, y) = w$ and $F(a, w) = z$. So $(F_a \circ F_x)(y) = z$. Then w, z are each generic over x, a, c , and $F_{I(a)}(z) = w$ and $F_{I(x)}(w) = y$, whereby $(F_{I(a)} \circ F_{I(x)})(z) = y$, hence $F_c(z) = y$, whereby $F_{I(c)}(y) = z$, proving the Claim.

(vi). — The fact that the inverse of the germ of $F_a \circ F_b$ is the germ of $F_{I(b)} \circ F_{I(a)}$ is proved as in the last part of the proof of (v), namely that for generic y over a, b , and $z = (F_a \circ F_b)(y)$, z is also generic over a, b and moreover $(F_{I(b)} \circ F_{I(a)})(z) = y$.

In order to prove that $\{F_a \circ F_b : a, b \in Y_0\}$ is a group it suffices to prove that for $a, b, c, d \in Y_0$, $F_a \circ F_b \circ F_c \circ F_d$ has the same germ as $F_x \circ F_y$ for some $x, y \in Y_0$.

Let $e \in Y_0$ be generic over a, b, c, d , and consider $F_a \circ F_b \circ F_e \circ F_{I(e)} \circ F_c \circ F_d$, which has the same germ as $F_a \circ F_b \circ F_c \circ F_d$.

Iterating (v) we see that $F_a \circ F_b \circ F_e$ has the same germ as some F_x (where x is generic), and likewise $F_{I(e)} \circ F_c \circ F_d$ has the same germ as some F_y . So putting it together, $F_a \circ F_b \circ F_c \circ F_d$ has the same germ as $F_x \circ F_y$. $\square$

From (vi) above we have constructed a group interpretable in RCF (over k) whose domain is the set of germs of $F_a \circ F_b$, $a, b \in Y_0$. The domain is the quotient of Y_0^2 by a k -definable equivalence relation E . We call this group G .

As RCF has elimination of imaginaries, this group is definably isomorphic over k to one whose underlying set is a definable subset of $\mathcal{U}^m$ some m . However, it will be convenient, also for other contexts where one does not have elimination of imaginaries, to have a finer statement which is what we do now.

Lemma 4.4. — *Let $N = \dim(Y_0)$, and let $\alpha > 2N$. Let $t_1, \dots, t_\alpha$ be independent generic elements of Y_0 . Then for any $a, b \in Y_0$, there is $d \in Y_0$ and $t \in \{t_1, \dots, t_\alpha\}$ such that $F_a \circ F_b$ has the same germ as $F_t \circ F_d$. Moreover d is determined by t and the germ of $F_t \circ F_d$.*

Proof. — Given a, b we can find, by dimension considerations, some $t \in \{t_1, \dots, t_\alpha\}$ such that t is generic over a, b . Consider $F_t \circ F_{I(t)} \circ F_a \circ F_b$ which has the same germ as $F_a \circ F_b$. But as in the proof of (vi) of the previous Lemma, $F_{I(t)} \circ F_a \circ F_b$ has the same germ as F_d for some (generic) $d \in Y_0$. Hence $F_a \circ F_b$ has the same germ as $F_t \circ F_d$.

The last part of the statement is Lemma 4.3(iv). $\square$

We can now write G naturally as a definable (rather than interpretable) group, as follows. By Lemma 4.4 we can think of G as covered by the charts $\{t_i\} \times Y_0$ (with (t_i, a) sent to the germ of $F_{t_i} \circ F_a$). In order to write this as a disjoint union, we remove from $\{t_2\} \times Y_0$ all germs which already appear in $\{t_1\} \times Y_0$, and then remove from $\{t_3\} \times Y_0$ all germs which already appear previously and so forth. This gives us our group living naturally on a collection of tuples of a given length.

On the face of it, the *definable* group G is defined over k together with parameters $t_1, \dots, t_\alpha$. However as the real closure of k is an elementary substructure of $\mathcal{U}$ and is also equal to the definable closure of k in $\mathcal{U}$ we can choose $t_1, \dots, t_\alpha$ in $\text{dcl}_L(k)$.

The end result is:

Lemma 4.5. — *The group of germs of $F_a \circ F_b$ for $a, b \in Y_0$ is an interpretable group G which is the same as the set of germs of $F_{t_i} \circ F_b$, $i = 1, \dots, \alpha$, $b \in Y_0$ which can be identified naturally as above with a definable (over $\text{dcl}_L(k)$) group.*

Remark 4.6. — We know ([22]) that a group definable in RCF has definably the structure of a Nash group. In the paper [20], we use this to get a Nash group structure on G , compatible with the way in which G originates from the L_∂ -definable group Γ .

We now return to consideration of the group Γ , L_∂ -definable in $(\mathcal{U}, \partial)$. So we revert to our notation distinguishing the different notions of generic etc.

The proof of Theorem 4.1 is completed by the following, where we again work over k , so suppress k from the notation.

Lemma 4.7. — *There is an L_∂ -definable group embedding h of Γ into G .*

Proof. — Let $\gamma \in \Gamma$. Choose $a \in \Gamma$ ∂ -generic over γ , and let $b = a^{-1}\gamma$ in Γ . So $\gamma = ab$ in Γ . Note that b is also ∂ -generic in Γ . Hence by Lemma 3.4 both a and b are L -generic in Y , so in Y_0 . Define $h(\gamma)$ to be the germ of $F_a \circ F_b \in G$. We have to check several things:

- (i) h is well-defined, i.e. $h(\gamma)$ does not depend on the choice of a above,
- (ii) h is ∂ -definable,
- (iii) h is a group homomorphism, and
- (iv) h is injective.

Proof of (i). — Suppose γ also equals cd where c, d are ∂ -generic in Γ , so L -generic in Y_0 . We want to show that $F_a \circ F_b$ and $F_c \circ F_d$ have the same germ. Choose $x \in Y_0$ L -generic over a, b, c, d . By Lemma 2.7 we may assume that $x \in \Gamma$. Hence working in Γ , $abx = cd x$. But for generic independent (in either sense, by Lemma 3.4) elements of Γ , multiplication in Γ is given by $*$, by Proposition 3.3(d). As b, x are independent generic elements of Γ , as well as a, bx, c, dx , it follows that $a * (b * x) = c * (d * x)$. But this means precisely that $F_a \circ F_b$ and $F_c \circ F_d$ agree at x . We have shown that $F_a \circ F_b \sim F_c \circ F_d$ as required. (We could alternatively have made use of the fact (Proposition 3.3) that $F(a, b)$ coincides with the group operation in Γ , together with associativity (Lemma 4.2).)

Proof of (ii). — It is enough, by compactness, to show that the graph of h is ∂ -type-definable. By the definition of h and part (i), we have, for $\gamma \in \Gamma$ and $g \in G$, $h(\gamma) = g$ if and only if for some $a \in \Gamma$ which is ∂ -generic and ∂ -independent from γ , g is the germ of $F_a \circ F_{a^{-1}\gamma}$. This suffices.

Proof of (iii). — Suppose that $\gamma = ab$ and $\delta = cd$ with a, b, c, d ∂ -generic in Γ . So $h(\gamma)$ is the germ of $F_a \circ F_b$ and $h(\delta)$ is the germ of $F_c \circ F_d$. We have to show that $h(\gamma\delta)$ is the germ of $F_a \circ F_b \circ F_c \circ F_d$. As in the proof of Lemma 4.3(v), let $f \in \Gamma$ be ∂ -generic and ∂ -independent of a, b, c, d . Then $\gamma\delta = (abf)(f^{-1}cd)$ (multiplication in Γ). Let $x = abf$ and $y = f^{-1}cd$. Then

x, y are ∂ -generic in Γ and $\gamma\delta = xy$. Finally one has to check that the germ of F_x is equal to the germ of $F_a \circ F_b \circ F_f$ and the germ of F_y equals the germ of $F_{f^{-1}} \circ F_c \circ F_d$, which is like in the proof of (i). So h is a homomorphism.

Proof of (iv). — Given (i), (ii), (iii), we have to prove that if $h(\gamma)$ is the identity then γ is the identity. Suppose that $\gamma = ab$ as before and $F_a \circ F_b$ is the identity. So $F_a \circ F_b$ has the same germ as $F_a \circ F_{a^{-1}}$. By Lemma (4.3) (iv), $b = a^{-1}$, so $\gamma = aa^{-1}$ is the identity of Γ . $\square$

Remark 4.8. — The embedding h of Γ in G from Lemma 4.7 also satisfies:

- (i) any L -generic type of G over k is realized by some element of (the image under h of) Γ .
- (ii) There is a covering of G by finitely many L_k -definable subsets X_i , and there are suitable L_k definable functions s_i on X_i , such that for any i , if $a \in X_i$ is L -generic in G over k then a is in (the image under h of) Γ iff $\partial(a) = s_i(a)$.

Proof. — Everything can be seen by inspecting the proofs above. However some precisions are carried out in the more general Theorem 5.4 in the next section. Anyway, with notation from the current (and previous) section, from Lemma 4.3 (iii) we have an injective map h_1 from Y_0 to G taking $a \in Y_0$ to the germ of F_a . We leave the reader to check that for $a \in \Gamma \cap Y_0$, $h_1(a) = h(a)$. By Theorem 5.4 (1) (and translating) $h_1(Y_0)$ is large in G , so $a \in Y_0$ is L -generic over k iff $h_1(a) \in G$ is L -generic over k . Also note that Y_0 is large in Y so contains all generics of Y . Now Proposition 3.3 and Lemma 3.4 give a covering of Y_0 by the V_i , and for $a \in V_i \cap Y_0$, $a \in \Gamma$ iff $\nabla(a) = s_i(a)$. Moreover any L -generic over k element of Y_0 has L -type realized by a (∂ -generic over k) element of Γ . Applying the injective map h_1 to Y_0 gives (i) and (ii) of the Remark. $\square$

5. Geometric structures and fields

In this section we present some straightforward generalizations of the methods and results from the earlier sections.

5.1. Geometric structures and generically defined groups. — We are here interested in generalizing that aspect of Section 4 which produces a semialgebraic group from the data (a semialgebraic pre-group) in Proposition 3.5. There are various kinds of generalizations, including just axiomatizing the proof. But a fairly clearly stated and useful generalization takes place in the context of geometric structures from [15]. We recall the definition and give a quick summary of properties from Section 2 of [15].

Definition 5.1. — A geometric structure is a structure M in a language L in which algebraic closure yields a pregeometry in any model of $\text{Th}(M)$ and which has elimination of the $\exists^\infty$ quantifier, namely for each formula $\phi(x, \bar{y})$ there is N_ϕ such that for any $\bar{b}$ in M , $\phi(x, \bar{b})(M)$ is infinite if it has cardinality $> N_\phi$.

Recently Johnson and Ye showed that if M is a pre-geometric structure, namely the algebraic closure has exchange, and M is a field then M is geometric, namely it eliminates $\exists^\infty$ (and it is perfect) [17, Theorem 5].

Geometric structures include strongly minimal structures, o-minimal structures, structures of SU -rank 1, as well as p -adically closed fields. Moreover in the case of real closed fields,

or p -adically closed fields, (model-theoretic) algebraic closure coincides with (relative) field-theoretic algebraic closure.

Let us now fix a saturated geometric structure M , where as usual A, B denote small subsets and $a, b, \dots$ will now denote finite tuples of elements of M .

We define $\dim(a/A)$ to be the (unique) cardinality of a maximal algebraically independent over A subtuple of a . Then a and b are said to be independent over A if $\dim(a, b/A) = \dim(a/A) + \dim(b/A)$. For $X \subseteq M^n$ definable (or even type-definable) over A , $\dim(X) = \max\{\dim(a/A) : a \in X\}$. $a \in X$ is said to be *generic over A* in X if $\dim(X) = \dim(a/A)$. Note that for p a complete n type over A , $\dim(p) = \dim(a/A)$ for some/any realization a of p . For tuples a, b , we have

$$\dim(a, b/A) = \dim(a/bA) + \dim(b/A).$$

Elimination of the $\exists^\infty$ quantifier implies that for any L -formula $\phi(x, y)$ and m , $\{b : \dim(\phi(x, b)) = m\}$ is definable over $\emptyset$ in M .

In the case of M being a real closed field, $\dim$ and independence coincide with what was described in Section 2.

Definition 5.2. — By a *generically defined group* (or a pre-group in M) we mean a definable set $X \subseteq M$ (defined over A say) together with A -definable partial operations $F : X \times X \rightarrow X$, and $I : X \rightarrow X$ such that:

- (1) For any $a, b \in X$, generic and independent over A , $c = F(a, b)$ is defined, and each of a, b, c is in the definable closure of A and the other two.
- (2) For $a, b, c \in X$, generic and independent over A , $F(F(a, b), c) = F(a, F(b, c))$.
- (3) For $a \in X$ generic over A , $I(a)$ is defined and generic in X over A . Moreover, if b is also generic in X over A and independent of a over A , we have $b = F(I(a), F(a, b))$, and $a = F(F(a, b), I(b))$.

Remark 5.3. —

- (i) We could replace (3) in the definition above, by
 - (3') there are A -definable partial functions μ, λ from $X \times X$ to X , which satisfy (1), and if $a, b, c \in X$ are generic and pairwise independent over A , then $F(a, b) = c$ iff $\mu(a, c) = b$ iff $\lambda(b, c) = a$.
- (ii) When X is an irreducible algebraic variety, and the ambient structure M is an algebraically closed field (of characteristic 0 say), then parts (1) and (2) of the definition above are precisely the definition of a “pre-group equipped with a normal law of composition f ” as in [30].

Explanation. —

- (i) (3') is easily a consequence of (1), (2), (3). And (3) follows from (1), (2), (3') by copying the proof of Proposition 1 in Weil's [30].
- (ii) In this context of ACF_0 for each n there is a unique n -type of n generic, independent elements of X . Then one deduces (3') as in the proof of Proposition 1 of [30]. $\square$

Theorem 5.4. — *Let (X, F, I) be a generically defined group in M , as in Definition 5.2. Then there is a group $(G, \cdot)$ definable in M over a set $B \supseteq A$, with $\dim(G) = \dim(X)$, and a partial B -definable function $h : X \rightarrow G$ such that:*

- (1) *h establishes a bijection between generic over B points of X and generic over B points of G , so in particular $h(X)$ is large in G ,*
- (2) *For $a, b \in X$ generic and independent over B , $h(F(a, b)) = h(a) \cdot h(b)$.*
- (3) *For $a \in X$ generic over B , $h(I(a)) = h(a)^{-1}$.*

Moreover when A is an elementary substructure of M we can choose $B = A$.

Proof. — This is given by the proofs in Section 4 which work under the more general assumptions. Here are some details. After changing some symbols, the definition of (X, F, I) being a generically defined group (over A), gives the statements in Proposition 3.5 above.

Our notation is now almost the same as in Section 4, except we have X in place of Y and A in place of the field k (over which everything is defined). By the definability properties of geometric structures, Lemma 4.2 goes through giving the suitable large subset X_0 of X defined over A (the “group chunk”).

Now we consider definable (with additional parameters) functions from X_0 to X_0 and their germs, where recall that f_1 and f_2 have the same germ if whenever $a \in X_0$ is generic over A together with the parameters in f_1 and f_2 , $f_1(a) = f_2(a)$. Equivalently there is a large definable subset of X_0 on which f_1 and f_2 agree.

Then Lemma 4.3 goes through, giving the interpretable group G_0 as the group of germs of $F_a \circ F_b$ for $a, b \in X_0$. Now Lemmas 4.4 and 4.5 go through to find elements $t_1, \dots, t_\alpha$ of X_0 such that taking $B = A \cup \{t_1, \dots, t_\alpha\}$, G_0 is definably over B isomorphic with a definable (over B) group G . As in the comments before Lemma 4.5, when A is an elementary substructure of M , the t_i can be found in A already. As G is covered by finitely many copies of X_0 , and X_0 is large in X we already see that $\dim(G) = \dim(X)$.

We now choose h . There is no harm assuming $A = B$. Let h be the A -definable function on X_0 taking $a \in X_0$ to the germ of F_a . So by (the general version of) Lemma 4.3(iii), h is an injective A -definable function from X_0 to G . (So also h is a partial A -definable function from X to G defined on the generic points of X .) As $\dim(X) = \dim(G)$, we see immediately that for $a \in X_0$ generic over A , $h(a)$ is generic in G over A .

Conversely, to finish the proof of (1) we want to check that every generic over A element of G is the germ of F_a for some generic over A element a of X_0 . From what we have just seen, $\dim(X) = \dim(G)$. Let g be generic over A in G . So we can write g as the germ of $F_u \circ F_v$ for $u, v \in X_0$. Let $b \in X$ be generic over A, u, v . Then g times the germ of F_b is the germ of F_a for some generic $a \in X_0$ over A by applying Lemma 4.3(v) twice. But then a dimension count shows that a and b are independent over A , as are a and $I(b)$. Then the germ of F_a times the germ of $F_{I(b)}$ equals g , but also equals the germ of F_c for c generic in X over A by Lemma 4.3(v). We have shown that g is the germ of F_c for c generic in X over A . This completes the proof of (1).

For (2), if a, b are generic and independent over A and are in X_0 then $F(a, b)$ is generic over A so in X_0 . $h(a)$ is the germ of F_a , $h(b)$ is the germ of F_b , and, since $F(a, F(b, c)) = F(F(a, b), c)$,

for c generic and independent over a, b, A , then the germ of $F_a \circ F_b$ is the germ of $F(a, b)$, giving (2).

(3) is immediate. $\square$

5.2. Geometric fields with a generic derivation. — We will now place the main results for finite-dimensional groups definable in RCF_∂ in a certain general context which subsumes other cases of interest.

We are concerned first with certain theories of fields in the ring language L possibly with additional constant symbols. Then we will add a “generic derivation” and work in L_∂ .

We will repeat some definitions from [15] and [4]. The notion of a “geometric field” was introduced in [15] in terms of “geometric substructures” of algebraically closed fields. But in Remark 2.10 of [15] the following characterization was given which we may take as a definition.

Definition 5.5. — A geometric field is a perfect field F , considered as an L -structure, where L is the language of rings with possibly additional constant symbols, such that in any model of $\text{Th}(F)$, model-theoretic algebraic closure coincides with field-theoretic (relative) algebraic closure, and for any L -formula $\phi(x, \bar{y})$ (x a single variable) there is N_ϕ such that in any model F' of $\text{Th}(F)$ and for any $\bar{b}$ in F' , $\phi(x, \bar{b})$ has finitely many solutions in F' iff it has at most N_ϕ solutions.

Note that a geometric field K is also a geometric structure. Moreover if $\bar{a}$ is an n -tuple from K and k a subfield, relatively algebraically closed in K if one wishes, then $\dim(\bar{a}/k) = \dim(V)$ where V is the variety over k with generic over k point $\bar{a}$.

We will restrict our attention to characteristic 0. We recall the strengthening of model-completeness from [4]:

Definition 5.6. —

- (i) Let T be a theory of fields in the language L of rings. We say that T has almost quantifier elimination (almost QE) if whenever F is a model of T , A is a relatively algebraically closed subset of F (in the field-theoretic sense) and $\bar{a}$ is an enumeration of A , then $T \cup \text{qftp}(\bar{a})$ implies a complete type $p(\bar{x})$ (over $\emptyset$) of T .
- (ii) Let T' be a theory of differential fields in the language L_∂ . We say that T' has almost QE, if whenever (K, ∂) is a model of T' , and k is a differential subfield of K which is relatively algebraically closed in K in the sense of fields, and $\bar{a}$ is an enumeration of k then $T' \cup \{\text{qftp}(\bar{a})\}$ implies a complete L_∂ -type (over $\emptyset$).

Note that model completeness of T is the special case of Definition 5.6(i) when $A = F$.

We have already defined large fields. Here is a summary of relevant results from [4] and [29].

Fact 5.7. — *Let T be a model complete theory of large fields in the language K . Then:*

- (i) $T \cup \{\partial \text{ is a derivation}\}$ has a model companion T_∂ in the language L_∂ .
- (ii) (The geometric axioms.) $(K, \partial) \models T_\partial$ iff $K \models T$, and whenever (V, s) is an irreducible rational D -variety over K with a smooth K -point, and U is a Zariski open subset of V defined over K then there is $a \in U(K)$ with $s(a) = \nabla(a)$.

- (iii) If T has quantifier elimination in a definitional expansion L^* , then T_∂ has quantifier elimination in L_∂^* .
- (iv) If T has almost QE, then so does T_∂ .

Explanation. — Everything can be extracted from Tressl's paper [29, 7.1 and 7.2]. But (ii) and (iv) are also proved in [4]. (See also Remark 2.4 in [4].)

We now generalize Lemma 2.7, even though in the proof of 2.7 we made use of RCF_∂ having QE in the ordered differential field language.

Lemma 5.8. — *Let T be a model-complete theory of large geometric fields. Let T_∂ be the model companion of $T \cup \{\partial \text{ is a derivation}\}$. Let $(K, \partial) \models T_\partial$ and (V, s) be a (irreducible) rational D -variety over K such that V has a Zariski-dense set of K -points. Let X be a definable (in K) subset of $V(K)$ with $\dim_L(X) = \dim(V) (= \dim_L(V(K)))$. Then there is some $a \in X$ with $s(a) = \nabla(a)$.*

Proof. — The proof is like that of Lemma 2.7, with the proviso mentioned earlier. Let F be a saturated elementary extension of the field K in the language L . Let $p(x)$ be a complete type over K containing the formula $x \in X$, and with $\dim(p) = \dim(X)$. Let a realize $p(x)$ in F . So a is a generic point of V over K . Let $s(a) = (a, b) \in F$, and as in the proof of Lemma 2.7 we can extend the derivation ∂ to a derivation ∂^* of F such that $\partial^*(a) = b$. So (F, ∂^*) is a model of T together with ∂^* is a derivation, so embeds in a model of T_∂ which by model-completeness is an elementary extension of (K, ∂) . This suffices. $\square$

Lemma 5.9. — *Let T be a model-complete theory of large geometric fields. Then in any model (K, ∂) of T_∂ , the model-theoretic algebraic closure of a set A equals the field-theoretic algebraic closure of the differential subfield of K generated by A .*

Proof. — It is enough to prove the following: Suppose (K, ∂) is model of T_∂ , k is a differential subfield of K , K is $|k|^+$ -saturated and $a \in K$ is not in the field-theoretic algebraic closure of k . Then $tp(a/k)$ in (K, ∂) has infinitely many realizations in K . This is a routine consequence of the previous lemma and the fact that $tp_{L_\partial}(a/k)$ is determined by $tp_L(a, \partial(a), \partial^{(2)}(a), \dots)/k$ (by Fact 5.7(iii)). $\square$

Now we bring in the almost QE hypothesis.

Proposition 5.10. — *Assume that T is a theory of large geometric fields which has almost QE. Let $(K, \partial) \models T_\partial$ and let k be a differential subfield of K . Then $\text{dcl}_{L_\partial}(k)$ is the same as $\text{dcl}_L(k)$.*

Proof. — Let $a \in \text{dcl}_{L_\partial}(k)$. By the previous lemma we know that a is in the field-theoretic algebraic closure of k . Hence it suffices to prove:

Claim. — Suppose a is in the field-theoretic algebraic closure of k , then $tp_L(a/k)$ implies $tp_{L_\partial}(a/k)$.

Proof of Claim. — Suppose $tp_L(b/k) = tp_L(a/k)$. So assuming (as we may) that K is $|k|^+$ -saturated, there is a field automorphism σ of K fixing k and taking a to b . Let k' be the algebraic closure of k in K (which by Lemma 5.9, equals the relative field-theoretic algebraic closure of k in K). So σ restricts to a permutation of k' over k taking a to b . Thus $\sigma|_{k'}$

preserves quantifier-free L -types over k . But ∂ has a unique extension to a derivation of k' , whereby $\sigma|_{k'}$ also preserves quantifier-free types over k in L_∂ . By Fact 5.7 (iv), a and b have the same type over k in (K, ∂) , proving the Claim and the Proposition. $\square$

Fix a model (K, ∂) of T_∂ , where T is a theory of large geometric fields with almost QE. There is no harm assuming (K, ∂) to be saturated. All of Section 2 goes through for T in place of RCF , using Fact 5.7, Lemmas 5.8 and 5.9, and Proposition 5.10. In particular, using Fact 5.7 (iii), any L_∂ -formula $\phi(\bar{x})$ is equivalent, modulo T_∂ to one of the form $\theta(\bar{x}, \partial(\bar{x}), \dots)$ where θ is an L -formula. Finite-dimensional sets definable in (K, ∂) are defined as in Definition 2.2. The various symbols for dimension, generics, independence, in T and T_∂ are as in Section 2.

Proposition 5.10 allows the proofs in Section 4 to go through, yielding:

Theorem 5.11. — *Let (K, ∂) be a model of T_∂ where T is a not necessarily complete theory of large geometric fields with almost QE. Then any finite-dimensional group Γ definable in (K, ∂) definably embeds in a group definable in the field K (i.e. in the ring language L). Moreover Remark 4.8 also holds.*

Proof. — We will again give just a brief guide to the proof, as the proof of Theorem 4.1 just goes through, given the comments in the previous two paragraphs. Let Γ be a finite-dimensional group definable in (K, ∂) over the small differential subfield k , which may be assumed to be an elementary substructure.

First Proposition 3.3 goes through for Γ : after replacing Γ by $\nabla^{(N)}(\Gamma)$ for suitable N , we obtain a set Y L -definable over k , satisfying (I), and (II) there. For (II) Proposition 5.10 is needed. Then Lemma 3.4 goes through with the same proof, using Lemma 5.8 to see that any L -generic type of Y can be realized by a realization of some ∂ -generic type of Γ . Proposition 3.5 follows as before. Now repeat Section 4 to get an L -definable over k group G into which Γ -definably embeds. Alternatively, $(Y, *, \text{inv})$ is a generically defined group over k in the L -structure K in the sense of Definition 5.2, so Theorem 5.4 applies to produce the L_k -definable group G . The proof of Lemma 4.7 gives a L_∂ -definable over k embedding of Γ in G .

Remark 4.8 goes through too by the same proof as at the end of Section 4. $\square$

Some examples of theories satisfying the assumptions of the last proposition are (the theories of) bounded pseudo-algebraically closed fields of characteristic 0 (including pseudo-finite fields of characteristic 0) [14], and (the theory of) $\mathbb{Q}_p$.

6. Other variants

We consider two other general contexts which go outside that of Section 5.2.

6.1. The Fornasiero–Kaplan setting: $\mathcal{o}$ -minimal structures with a generic derivation. — Here we discuss $\mathcal{o}$ -minimal expansions of real closed fields with a generic derivation, as appearing in [10].

Formally T is taken to be a model-complete $\mathcal{o}$ -minimal expansion of RCF . $T \cup \{\partial \text{ is a “compatible” derivation}\}$ is shown to have a model companion T_∂ . Compatibility will be defined below, but when the language is just that of rings, any derivation is compatible. So for $T = RCF$, we obtain RCF_∂ . We take now L to be the language of T (which is in general an expansion of the language of rings) and L_∂ to be $L \cup \{\partial\}$.

We will again prove that finite-dimensional groups definable in models (K, ∂) of T_∂ are definably embeddable in groups definable in the language L . The proof will be like the earlier ones except for the first steps where cell-decomposition in models of T replaces algebraic varieties and their “prolongations”. More accurately the only algebraic varieties that need to be considered are the affine spaces K^n . These o -minimal methods could also have been used in the RCF_∂ case but we chose instead to use techniques which could be generalized as we did in Section 5.2.

We will give a brief account of the theory from [10], but with notation consistent with the current paper. Then we sketch again the proof of the desired result.

For concrete applications T can be an arbitrary model-complete (and complete) o -minimal theory in a language L expanding RCF . We now assume that a symbol for the ordering $<$ is in the language L . For the kind of general results we have in mind there is no harm in assuming that T has quantifier elimination (and is universally axiomatized). For models M of T we will freely use basic notions from o -minimality. “ L -algebraic” now means in the sense of the o -minimal structure M .

The paper [10] defines a derivation ∂ of the underlying field M to be *compatible* with the o -minimal structure on M if for every n , open definable subset U of M^n , and C^1 -function $f : U \rightarrow M$ which is $\emptyset$ -definable in M , we have that for all $\bar{a} \in U$, $\partial(f(\bar{a})) = \sum_{i=1, \dots, n} (\partial f / \partial x_i)(\bar{a}) \partial(a_i)$. We assume now that ∂ is compatible with M (so with any model of T).

Note that compatibility follows from the definition of a derivation in the case that f is a polynomial over the constants of M . We will discuss later the case when f is a C^1 -definable function with parameters.

From Section 4 of [10] we have:

Fact 6.1. —

- (1) $T \cup \{\partial \text{ is a compatible derivation}\}$ has a model companion T_∂ which is complete, and moreover has quantifier elimination in L_∂ assuming T does.
- (2) T_∂ can be axiomatized (modulo $T \cup \{\partial \text{ is a compatible derivation}\}$) by the following: Suppose $X \subseteq M^{2n}$ is L -definable (with parameters) and its projection to M^n has $\dim = n$ (i.e. contains an open set) then there is $\bar{a} \in M^n$ such that $(\bar{a}, \partial(\bar{a})) \in X$.
- (3) For any L_∂ -formula $\phi(\bar{x})$ there is an L -formula $\theta(\bar{x}_0, \dots, \bar{x}_N)$ (with $\text{length}(\bar{x}) = \text{length}(\bar{x}_0)$) such that for any $(M, \partial) \models T_\partial$, and $\bar{a}$ from M , we have $(M, \partial) \models \phi(\bar{a}) \leftrightarrow \theta(\bar{a}, \dots, \partial^{(N)}(\bar{a}))$.

Given $(M, \partial) \models T_\partial$, we let dcl_{L_∂} denote definable closure in the sense of the structure (M, ∂) and dcl_L for definable closure in the sense of the o -minimal structure $M \models T$.

From Corollary 5.14 of [10] we have:

Fact 6.2. — Let $A \subset M$ where $(M, \partial) \models T_\partial$. Then $\text{dcl}_{L_\partial}(A) = \text{dcl}_L(\{\partial^{(n)}(a) : a \in A : n = 0, 1, 2, \dots\})$. In other words $\text{dcl}_{L_\partial}(A)$ is the definable closure in the L -structure M of the differential field generated by A .

We now work in a saturated model (M, ∂) of T_∂ . k typically denotes a (small) differential subfield of (M, ∂) . We will use $\dim_L$, L -generic etc. to mean in the sense of o -minimal structure M .

As in the earlier sections we will use the notation $\nabla^{(N)}(\bar{a})$ for $(\bar{a}, \partial\bar{a}, \dots, \partial^{(N)}(\bar{a}))$. Similarly for X a definable set in (M, ∂) , $\nabla^{(N)}(X) = \{\nabla^{(N)}(\bar{a}) : \bar{a} \in X\}$.

Definition 6.3. — By $\dim_{\partial}(\bar{a}/k)$ we mean ∞ or $\max\{\dim(\nabla^{(N)}(\bar{a})/k) : N = 0, 1, \dots\}$ if there is a maximum.

Note that $\dim_{\partial}(\bar{a}/k)$ is an invariant of $\text{dcl}_{L_{\partial}}(k(\bar{a}))$ (and k). So if $\bar{a}$ and $\bar{b}$ are L_{∂} -interdefinable over k then $\dim_{\partial}(\bar{a}/k) = \dim_{\partial}(\bar{b}/k)$.

As in the earlier sections we will define an $(L_{\partial})_k$ definable set to be *finite-dimensional* if there is a finite bound on $\dim_{\partial}(\bar{a}/k)$ for $\bar{a} \in X$ and will define $\dim_{\partial}(X)$ to be this finite bound.

Definition 6.4. — For X an $(L_{\partial})_k$ -definable set of finite-dimension, and $\bar{a} \in X$ we say that $\bar{a}$ is ∂ -generic in X over k , or $tp_{L_{\partial}}(\bar{a}/k)$ is a ∂ -generic type of X , if $\dim_{\partial}(\bar{a}/k) = \dim_{\partial}(X)$.

We have to discuss the application of ∂ to elements of the form $f(\bar{a})$ where f is an L -definable *with parameters* function from some open subset U of M^n to M . First we can write $f(\bar{x})$ as $g(\bar{x}, \bar{b})$ where $g(\bar{x}, \bar{y})$ is some L -definable over $\emptyset$ function from an open subset of M^{n+m} to M and $\bar{b} = (b_1, \dots, b_m)$ consists of independent nonalgebraic over $\emptyset$ elements of M . With this notation and using the compatibility of ∂ , we have:

Remark 6.5. — With the conventions above, for $\bar{a} \in U$,

$$\partial(f(\bar{a})) = \sum_{i=1, \dots, n} (\partial f / \partial x_i)(\bar{a}) \partial(a_i) + f^{\partial}(\bar{a}),$$

where $f^{\partial}(\bar{a})$ is $\sum_{j=1, \dots, m} (\partial g / \partial y_j)(\bar{a}, \bar{b}) \partial(b_j)$.

Corollary 6.6. — Suppose $\partial^{(n+1)}(\bar{a}) \in \text{dcl}_L(k(\nabla^{(n)}(\bar{a})))$. Then, for all m , $\partial^{(m)}(\bar{a}) \in \text{dcl}_L(k(\nabla^{(n)}(\bar{a})))$.

Finally we need the analogue of Lemma 2.7:

Lemma 6.7. — Let $p(\bar{x})$ be an L -generic type of M^n over k (which just means that $p = tp_L(\bar{a}/k)$ where $\dim_L(\bar{a}/k) = n$), and let $s(\bar{x})$ be a partial function from K^n to K^n definable in L over k and defined at $\bar{a}$. Then there is a realization $\bar{b}$ of p such that $\partial(\bar{b}) = s(\bar{b})$.

Proof. — Let $U \subseteq K^n$ be open and L -definable over k such that $\bar{a} \in U$ and s is defined and C^1 on U . By Fact 6.1 (2), for every formula $\phi(\bar{x}) \in p(\bar{x})$ there is a realization $\bar{b}$ of ϕ such that $\partial(\bar{b}) = s(\bar{b})$. Apply compactness (in the saturated model (M, ∂)). $\square$

We also have of course o -minimal independence in M and corresponding notion of ∂ -independence in (M, ∂) where $\bar{a}$ and $\bar{b}$ are ∂ -independent over A if the L_{∂} -definable closures of $\bar{a}$ and $\bar{b}$ are L -independent over the L_{∂} -definable closure of A . We will use freely the obvious properties of ∂ -independence, as in the RCF_{∂} case.

We can now give the proof of the following, where items (ii) and (iii) are useful for further work.

Theorem 6.8. — Assume T is a model-complete, complete, o -minimal expansion of RCF in language L , and let (M, ∂) be a saturated model of T_{∂} . Suppose that Γ is a finite-dimensional group definable over k in (M, ∂) . Assume k is a (small) elementary substructure of (M, ∂) . Then there is a group G , L -definable over k in M such that

- (i) *there is an L_{∂} -definable (over k) embedding h of Γ into G .*
- (ii) *Every L -generic over k type of G is realized by an element of $h(\Gamma)$.*
- (iii) *There are L -definable over k sets $X_1, \dots, X_r \subseteq G$ L -definable functions $s_i : X_i \rightarrow M^n$ for suitable n , such that $\bigcup_i X_i = G$, and for each L -generic over k element $a \in G$, if $a \in X_i$ then $a \in \Gamma$ iff $s_i(a) = \partial(a)$.*

Proof. — As remarked above, the only change from the proof of Theorem 4.1 concerns some of the preparatory material in Section 3. This approach will also work for RCF_{∂} which will be a useful observation for later work.

Fix the group Γ , which by Fact 6.1(3) is defined by a $(L_{\partial})_k$ -formula of the form $\theta(\bar{x}, \partial(\bar{x}), \dots, \partial^{(N)}(\bar{x}))$ for some L -formula θ over k and some N . By finite-dimensionality of Γ , we may assume that for all $\bar{a}$, $\theta(\bar{a}, \partial(\bar{a}), \dots, \partial^{(N)}(\bar{a}))$ implies $\partial^{(N)}(\bar{a}) \in \text{dcl}_L(k((\nabla^{(N-1)}(\bar{a})))$. By Corollary 6.6 for each $\bar{a} \in \Gamma$, $\partial^{(N+1)}(\bar{a}) \in \text{dcl}_L(\nabla^{(N)}(\bar{a}))$, so equals $f(\nabla^{(N)}(\bar{a}))$ for some L_k -definable partial function f defined at $\nabla^{(N)}(\bar{a})$ so defined on the realizations of some L_k -formula $\psi_{\bar{a}}(x_0, \dots, x_N)$ in $tp_L(\nabla^{(N)}(\bar{a})/k)$. Replace Γ by $\nabla^{(N)}\Gamma$. So the new Γ is defined by $\theta(x_0, \dots, x_N)$ together with $\partial(x_i) = x_{i+1}$ for $i = 0, \dots, N-1$. The new $\bar{x}$ is $(\bar{x}_0, \dots, \bar{x}_N)$.

We let Y be the L -definable over k set defined by $\theta(\bar{x})$.

$\bar{a}$ will now denote a tuple from Γ , so of the form $\bar{a} = (\bar{a}_0, \dots, \bar{a}_N)$. And for $\bar{a}$ in Γ , $\partial(\bar{a}) = (\bar{a}_1, \dots, \bar{a}_N, \partial(\bar{a}_N))$. Moreover by the previous paragraph, for each such $\bar{a} \in \Gamma$, $\partial(\bar{a}) = s_{\bar{a}}(\bar{a})$ for some L -definable over k function $s_{\bar{a}}$ which is defined on some formula $\phi_{\bar{a}}(\bar{x}) \in tp_L(\bar{a}/k)$.

We may assume $\dim_L(\phi_{\bar{a}}(\bar{x}))$ equals $\dim_L(\bar{a}/k)$.

Note that for $\bar{a} \in \Gamma$, $\phi_{\bar{a}}$ and $s_{\bar{a}}$ depend only on $p = tp_{L_{\partial}}(\bar{a}/k)$ so can be written as ϕ_p, s_p . We let P be the collection of $tp_{L_{\partial}}(\bar{a}/k)$, for $\bar{a} \in \Gamma$.

As in the proof of the first part of Proposition 3.3, we have:

Claim 1. — For $p \in P$, and $\bar{b}$ satisfying $\theta(\bar{x}) \wedge \phi_p(\bar{x}) \wedge “\partial(\bar{x}) = s_p(\bar{x})”$, we have that $\bar{b} \in \Gamma$.

By compactness, we conclude,

Claim 2. — There are $p_1, \dots, p_s \in P$, such that

$$\Gamma = \bigcup_{i=1, \dots, s} \left\{ \bar{b} : \models \theta(\bar{b}) \wedge \phi_{p_i}(\bar{b}) \wedge “\partial(\bar{b}) = s_{p_i}(\bar{b})” \right\}.$$

This already gives a kind of analogue of part (I) of Proposition 3.3. The analogue of part (II) follows by the same proof.

Claim 3. — There are partial L -definable over k functions $*$: $Y \times Y \rightarrow Y$, and $\text{inv} : Y \rightarrow Y$ such that $*$ is defined on $\Gamma \times \Gamma$ and is precisely multiplication in Γ , and such that inv is defined on Γ and is precisely inversion on Γ .

Note that:

Claim 4. — For $\bar{a} \in \Gamma$, $\dim_L(\bar{a}/k) = \dim_{\partial}(\bar{a}/k)$.

We now will make use of cell decomposition, focusing on the V_{p_i} (and p_i) of maximal ∂ -dimension. Let $n = \dim_{\partial}(\Gamma) = \dim_L(Y)$.

Take a C^1 -cell decomposition of Y by C^1 -cells defined over k which is compatible with the collection of formulas $\phi_{p_i}(\bar{x})$ and definable functions s_{p_i} . Let $C_1, \dots, C_r$ be the cells of maximal

dimension n . So by Claim 4, each p_i of maximal ∂ -dimension n is in some C_j , and also s_{p_i} is a C^1 -function on C_j . Moreover each C_j contains one of the types p_i of maximal ∂ -dimension. Let us fix for each $j = 1, \dots, r$, some p_i which is in C_j . So s_{p_i} is a C^1 -function on C_j , and we rewrite s_{p_i} as s_j .

We conclude

Claim 5. — $\bar{a} \in \Gamma$ is ∂ -generic in Γ over k iff for some $j = 1, \dots, r$, $\bar{a} \in C_j$, $\dim_L(\bar{a}/k) = n$ and $\partial(\bar{a}) = s_j(\bar{a})$.

Fix $j \in \{1, \dots, r\}$. As C_j is an n -dimensional cell, let π^j be a suitable projection from the ambient space to M^n giving a bijection between C_j and some open L -definable over k subset U_j of M^n (i.e., such that the map $(\pi^j)^{-1}|_{U_j}$ is C^1). By reordering coordinates, for $\bar{a} \in U_j$, write $\pi^j(\bar{a})$ as $(a_1, \dots, a_n)$. We let $\pi^j(s_j) : U_j \rightarrow M^n$ denote the C^1 -function on U_j taking $(a_1, \dots, a_n)$ to the first n -coordinates of $s_j(\bar{a})$ where $\bar{a} \in C_j$ and $\pi^j(\bar{a}) = (a_1, \dots, a_n)$. With this notation, for $\bar{a} \in C_j$, $s_j(\bar{a})$ is determined by $\pi(s_j)(a_1, \dots, a_n)$, using the formula in Remark 6.5. We now identify C_j with U_j via π^j and also identify s_j with $\pi^j(s_j) : U_j \rightarrow M^n$. We conclude:

Claim 6. — Let U be the disjoint union of the U_j , and let s be the disjoint union of the s_j . So (with the identification of C_j and U_j as above) the ∂ -generic over k elements of Γ are precisely the elements $\bar{a} \in U$ such that $\dim_L(\bar{a}/k) = n$ and $\partial(\bar{a}) = s(\bar{a})$.

Both $*$ and inv from Claim 3 induce partial L_k -definable operations on $U \times U$ and U respectively, which we still call $*$ and inv .

Note that, by the definition of ∂ -independence we have that:

Claim 7. — If $g_1, g_2 \in \Gamma$ are ∂ -generic over k and ∂ -independent over k then the product $g_1 g_2$ is ∂ -generic in Γ over k and ∂ -independent from each of g_1, g_2 over k . Likewise $g_1^{-1} \in \Gamma$ is ∂ -generic over k .

Already Claims 6 and 7 give part of the current analogue of Lemma 3.4. The rest is given by:

Claim 8. — Let $a, b \in U$ be such that $\dim_L(a, b/k) = 2n$ (so a, b are L -independent L -generics of U over k). Then there are ∂ -generic and ∂ -independent over k , $g_1, g_2 \in \Gamma$, such that $tp_L(a, b/k) = tp_L(g_1, g_2/k)$.

Proof. — Suppose $a \in C_j$ and $b \in C_\ell$. By Lemma 6.7, there is (g_1, g_2) realizing $tp_L(a, b/k)$ with $\partial(g_1) = s_j(g_1)$ and $\partial(g_2) = s_\ell(g_2)$. So $(g_1, g_2) \in \Gamma \times \Gamma$, and g_1, g_2 are ∂ -generic, ∂ -independent over k (as these properties depend only on $tp_L(g_1, g_2/k)$ once we know $g_1, g_2 \in \Gamma$). $\square$

From Claims 6, 7, 8, we can prove the analogue of Proposition 3.5 in our current situation. The construction of an enveloping L -definable group G proceeds as in Section 4, or is given directly by Section 5.1. The definable embeddability of Γ in G is as in Lemma 4.7. This completes the proof of Theorem 6.8(i).

For the rest, by Theorem 5.4, the image of U in G is large, so by Lemma 6.7 and Claim 6 we get (ii) and (iii). $\square$

6.2. Open theories of topological fields with a generic derivation. — This is a context developed by the third author and collaborators for which the reader is referred to [12] and [7]. It subsumes some new examples such as algebraically closed and real closed valued fields.

We first give a brief description of the relevant theories and then proceed to the main theorem and proof in this new context. It will be closer to the material in Sections 2, 3, 4, and 5.2 than Section 6.1.

We first fix a (one-sorted) language L , an extension of the language of rings by relation symbols and constant symbols. Let T be a complete L -theory extending the theory of large fields of characteristic 0 such that:

- (1) For some L -formula $\phi(x, \bar{y})$, in any model K of T , the set of formulas $\phi(x, \bar{a})$ as $\bar{a}$ ranges over tuples from K of the right length, defines a basis for a (Hausdorff) topology on K which makes K into a topological field. When we say open, closed, etc we mean with respect to this topology.
- (2) For any model K of T , and subfield k , any k -definable subset of K^n is a finite union of sets of the form $Z \cap U$ where Z is a Zariski closed subset of K^n defined by polynomials over k and U is an open subset of K^n also defined over k .

A certain many-sorted version of this setup-and the assumptions (1), (2), with a distinguished topological field sort, was defined in [7], and called an “open theory of topological fields”. Already the one-sorted context includes the complete theories $ACVF$ and $RCVF$ (treated as one-sorted structures).

We fix such an open theory of (large) topological fields T . There is no harm assuming quantifier elimination although this already holds in $ACVF$, $RCVF$ in the natural languages.

It is important to note that T is a “geometric theory” (its models are geometric structures), by virtue of definability of dimension. And T is “almost” a theory of geometric fields, except that the language L may contain more than the ring language.

We will typically denote models of T by K (although the language L may be an expansion of the ring language), and a saturated model by $\mathcal{U}$. Nevertheless we see that in models of T , model-theoretic algebraic closure coincides with field-theoretic algebraic closure. The dimension we use in models of T will of course be algebraic dimension $\dim(\bar{a}/k) = \text{trdeg}(k(\bar{a})/k)$. Notice that by our assumptions on T , especially (2) above we have:

Remark 6.9. — Let $K \models T$, k a subfield and $\bar{a}$ an n -tuple from K . Then $tp(\bar{a}/k)$ is axiomatized by the following information $\bar{x} \in V$, where V is the variety over k with k -generic point $\bar{a}$, and $\{\bar{x} \in U\}$ where U ranges over all open k -definable subsets of $V(K)$ containing $\bar{a}$.

We have independence, genericity, etc. in models of T , coming from $\dim$. So for example, if $K \models T$, and V is an irreducible variety over $k \leq K$ with a Zariski-dense set of K -points, then $tp(\bar{a}/k)$ is a generic type of $V(K)$ if $\dim(\bar{a}/k) = \dim(V)$.

In our account of the main theorem below (Theorem 6.12) in the context of models of T with a generic derivation, the topology on models of T will not play much of a role.

We have the field-theoretic algebraic closure operation on K giving rise to dimension, independence, on top of which is some additional structure which does not affect dimension, algebraic closure etc.

It is proved in [7] and [12], under additional assumptions to get consistency, that the class of existentially closed models of $T \cup \{\partial \text{ is a derivation}\}$ is elementary with additional axioms: If $p(x_0, \dots, x_n)$ is a polynomial over k (without loss, irreducible) and $\bar{a}$ is a K -point on the hypersurface H_p defined by $p(\bar{x}) = 0$ such that $(\partial p / \partial x_n)(\bar{a}) \neq 0$, then for any open neighbourhood U of $\bar{a}$ in K^{n+1} there is $b \in K$ such that $\nabla^{(n)}(b) \in U \cap H_p$. If T is a complete theory of topological fields for which the topology is induced by an henselian valuation (not necessarily definable) then T_∂ is consistent. See Section 2.3 of [7] where this is discussed. Here $\nabla^{(n)}$ is as we defined earlier.

We call this theory T_∂ . T_∂ is a complete theory in L_∂ which has QE if T does (which we will assume).

To prove our main theorem on finite-dimensional definable groups in this context (namely a finite-dimensional definable group Γ in a model (K, ∂) of T_∂ definably embeds in a group G definable in the L -structure K) we need as before a couple of additional facts:

Fact 6.10 ([7, Lemma 3.1.9]). — *In models of T_∂ the algebraic closure of a set A coincides with the field-theoretic algebraic closure (equals L -algebraic closure) of the differential field generated by A , and the definable closure of a set A coincides with the L -definable closure of the differential field generated by A .*

For the next lemma we make use of rational D -varieties as defined above. From now on we let $(\mathcal{U}, \partial)$ be a saturated model of T_∂ . $k, K, \dots$ will typically denote small differential subfields.

Lemma 6.11. — *Let (V, s) be a rational D -variety over k (V affine, irreducible, with a Zariski-dense set of $\mathcal{U}$ -points) over k . Let $p(\bar{x})$ be a complete generic type of V over k , in the sense of $\mathcal{U} \models T$. Then there is a realization $\bar{a}$ of p such that $\nabla(\bar{a}) = s(\bar{a})$.*

Proof. — The proof is like that of Lemma 2.7. Let $\bar{c}$ realize p in some model K of T containing k . So $\bar{c}$ is a generic point of V over k . Apply s to $\bar{c}$ to obtain $(\bar{c}, \bar{d}) \in T_\partial(V)$. The extension theorem for derivations allows us to extend $\partial|_k$ to a derivation ∂^* of $k(\bar{c})$ with $\partial^*(\bar{c}) = \bar{d}$. By quantifier elimination of T_∂ there is an embedding of the L_∂ -structure $(k(\bar{c}), \partial^*)$ into $(\mathcal{U}, \partial)$ over (k, ∂) . Let $\bar{a}$ be the image of $\bar{c}$. $\square$

All the earlier notions of finite-dimensional, ∂ -independence, ∂ -generics go over to the new context.

Again we have:

Theorem 6.12. — *Let Γ be a finite-dimensional group definable over k in $(\mathcal{U}, \partial) \models T_\partial$ where T is an open theory of large topological fields. Then Γ is definably (over k) embeddable in a group G definable over k in the L -structure $\mathcal{U}$. Remark 4.8 also holds.*

Discussion. — The proof is identical to that of Theorem 4.1 (or Theorem 5.11) going through the same steps, except we replace “semialgebraic” everywhere by L -definable (where we of course still have $\dim_L$). $\square$

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