PUBLICATIONS MATHÉMATIQUES DE BESANÇON
Algèbre et Théorie des nombres

Guy Casale, Matthew DeVilbiss, James Freitag, and Joel Nagloo

Strong minimality of triangle functions

2026, p. 53-59.
https://doi.org/10.5802/pmb.69

PRESSES UNIVERSITAIRES DE FRANCHE-COMTÉ

Les Publications mathématiques de Besançon *sont membres du Centre Mersenne pour l'édition scientifique ouverte*
http://www.centre-mersenne.org/
e-ISSN : 2592-6616

Publications mathématiques de Besançon – 2026, 53-59

STRONG MINIMALITY OF TRIANGLE FUNCTIONS

by

Guy Casale, Matthew DeVilbiss, James Freitag and Joel Nagloo

Abstract. — In this manuscript, we give a new proof of strong minimality of certain automorphic functions, originally results of Freitag and Scanlon (2017), Casale, Freitag and Nagloo (2020), Blázquez-Sanz, Casale, Freitag and Nagloo (2020). Our proof is shorter and conceptually different than those presently in the literature.

Résumé. — Dans ce manuscrit, nous présentons une nouvelle preuve de la forte minimalité de certaines fonctions automorphes, résultats établis à l'origine par Freitag et Scanlon (2017), Casale, Freitag et Nagloo (2020), ainsi que Blázquez-Sanz, Casale, Freitag et Nagloo (2020). Notre preuve est plus courte et conceptuellement différente de celles qui figurent actuellement dans la littérature.

1. Introduction

In [13], Mahler proved that $j\left(\frac{\log q}{\pi i}\right)$ satisfies no first or second order differential equation over $\mathbb{C}(q)$, while conjecturing that any automorphic function of a Fuchsian subgroup of $\mathrm{SL}_2(\mathbb{R})$ with at least 3 limit points with respect to the action on the upper half plane should also have this property.[1] Mahler established the result under additional restrictions on the Fuchsian group $G \leq \mathrm{SL}_2(\mathbb{R})$, and Nishioka [16] proved Mahler's conjecture, showing:

Theorem 1.1 **([16]).** — *Let G be a discontinuous subgroup of* $\mathrm{SL}_2(\mathbb{R})$ *which possesses at least three limit points. Let u be a nonzero complex number. Then any automorphic function of G satisfies no algebraic differential equation of order two over* $\mathbb{C}(z, e^{uz})$.

Given a hyperbolic triangle in the upper half plane with angles 0 or π/n for $n \in \mathbb{N}_{>1}$, through successive hyperbolic reflections over the sides of the triangle, one can tile the upper half plane. The group of transformations generated by the reflections is an example of a

2020 ***Mathematics Subject Classification.*** — 11F03, 12H05, 03C60.
Acknowledgements. — G. Casale is partially supported by CAPES-COFECUB project MA 932/19 and Centre Henri Lebesgue, program ANR-11-LABX-0020-0. J. Freitag is partially supported by NSF CAREER award 1945251. J. Nagloo is partially supported by NSF grants DMS-2203508 and DMS-2348885.

[1]Any of these automorphic functions does satisfy a differential equation of third order over $\mathbb{C}(q)$.

(hyperbolic) triangle group; such groups are discrete Zariski-dense subgroups of $\mathrm{SL}_2(\mathbb{R})$ and their collection of limit points is $\mathbb{R} \cup \{\infty\}$. They are thus are a special case of the groups considered in Theorem 1.1. Schwarz triangle functions are automorphic functions for these hyperbolic triangle groups, and Nishioka proved a strengthening of Theorem 1.1 for Schwarz triangle functions:

Theorem 1.2. — *Schwarz triangle functions are not 2-reducible*[2] *over $\mathbb{C}$.*

Nishioka's above theorems are special cases of a conjecture of Painlevé [19, pp. 444–445 and p. 519] proved in [3], in which the field $\mathbb{C}(z, e^{uz})$ can be replaced with an arbitrary differential field extension. This generalization is equivalent to the *strong minimality* of the equation, a notion with origins in model theory. In this short article we give a new proof of the strong minimality of Schwarz triangle functions (originally results of [8], later [1] for the j-function and [2] in full generality) using techniques from the recent work of [7, 14].
Our proof utilizes the connection between solutions of a Schwarzian equation and a certain associated linear differential equation, via Puiseux series. From there, we use Nishioka's theorem 1.2 with a model theoretic study of fibrations inspired by the work of [15] to prove the strong minimality of Schwarz triangle functions. The motivation for establishing the strong minimality of certain differential equations has been discussed many places and ranges from functional transcendence theorems [3] to diophantine results [8, 9, 10] to understanding solutions of equations from special classes of functions [4, 5].

1.1. Organization of this paper. — In Section 2, we give a brief explanation of the work of Freitag and Moosa [7] and its specific consequences in our context. In Section 3, we develop a connection between order two subvarieties of our nonlinear equations and algebraic solutions of certain associated Riccati equations. In Section 4 we finish the proof of the strong minimality of Schwarz triangle functions using Section 3.

2. No proper fibrarions

We begin with a definition of [14]:

Definition 2.1. — A stationary type $p = \mathrm{tp}(a/A)$ *admits no proper fibrations* if whenever $c \in \mathrm{dcl}(Aa) \setminus \mathrm{acl}(A)$, we must have $a \in \mathrm{acl}(Ac)$.

The following result [14, Proposition 2.3] gives a very useful restriction on those types with no proper fibrations.

Proposition 2.2. — *Suppose that $p = \mathrm{tp}(a/A)$ is stationary and admits no proper fibrations. Then either*

(1) *p is almost internal to a non locally modular minimal type, or*

(2) *a is interalgebraic over A with a finite tuple of independent realizations of a locally modular minimal type over A.*

We aim to use Proposition 2.2 in conjunction with the following notion of Nishioka, developed in the series of papers [17, 18].

[2] For the definition of n-reducibility, see Section 2.

Definition 2.3. — Let y be differentially algebraic over a differential field k. We say a is *r-reducible over k* if there exists a finite chain of differential field extensions,

$$k = R_0 \subset R_1 \subset \cdots \subset R_m$$

such that $a \in R_m$ and $\operatorname{trdeg} R_i/R_{i-1} \leq r$.

Lemma 2.4. — *Suppose $p = \operatorname{tp}(a/k)$ is the generic type of an order 3 differential equation over k. If a is not 2-reducible over k, then $p = \operatorname{tp}(a/k)$ has no proper fibrations.*

Proof. — We will prove the contrapositive statement. Suppose that $\operatorname{tp}(a/k)$ has a proper fibration. Let $b \in \operatorname{dcl}(a/k)$ be such that $a \notin \operatorname{acl}(b/k)$. Then note that $3 = \operatorname{trdeg} k\langle a\rangle/k = \operatorname{trdeg} k\langle a, b\rangle/k$, but since $a \notin \operatorname{acl}(b/k)$ and $b \notin \operatorname{acl}(k)$, we must have $1 \leq \operatorname{trdeg} k\langle b\rangle/k \leq 2$. It follows that $k \subset k\langle b\rangle \subset k\langle a\rangle$ implies that a is 2-reducible. □

As mentioned in the introduction, in [18], Nishioka proved that Schwarz triangle functions are not 2-reducible over $\mathbb{C}$. Though most of the above conditions deal only with the generic solutions of the differential equations we consider, a beautiful elementary argument of Nishioka [16] shows that the differential equations satisfied by Fuchsian automorphic functions have no proper infinite differential subvarieties over $\mathbb{C}$, so any nonalgebraic solution is generic over $\mathbb{C}$. Thus, in order to establish that these equations are strongly minimal, one needs only to verify that their generic types have no nonalgebraic forking extensions.

3. No order two subvarieties

In this section, we will work with a third order differential equation for derivation δ over $\mathbb{C}$ with δ constant on $\mathbb{C}$. For shorthand, we write $y' := \delta(y)$. We consider the Schwarzian equation

$$S_\delta(y) + (y')^2 \cdot R(y) = 0, \tag{$\star$}$$

where R is a rational function in $\mathbb{C}(y)$ and $S_\delta(y) := \left(\frac{y''}{y'}\right)' - \frac{1}{2}\left(\frac{y''}{y'}\right)^2$ is the Schwarzian derivative. As we will soon see, the triangle functions satisfy such a Schwarzian equation for some specific choice of the rational function $R(y)$ (cf. [3]). We now aim to show that modulo a certain condition, the Schwarzian equation ($\star$) has no order two differential subvarieties over *any* differential field (k, δ) extending $\mathbb{C}$. Our proof is inspired by Nishioka [18].

For any field k extending $\mathbb{C}$, by $\frac{\partial}{\partial y}$ we denote the derivation[3] on $k(y)^{alg}$ uniquely determined by $\frac{\partial y}{\partial y} = 1$ and $\frac{\partial}{\partial y}|_k = 0$. Over $\mathbb{C}(y)$, we have a Riccati equation (in differential variable u) associated to Equation ($\star$) given by:

$$\frac{\partial u}{\partial y} + u^2 + \frac{1}{2}R(y) = 0. \tag{$\star\star$}$$

The following condition might or might not hold, depending on how $R(y)$ is chosen:

Condition 3.1. — *The Riccati equation ($\star\star$) has no solution in $\mathbb{C}(y)^{alg}$.*

[3]As usual we will write $\frac{\partial a}{\partial y}$ instead of $\frac{\partial}{\partial y}(a)$.

Lemma 3.2. — *Assume that Condition 3.1 holds. Then let (k,δ) be any differential field extension of $\mathbb{C}$. If y is a solution of the Schwarzian equation $(\star)$ in some differential field extending k, then*

$$\operatorname{trdeg} k\langle y\rangle/k \neq 2.$$

In other words, Equation $(\star)$ has no order two differential subvarieties.

Proof. — Let y be a solution of the Schwarzian equation $(\star)$ in some differential field extending k. Suppose $\operatorname{trdeg} k\langle y\rangle/k = 2$ and let $L = k(y)^{alg}$. Since $y' := \delta(y)$ is transcendental over L we can work in the field $L\langle\langle 1/y'\rangle\rangle$ of Puiseux series in $1/y'$ over L. We have an extension of δ to $L\langle\langle 1/y'\rangle\rangle$ given by:

$$\delta\left(\sum a_i y'^{\lambda_i}\right) := \sum \delta(a_i) y'^{\lambda_i} + \sum \frac{\partial a_i}{\partial y} y'^{\lambda_i+1} + \sum \lambda_i a_i y'^{\lambda_i-1} y''$$

where λ_i, $i \geq 0$, are descending rational numbers with a common denominator, $a_i \in L$ and $a_0 \neq 0$. Note here that by the transcendence assumption, y'' is an algebraic function of $y, \frac{1}{y'}$ over k. It follows that L is a differential subfield of $L\langle\langle 1/y'\rangle\rangle$.
Now let $u = \frac{y''}{y'^2}$ and by replacing in equation $(\star)$ we get

$$\frac{u'}{y'} + \frac{1}{2}u^2 + R(y) = 0.$$

Writing $u = \sum a_i y'^{\lambda_i}$ with $i \geq 0$ and $a_0 \neq 0$ and differentiating

$$\frac{u'}{y'} = \sum \partial(a_i) y'^{\lambda_i - 1} + \sum \frac{\partial a_i}{\partial y} y'^{\lambda_i} + \sum \lambda_i a_i y'^{\lambda_i} \sum a_i y'^{\lambda_i} = -\frac{1}{2}\left(\sum a_i y'^{\lambda_i}\right)^2 - R(y).$$

Since $R(y)$ (a coefficient of y'^0) appear non-trivially in the above and since $R(y) \notin k^{alg}$, it is easily seen that $\lambda_0 = 0$ and

$$\frac{\partial a_0}{\partial y} + \frac{1}{2}a_0^2 + R(y) = 0.$$

The field $(k(y)^{alg}, \frac{\partial}{\partial y})$ is a differential field extention of $(\mathbb{C}(y)^{alg}, \frac{\partial}{\partial y})$ and $R(y) \in \mathbb{C}(y)$. We can find a $\aleph_0$-saturated model $\mathcal{U}$ of the theory of differentially closed fields extending $\mathbb{C}(y)^{alg}$ having $\mathbb{C}$ as its constant field, so if there is a solution of $(\star\star)$ in $k(y)^{alg}$ (here $\frac{a_0}{2}$), then already there is a solution in $\mathbb{C}(y)^{alg}$, contradicting Condition 3.1. $\square$

Let us say a few words about when Condition 3.1 holds. First, it is well-known that the Riccati equation $(\star\star)$ has an algebraic solution if and only if a certain associated order two linear equation has Liouvillian solutions (see [12, 1.3]). In [12], Kovacic gives an algorithm to determine when an order two linear equation has Liouvillian solutions, and so, by the above equivalence, Kovacic's algorithm can be used to determine when Condition 3.1 holds. Moreover, when the rational function $R(y)$ is of "triangular form"

$$R(y) = R_{\alpha,\beta,\gamma}(y) = \frac{1}{2}\left(\frac{1-\beta^{-2}}{y^2} + \frac{1-\gamma^{-2}}{(y-1)^2} + \frac{\beta^{-2}+\gamma^{-2}-\alpha^{-2}-1}{y(y-1)}\right)$$

with $\alpha, \beta, \gamma \in \mathbb{C} \cup \{\infty\}$, one can give precise conditions under which Condition 3.1 holds. The particular rational function $R_{\alpha,\beta,\gamma}(y)$ given above is referred to as being in triangular form because a Schwarz triangle function with angles of the associated hyperbolic triangle given by $(\frac{\pi}{\alpha}, \frac{\pi}{\beta}, \frac{\pi}{\gamma})$, for integers $\alpha, \beta, \gamma \in \mathbb{N} \cup \{\infty\}$ satisfies $S_{\frac{d}{dt}}(y) + (y')^2 \cdot R_{\alpha,\beta,\gamma}(y) = 0$. This classification of algebraic solutions to 3.1 applies more generally to the case where $\alpha, \beta, \gamma \in \mathbb{C} \cup \{\infty\}$ and comes ultimately from Theorem I of [11], but is explained in detail in [2, Proposition 4.4].

Fact 3.3. — *Condition 3.1 holds of*

$$\frac{\partial u}{\partial y} + u^2 + \frac{1}{2} R_{\alpha,\beta,\gamma}(y) = 0$$

as long as none of the following conditions hold.

(1) *The quantities α^{-1} or $-\alpha^{-1}$, β^{-1} or $-\beta^{-1}$ and γ^{-1} or $-\gamma^{-1}$ take, in an arbitrary order, values given in the following table:*

	$\pm\alpha^{-1}$	$\pm\beta^{-1}$	$\pm\gamma^{-1}$	
1	$\frac{1}{2}+\ell$	$\frac{1}{2}+m$	*arbitrary*	
2	$\frac{1}{2}+\ell$	$\frac{1}{2}+m$	$\frac{1}{2}+n$	
3	$\frac{2}{3}+\ell$	$\frac{1}{3}+m$	$\frac{1}{4}+n$	$\ell+m+n$ *even*
4	$\frac{1}{2}+\ell$	$\frac{1}{3}+m$	$\frac{1}{4}+n$	
5	$\frac{2}{3}+\ell$	$\frac{1}{4}+m$	$\frac{1}{4}+n$	$\ell+m+n$ *even*
6	$\frac{1}{2}+\ell$	$\frac{1}{3}+m$	$\frac{1}{5}+n$	
7	$\frac{2}{5}+\ell$	$\frac{1}{3}+m$	$\frac{1}{3}+n$	$\ell+m+n$ *even*
8	$\frac{2}{3}+\ell$	$\frac{1}{5}+m$	$\frac{1}{5}+n$	$\ell+m+n$ *even*
9	$\frac{1}{2}+\ell$	$\frac{2}{5}+m$	$\frac{1}{5}+n$	$\ell+m+n$ *even*
10	$\frac{3}{5}+\ell$	$\frac{1}{3}+m$	$\frac{1}{5}+n$	$\ell+m+n$ *even*
11	$\frac{2}{5}+\ell$	$\frac{2}{5}+m$	$\frac{2}{5}+n$	$\ell+m+n$ *even*
12	$\frac{2}{3}+\ell$	$\frac{1}{3}+m$	$\frac{1}{5}+n$	$\ell+m+n$ *even*
13	$\frac{4}{5}+\ell$	$\frac{1}{5}+m$	$\frac{1}{5}+n$	$\ell+m+n$ *even*
14	$\frac{1}{2}+\ell$	$\frac{2}{5}+m$	$\frac{1}{3}+n$	$\ell+m+n$ *even*
15	$\frac{3}{5}+\ell$	$\frac{2}{5}+m$	$\frac{1}{3}+n$	$\ell+m+n$ *even*

where ℓ, m, n stand for arbitrary integers.

(2) *At least one of the four complex numbers, $\alpha^{-1}+\beta^{-1}+\gamma^{-1}$, $-\alpha^{-1}+\beta^{-1}+\gamma^{-1}$, $\alpha^{-1}-\beta^{-1}+\gamma^{-1}$, $\alpha^{-1}+\beta^{-1}-\gamma^{-1}$ is an odd integer.*

As is well-known (cf. [3, 17]), triangle functions satisfy the Schwarzian equation ($\star$) where the rational function $R(y) = R_{\alpha,\beta,\gamma}(y)$ is such that $\alpha, \beta, \gamma \in \mathbb{N}$ with $2 \leq \gamma \leq \beta \leq \alpha \leq \infty$ and $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} < 1$, that is such that α, β, γ satisfying the hyperbolicity conditions for Fuchsian triangle groups. We have the following

Proposition 3.4. — *The differential equations ($\star$) satisfied by Schwarz triangle functions have no order two subvarieties.*

Proof. — Condition (2) of Fact 3.3 can never hold for $\alpha, \beta, \gamma \in \mathbb{N}$ such that $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} < 1$. Indeed, with α, β integers and $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} < 1$, we can never have rows 1,2 of Condition (1), since the conditions on α and β would require both to be 2, violating hyperbolicity. Rows 3, 5, 7, 8, 10, 11, 12, 13, and 15 can never occur for an integer α. For integers α, β, γ, row 4 would

require $\alpha = 2, \beta = 3, \gamma = 4$, violating hyperbolicity. Similarly, we would have a violation of hyperbolicity if row 6 holds. The conditions on β in rows 9 and 14 can not hold for an integer β. So we have now ruled out Conditions (1) and (2) of Fact 3.3.
Thus, Condition 3.1 holds when $R(y) = R_{\alpha,\beta,\gamma}(y)$ with α, β, γ satisfying the hyperbolicity conditions for Fuchsian triangle groups. Now, by Lemma 3.2, the equation $S_{\frac{d}{dt}}(y) + (y')^2 \cdot R_{\alpha,\beta,\gamma}(y) = 0$ satisfied by the Schwarz triangle equation has no order two subvarieties. □

4. Strong minimality in the style of Nishioka

In this section, we use the results of Section 3 with the model theoretic notions of Section 2 to give a new proof of the strong minimality of the equations satisfied by Schwarz triangle functions (and so including the j-function). In particular these results strengthen the main theorems of [17, 18], generalize the main theorem of [8], and give new proofs of results of [2, 3] in the case of Schwarz triangle functions.

Proposition 4.1. — *The differential equations ($\star$) satisfied by Schwarz triangle functions are strongly minimal.*

Proof. — Denote by X the solution set of such an equation. By the remarks concluding Section 2, we need only establish the minimality of the generic type of X. By Theorem 1.2 and Lemma 2.4, the generic type $p = \mathrm{tp}(a/\mathbb{C})$ of such an equation has no proper fibrations. So, it follows by Proposition 2.2 that p is either almost internal to $\mathbb{C}$ or is interalgebraic over $\mathbb{C}$ with the product of a locally modular minimal type with itself.
If p is almost internal to the constants, then it follows that p is nonorthogonal to $\mathbb{C}$ and so (by the main result of [6]) there is a nonconstant definable map from X to $\mathbb{C}$. A generic fiber of this map gives an order 2 subvariety of X, which would contradict Proposition 3.4.
If p is interalgebraic with $q \otimes q \otimes \cdots \otimes q$, where q is locally modular, then either q is an order one type and the product is $q \otimes q \otimes q$ or q is order three and p is minimal. In the first case, we have an algebraic correspondence: $\phi : p \to q \otimes q \otimes q$ and a natural map $\pi_1 : q \otimes q \otimes q \to q$. Let a be a $\mathbb{C}$-generic realization of q. Then $\phi^{-1}(\pi_1^{-1}(a))$ has a generic type of order 2 over $\mathbb{C}\langle a \rangle$ and is contained in X. This is impossible by Proposition 3.4. So, it must be that p is minimal; it follows by Theorem 1.1 that the equation is strongly minimal. □

Since X is defined over $\mathbb{C}$, it follows from Proposition 5.8 of [3] that X is geometrically trivial. Generalizing Nishioka's result, Theorem 1.2, to larger classes of automorphic functions, would similarly yield a proof the strong minimality of the associated equations - can one give a proof in the style of [18]? Doing so for Fuchsian functions of the first kind would allow one to recover the main strong minimality results of [3].

References

[1] V. Aslanyan, "Ax–Schanuel and strong minimality for the j-function", *Ann. Pure Appl. Logic* **172** (2021), no. 1, article no. 102871 (24 pages).

[2] D. Blázquez-Sanz, G. Casale, J. Freitag & J. Nagloo, "Some functional transcendence results around the Schwarzian differential equation", *Ann. Fac. Sci. Toulouse, Math. (6)* **29** (2020), no. 5, p. 1265-1300.

[3] G. Casale, J. Freitag & J. Nagloo, "Ax–Lindemann–Weierstrass with derivatives and the genus 0 Fuchsian groups", *Ann. Math. (2)* **192** (2020), no. 3, p. 721-765.

[4] J. Freitag, "Not Pfaffian", 2021, `https://arxiv.org/abs/2109.09230`.

[5] J. Freitag, R. Jaoui, D. Marker & J. Nagloo, "On the equations of Poizat and Liénard", *Int. Math. Res. Not.* **2023** (2023), no. 19, p. 16478-16539.

[6] J. Freitag & R. Moosa, "Finiteness theorems on hypersurfaces in partial differential-algebraic geometry", *Adv. Math.* **314** (2017), p. 726-755.

[7] ———, "Bounding nonminimality and a conjecture of Borovik–Cherlin", *J. Eur. Math. Soc.* **27** (2025), no. 2, p. 589-613.

[8] J. Freitag & T. Scanlon, "Strong minimality and the j-function", *J. Eur. Math. Soc.* **20** (2017), no. 1, p. 119-136.

[9] E. Hrushovski, "The Mordell–Lang conjecture for function fields", *J. Am. Math. Soc.* **9** (1996), no. 3, p. 667-690.

[10] E. Hrushovski & A. Pillay, "Effective bounds for the number of transcendental points on subvarieties of semi-abelian varieties", *Am. J. Math.* **122** (2000), no. 3, p. 439-450.

[11] T. Kimura, "On Riemann's equations which are solvable by quadratures", *Funkc. Ekvacioj, Ser. Int.* **12** (1969), p. 269-281.

[12] J. J. Kovacic, "An algorithm for solving second order linear homogeneous differential equations", *J. Symb. Comput.* **2** (1986), no. 1, p. 3-43.

[13] K. Mahler, "On algebraic differential equations satisfied by automorphic functions", *J. Aust. Math. Soc.* **10** (1969), no. 3-4, p. 445-450.

[14] R. Moosa & A. Pillay, "Some model theory of fibrations and algebraic reductions", *Sel. Math., New Ser.* **20** (2014), no. 4, p. 1067-1082.

[15] R. Moosa & T. Scanlon, "Model theory of fields with free operators in characteristic zero", *J. Math. Log.* **14** (2014), no. 2, article no. 1450009 (43 pages).

[16] K. Nishioka, "A conjecture of Mahler on automorphic functions", *Arch. Math.* **53** (1989), no. 1, p. 46-51.

[17] ———, "Painlevé's theorem on automorphic functions", *Manuscr. Math.* **66** (1990), no. 4, p. 341-349.

[18] ———, "Painlevé's Theorem on Automorphic Functions II", *Funkc. Ekvacioj, Ser. Int.* **35** (1992), p. 597-602.

[19] P. Painlevé, "Leçons sur la théorie analytique des équations différentielles (Leçons de Stockholm, 1895) (Hermann, Paris, 1897)", in *Œuvres de Paul Painlevé. Tome I*, Éditions du Centre National de la Recherche Scientifique, 1973.

Guy Casale, Univ Rennes, CNRS, IRMAR-UMR 6625, F-35000 Rennes, France
E-mail : `guy.casale@univ-rennes1.fr`

Matthew DeVilbiss, Saint Xavier University, Department of Mathematics, 3700 W 103rd St, Chicago, IL 60655, USA • *E-mail :* `devilbiss@sxu.edu`

James Freitag, University of Illinois Chicago, Department of Mathematics, Statistics, and Computer Science, 851 S. Morgan Street, Chicago, IL 60607-7045, USA • *E-mail :* `jfreitag@uic.edu`

Joel Nagloo, University of Illinois Chicago, Department of Mathematics, Statistics, and Computer Science, 851 S. Morgan Street, Chicago, IL 60607-7045, USA • *E-mail :* `jnagloo@uic.edu`