



PUBLICATIONS MATHÉMATIQUES DE BESANÇON

Algèbre et Théorie des nombres

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Resolution of singularities for the dynamical mathematician

2026, p. 87-107.

<https://doi.org/10.5802/pmb.72>

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PRESSES UNIVERSITAIRES DE FRANCHE-COMTÉ



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<http://www.centre-mersenne.org/>

e-ISSN : 2592-6616

RESOLUTION OF SINGULARITIES FOR THE DYNAMICAL MATHEMATICIAN

by

Dan Abramovich

Abstract. — I begin by explaining to non-specialists why resolution of singularities in characteristic 0 works. Then I go into some ideas telling how it actually works. I finish with a brief discussion of related results on foliations.

Résumé. — Nous commençons par expliquer aux non-spécialistes pourquoi la résolution des singularités en caractéristique 0 fonctionne. Nous exposons ensuite quelques idées pour expliquer la manière dont elle fonctionne vraiment. Nous terminons par une brève discussion des résultats associés concernant les feuilletages.

1. Introduction

These are lecture notes for a minicourse by the same title delivered at the CIRM conference *Foliations, birational geometry and applications*, 3-7 February, 2025.

We work in characteristic 0. The main result is due to Hironaka: resolution of singularities in characteristic 0 exists.

I report on work with André Belotto da Silva, Michael Temkin, and Jarosław Włodarczyk; any claim to originality is joint with them and appears in the paper [4]. There is also related work with Ming Hao Quek and Bernd Schober which influenced my presentation but is not explicitly described here.

There is a whole lot of other work on the subject, to some of which I will hopefully refer at appropriate junctures. Some reference are [6, 9, 13, 15, 17, 18, 20, 21, 24, 33, 35].

2020 Mathematics Subject Classification. — 14E15, 32S65, 32S45, 14A20, 14A21.

Acknowledgements. — This is a report on research supported by BSF grants 2018193, 2022230, ERC Consolidator Grant 770922 - BirNonArchGeom, NSF grants DMS-2100548, DMS-2401358, the Plan d'Investissements France 2030, IDEX UP ANR-18-IDEX-0001, and Simon foundation grants MPS-SFM-00006274, MPS-TSM-00008103.

2. Blowing up

I describe an approach to blowing up that originates with Rees's $\mathfrak{a}$ -transform of a ring A with ideal $\mathfrak{a}$. It was rediscovered many times, each time with new angles and insights: Fulton and MacPherson use it under the name *degeneration to the normal cone*, a key construction in intersection theory, a term I stick by [16]. Miles Reid showed its relevance for GIT and the minimal model program, bringing in flips to the picture [31]. For Cox it is a case of the Cox construction [12]. Włodarczyk used it first for factorization of birational maps [34] and more recently for resolution [36]. Quek and Rydh used it specifically to describe weighted blowups [30].

2.1. Classical blowup in concrete terms: the cox construction. — Let us blow up the vanishing locus of the ideal $\bar{J} = (x_1, \dots, x_k)$ on the variety $Y = \operatorname{Spec} k[x_1, \dots, x_n]$, the affine space.

Define the *extended Rees algebra*

$$\tilde{R} := \mathcal{O}_Y[s, x'_1, \dots, x'_k] / (x_1 - sx'_1, \dots, x_k - sx'_k),$$

and denote

$$B := \operatorname{Spec}_Y \tilde{R}.$$

The variable s is the *exceptional variable*, and it has global meaning: the Y -variety B is the *deformation to the normal cone* of $\bar{J}$, and s is the deformation parameter. The variables x'_i are the *transformed variables*, associated to the generators x_i of $\bar{J}$. These depend on the choice of generators.

We denote by

$$V := V(x'_1, \dots, x'_k)$$

the *vertex* of B .

The algebra $\tilde{R}$ is graded, with x'_i in degree 1 and s in degree -1 . This corresponds to the natural action of the multiplicative group $\mathbb{G}_m$ on B relative to Y , given by

$$t \cdot (s, x'_1, \dots, x'_k) = (t^{-1}s, tx'_1, \dots, tx'_k).$$

This action leaves the vertex invariant, providing an action on $B_+ := B \setminus V$. The action on B_+ is free, since if $x'_i \neq 0$ and $t \neq 1$ we have $tx'_i \neq x'_i$.

We can therefore consider the quotient

$$Bl_{\bar{J}}(Y) := B_+ / \mathbb{G}_m,$$

the *blowup* of Y along $\bar{J}$. This is in fact a variety, as we recall below.

2.2. Classical blowup in concrete terms: old charts. — The construction above, using extended Rees algebras, deformation to the normal cone, and $\mathbb{G}_m$ -actions, is periodically rediscovered every few years, each time shedding new light. I contend that it is the *right* way to introduce blowups, albeit requiring the sophistication of group actions.

Since we did learn another presentation of a blowup in a basic course, let us see that it is the same thing, and at the same time be convinced that $Bl_{\bar{J}}(Y)$ is a variety.

The variety B_+ is covered by the affine opens B_i where $x'_i \neq 0$. For instance

$$B_1 = \operatorname{Spec}_Y \mathcal{O}_Y[s, x'_1, \dots, x'_k, x_1^{-1}] / (x_1 - sx'_1, \dots, x_k - sx'_k).$$

The action of $\mathbb{G}_m$ on B_1 admits a *slice* Y_1 where $x'_1 = 1$, on which $s = x_1$ and $x_j = x_1 x'_j$. In other words $B_i/\mathbb{G}_m \simeq Y_i$ is an affine variety, and the transition from Y to Y_i , and between the different charts Y_i , is given by the equations we learned in a standard algebraic geometry course.

Exercise 2.1. — Show directly that the invariants of the action are given by x_i and $y_i = x'_i/x'_1$, so that

$$B_1 = \operatorname{Spec}_Y \mathcal{O}_Y[y_2, \dots, y_k]/(x_2 - x_1 y_2, \dots, x_k - x_1 y_k)[x'_1, x'^{-1}_1] = \operatorname{Spec}_{Y_1} \mathcal{O}_{Y_1}[x'_1, x'^{-1}_1]$$

and the blowup is given by the standard affine charts

$$Y_1 = \operatorname{Spec}_Y \mathcal{O}_Y[y_2, \dots, y_k]/(x_2 - x_1 y_2, \dots, x_k - x_1 y_k).$$

2.3. Classical blowup in concrete terms: global interpretation. — If Y is general and $\bar{J}$ is an ideal, then

$$\tilde{R} = \bigoplus_{a \in \mathbb{Z}} R_a,$$

where $R_a = \bar{J}^a$ for $a \geq 0$ and $\mathcal{O}_Y$ for $a \leq 0$. In other words, just as the blowup is given as the Proj of the standard Rees algebra of $\bar{J}$, we have given it here a variant quotient construction in terms of the *extended* Rees algebra.

The element s corresponds to the generator 1 of the ideal $\bar{J}_{-1} = \mathcal{O}_Y$ in degree -1 .

Rees actually mainly introduced the extended Rees algebra, and used it to great advantage to study ideals. This continued through history — many of the deeper properties of blowups are studied through the deformation to the normal cone.

2.4. Weighted blowup: the cox construction. — I follow Quek and Rydh [30] and Włodarczyk [36].

Let us now blow up the object $\bar{J} = (x_1^{1/w_1}, \dots, x_k^{1/w_k})$ on the same variety $Y = \operatorname{Spec} k[x_1, \dots, x_n]$. I will say what $\bar{J}$ is — it is no longer an ideal — but first let us see it in action.

The extended Rees algebra is now

$$\tilde{R} := \mathcal{O}_Y[s, x'_1, \dots, x'_k] / (x_1 - s^{w_1} x'_1, \dots, x_k - s^{w_k} x'_k),$$

note the powers of s in the ideal, and denote

$$B := \operatorname{Spec}_Y \tilde{R}.$$

The Y -variety B is now the deformation of Y to the *weighted* normal cone of $\bar{J}$. We still have *transformed variables* x'_i associated to the generators x_i of $\bar{J}$.

We still denote by $V := V(x'_1, \dots, x'_k)$ the *vertex* of B .

The algebra $\tilde{R}$ is graded, with s in degree -1 , but now x'_i have degree w_i . This corresponds to the natural action of the multiplicative group $\mathbb{G}_m$ on B relative to Y , given by

$$t \cdot (s, x'_1, \dots, x'_k) = (t^{-1}s, t^{w_1}x'_1, \dots, t^{w_k}x'_k).$$

This action still leaves the vertex invariant, providing an action on $B_+ := B \setminus V$.

The action on B_+ is no longer free, since if $x'_i \neq 0$ the group μ_{w_i} fixes x'_i and, when $w_i > 1$, always has fixed points where the other x'_j vanish.

It is therefore natural to consider the quotient *stack*

$$Bl_{\bar{J}}(Y) := [B_+/\mathbb{G}_m],$$

the stack-theoretic *weighted blowup* of Y along $\bar{J}$. This is a Deligne–Mumford stack.¹

2.5. What is $\bar{J}$? What is $\tilde{R}$? For the discussion here, which is mostly local, we can view $\bar{J}$ as a *monomial valuation* $v_{\bar{J}}$, where $v_{\bar{J}}(x_i) = w_i$. A global interpretation is that $\bar{J}$ is a $\mathbb{Q}$ -ideal of a particularly nice shape.

A $\mathbb{Q}$ -ideal is a notion that in this discussion is equivalent to Hironaka’s *idealistic exponent* [22]. A $\mathbb{Q}$ -ideal is an object of the form $\mathcal{I}^{1/b}$, an equivalence class given by ideal $\mathcal{I}$ and positive integer b , where $\mathcal{I}_1^{1/b_1}$ is equivalent to $\mathcal{I}_2^{1/b_2}$ if $\mathcal{I}_1^{b_2}$ and $\mathcal{I}_2^{b_1}$ have the same integral closure. One can add $\mathbb{Q}$ -ideals, and $\bar{J} = (x_1)^{1/w_1} + \cdots + (x_k)^{1/w_k}$.

The extended Rees algebra $\tilde{R} = \bigoplus_{a \in \mathbb{Z}} R_a$ is determined by the valuation as follows:

$$R_a = \{f \in \mathcal{O}_Y : v_{\bar{J}}(f) \geq a\}.$$

In these terms $B = \operatorname{Spec}_Y \tilde{R}$, the vertex is $V = V(\bigoplus_{a > 0} R_a)$, and $B_+ = B \setminus V$.

2.6. Deformation to the normal cone: the big picture. — First note that we have a globally defined morphism $B \rightarrow \operatorname{Spec} k[s]$.

Where $s = 1$ we have $x_i = x'_i$ so the fiber is just Y . A similar picture holds when we fix any $s \neq 0$.

Over $s = 0$ we have that $x_i = 0$, so the fiber lies over $Z := V(\bar{J}) \subset Y$. Over a point of Z the variables x'_i are free to vary arbitrarily, in other words the fiber is an affine-space bundle over Z — that’s the weighted normal cone.²

The action of $\mathbb{G}_m$ is a product action over $V \simeq Z \times \mathbb{A}^1$, pulling towards $s = 0$. The action on the normal cone expands x'_i with weight w_i . It is instructive to draw the picture (as I do in my lectures), and the picture remains important later on.

Exercise 2.2. — Draw the picture!

Don’t proceed any further until you either solve the exercise or peek in its solution, see Section 7.1. Ignore the green X , BX and NX as they arrive later in the game.

3. Resolution and invariants

3.1. Resolution. — We are interested in *embedded* resolution of singularities: we have a singular, say irreducible, subvariety $X \subset Y$ where Y is smooth. The procedure we describe is canonical enough so that one can resolve any variety using local embeddings, but I will not discuss this reduction.

An embedded *resolution of singularities* of $X \subset Y$ is a modification $\pi : Y' \rightarrow Y$ where

- Y' is still smooth,
- π is an isomorphism over the generic point of X , and
- the strict transform X' of X is smooth.

¹In positive characteristic it is only a tame algebraic stack.

²In the truly weighted case the fiber over $s = 0$ is not necessarily a vector bundle, but the action of $\mathbb{G}_m$ provides enough structure to work with it.

One important note about my exposition: *I allow Y' , and hence X' , to be a stack.* I hope you will allow me this transgression. I will just note that, if you do want to arrive at a resolution with a variety at the end, the remaining problem is much easier — it has to do with resolving abelian quotient singularities, for which toric methods apply. A more sophisticated approach uses *Bergh's destackification algorithm* [8].

3.2. Invariants. —

- By a *singularity invariant* we mean a rule³ that, given $X \subset Y$ gives a function $\text{inv}_X : |X| \rightarrow \Gamma$, where
 - Γ is a well-ordered set,
 - inv_X is upper-semicontinuous, and
 - $\text{inv}_X(p) = \min \Gamma \iff p \in X$ is nonsingular.

- The invariant is *functorial* if whenever $\phi : Y_1 \rightarrow Y$ is smooth, with $X_1 = \phi^{-1}X$, we have

$$\text{inv}_{X_1}(p_1) = \text{inv}_X(\phi(p_1)).$$

- The invariant is *smooth* if the maximal locus

$$\{p \in X : \text{inv}_X(p) \text{ is maximal}\}$$

in X is smooth.

We note that a maximal value exists since upper-semicontinuity implies that inv is constructible, so by noetherianity of X a maximum exists.

Also note that upper-semicontinuity implies that the maximal locus is closed.

4. Resolution in characteristic 0: why does it work?

I now explain *why* resolution of singularities in characteristic 0 works, following [36].

4.1. Statement. — For a center $\bar{J}$ on Y we defined in Section 2.4 the deformation to the weighted normal cone B .

If $X \subset Y$ is a subvariety, we denote by $BX \subset B$ the proper transform of X : it is the closure in B of $X \times \mathbb{G}_m$ which sits inside $\{s \neq 0\} = Y \times \mathbb{G}_m \subset B$. Its intersection with $\{s = 0\}$ is the normal cone $N_{Z \subset X}$ of Z in X (that is, the weighted normal cone).

Similarly, the *weighted blowup* of X is $Bl_{\bar{J}}(X) := [(BX \setminus V)/\mathbb{G}_m] \subset Bl_{\bar{J}}(Y)$. It is the proper transform of X in $Bl_{\bar{J}}(Y)$. The picture can be viewed in your solution of Exercise 2.2 or in Section 7.1 (and now the green items X, BX, NX should make sense).

Now assume we have a smooth, functorial singularity invariant inv .

Lemma 4.1. —

- (1) Fix $X \subset Y$ and assume we have a center

$$\bar{J} = (x_1^{1/w_1}, \dots, x_k^{1/w_k})$$

on Y which satisfies

³Really a functor, see below.

- (a) $V(\bar{J})$ is the maximal locus of inv_X on X , and
- (b) $\max \text{inv}_X = \max \text{inv}_{BX}$.

Then

$$\max \text{inv}_{Bl_{\bar{J}}(X)} < \max \text{inv}_X .$$

- (2) Assume a center $\bar{J}$ satisfying these conditions exists for every resolution situation $X \subset Y$. Then resolution of singularities holds.

Exercise 4.2. — Try to prove the lemma without looking below.

4.2. Proof of part (2). — We have $\max \text{inv}_{Bl_{\bar{J}}(X)} < \max \text{inv}_X$, and by assumption $X' := Bl_{\bar{J}}(X)$ has its own center and blowup with the same property, giving rise to a sequence of birational morphisms $X^{(k)} \rightarrow \cdots \rightarrow X' \rightarrow X$ where $\max \text{inv}_{X^{(k)}} < \max \text{inv}_{X^{(k-1)}}$. Since Γ is well ordered, this sequence of maximal values terminates, and the process only terminates if $X^{(k)}$ is smooth.

4.3. The maximal locus. — Before we show part (1), a claim. You want to look at the big picture of B you drew in Exercise 2.2.

Claim. — The maximal locus W of inv_{BX} is V .

Proof of claim. — By smoothness of the invariant, W is smooth.

First we have by assumption (a) that the maximal locus of inv_X is Z .

Since the locus in BX where $s \neq 0$ is just $X \times \mathbb{G}_m$, and $\mathbb{G}_m$ is smooth, we have by functoriality that the invariant in this locus coincides with that on the image in X . So the maximal locus of inv_{BX} on the locus $s \neq 0$ is just $Z \times \mathbb{G}_m$.

By assumption (b) the maximal value on this locus coincides with the maximal value on the whole of BX , which is attained along W . Since the maximal locus $W \subset BX$ is closed, it contains V as an irreducible component. Since V and W are smooth,

V is a connected component of W .

Suppose there is a point $b \in N_{Z \subset X} \setminus Z$ where the maximal value is also attained. Again by functoriality, the invariant attains its maximum along the whole orbit $C = \{t \cdot b : t \in \mathbb{G}_m\}$. Since the maximal locus is closed, the same is true of the closure $\bar{C}$ of C , which meets Z at a point b_0 . In other words, V is not a connected component of W , contradicting the highlighted conclusion above.

Thus the claim holds, and the maximal locus of inv_{BX} is V . □

4.4. Proof of part (1). — Continuing with part (1) of the lemma, the claim means that

$$\max_{b \notin V} \text{inv}_{BX} < \max \text{inv}_{BX} = \max \text{inv}_X ,$$

where the last equality is assumption (2) of the lemma used once more.

Note that $B \setminus V \rightarrow Bl_{\bar{J}}(Y)$ is a smooth morphism, being the quotient by a smooth group-scheme. The preimage of $X' = Bl_{\bar{J}}(X)$ is precisely $BX \setminus V$. Using functoriality again, we have that

$$\max \text{inv}_{Bl_{\bar{J}}(X)} = \max_{b \notin V} \text{inv}_{BX} .$$

Combining with the inequality above we obtain

$$\max \operatorname{inv}_{Bl_{\bar{J}}(X)} < \max \operatorname{inv}_X,$$

as required. $\square$

5. Resolution in characteristic 0: how does it work?

5.1. Statement. — From here on we assume our base field has characteristic 0. Here is our main task:

Theorem 5.1 ([6], see also [25]). — *There is a smooth, functorial singularity invariant inv and, for every $X \subset Y$, there is a center $\bar{J}$ satisfying (a) and (b) of Lemma 4.1 (1).*

So by Lemma 4.1 (2) resolution of singularities in characteristic 0 holds true.

The main purpose of this section is therefore to explain *how* resolution of singularities in characteristic 0 works. I hope to provide enough of a picture to convince the reader that all the tools are there. I provide many of the proofs, hoping to make the picture convincing, but I do not see them as a critical part of this exposition. A number of exercises, with some suggestions for their solutions in Section 7, will hopefully guide the reader through some of the material.

5.2. Preview examples. — Before we dive into a series of definitions, examples will show the invariant and center in action.

Example 5.2. — Let $X = V(x^2 - y^2z)$. The most singular point is the origin $V(x, y, z)$, where the variables x, y, z appear in degrees 2, 3, 3 respectively. So the invariant we will define has value $(2, 3, 3)$ at the origin. One writes a corresponding center $J = (x^2, y^3, z^3)$, which gives x valuation $1/2$, and gives y and z valuation $1/3$. Blowups require integer valuations, so we rescale the valuation by a factor of 6, with associated *reduced* center $\bar{J} = (x^{1/3}, y^{1/2}, z^{1/2}) = J^{1/6}$ which we can actually blow up.

On the blowup we have $x = s^3x', y = s^2y', z = s^2z'$. Plugging in, the equation becomes $s^6(x'^2 - y'^2z')$ on B . The proper transform BX is given by $V(x'^2 - y'^2z')$, and indeed its maximal locus is again the locus $V(x', y', z')$. This locus is removed in B_+ , hence the invariant drops.

Exercise 5.3. — Draw the Newton polygon of $X = V(x^2 - y^2z)$ and of the center (x^2, y^3, z^3) . What can you surmise on their relationship? (See Section 7.2.)

Exercise 5.4. — Do your best at drawing a picture of the blowup above, in a way that shows how it resolves the pinch point. (See Section 7.3.)

Example 5.5. — Consider the same X but with the blowup but with $\bar{J} = (x, y, z)$. In this case the proper transform BX is defined by the equation

$$x'^2 - y'^2z's.$$

we notice that at the point where $x' = y' = z' = s = 0$ has these variables appearing in degrees 2, 4, 4, 4 which is larger than the original 2, 3, 3. In other words, it does not satisfy the criterion. In this case, as is well-known, the standard blowup of X has an isomorphic

singularity: for any $z' \neq 0$ the equation above is isomorphic to X , so the invariant stays the same after blowup.

Exercise 5.6. — Draw the Newton polygon of $X = V(x^5 + x^3y^3 + y^{100})$. (See Section 7.4.)

Example 5.7 (Sketch). — $X = V(x^5 + x^3y^3 + y^{100})$. The Newton polygon suggests that we want to give x^5 and x^3y^3 valuation 1. This means $\text{inv}(0) = (5, 7.5)$, $J = (x^5, y^{7.5})$, and rescaling we get $\bar{J} = (x^{1/3}, y^{1/2})$. The proper transform is $x'^5 + x'^3y'^3 + y'^{100}s^{185}$ and the criterion holds. The invariant of the blowup is $(3, 185) < (5, 7.5)$ lexicographically.

Exercise 5.8. — Carry out the example in detail.

Both examples bring about a choice of variables — x, y and z in the first, x and y in the second. A central point of the process below is a good choice of these variables.

5.3. Derivatives. — From here on there is a change of perspective, where we replace $X \subset Y$ by its ideal $\mathcal{I} := \mathcal{I}_X \subset \mathcal{O}_Y$.

Consider an ideal $\mathcal{I} \subset \mathcal{O}_Y$. We write $\mathcal{D}_Y^{\leq a}$ for the sheaf of differential operators of order at most a .

Definition 5.9. — $\mathcal{D}^a(\mathcal{I}) = \{\nabla f : f \in \mathcal{I}, \nabla \in \mathcal{D}_Y^{\leq a}\}$.

In characteristic 0 we have $\mathcal{D}^a(\mathcal{I}) = \mathcal{D}^1 \mathcal{D}^{(a-1)}(\mathcal{I})$. In a sense, the failure of this is the source of trouble in positive characteristics.

Example 5.10. — $\mathcal{D}((x^2 - y^2z)) = (x, yz, y^2)$.

Exercise 5.11. — Compute $\mathcal{D}(x^5 + x^3y^3 + y^{100})$.

5.4. Order. —

Definition 5.12. — $\text{ord}_{\mathcal{I}}(p) = \min\{a : (\mathcal{D}^a(\mathcal{I}))_p = \mathcal{O}_{Y,p}\}$ (and the order of the zero ideal is ∞).

Note: $\text{ord}_{\mathcal{I}}(p) \geq a \iff p \in V(\mathcal{D}^{a-1}(\mathcal{I}))$.

This implies:

Observation. — The function $\text{ord}_{\mathcal{I}}$ is upper semicontinuous.

Note also that

Observation. — The function $\text{ord}_{\mathcal{I}}$ is functorial for smooth morphisms.

This follows because derivatives are compatible with étale morphisms; up to étale morphisms a smooth morphism is like a product with affine space; and $\text{ord}_{\mathcal{I}}$ is compatible with a product with affine space by direct computation.

Exercise 5.13. — Find the order of $x^2 - y^2z$ and of $x^5 + x^3y^3 + y^{100}$ at the origin.

The order is our first step towards a singularity invariant. In Section 5.5 we build a parallel step towards a blowup center.

The order cannot suffice: while $\text{ord}_{(xy,z)}(0, 0, 0) = 1$, the zero locus $xy = z = 0$ is singular.

5.5. Maximal contact and the question of induction. —

Definition 5.14 ([18]). — Consider a nonzero ideal $\mathcal{I}$ with $\text{ord}_{\mathcal{I}}(p) = a > 0$. An element $x \in \mathcal{O}_p$ is a *maximal contact element*, and its vanishing locus $H = V(x)$ is a *maximal contact hypersurface* at p , if $x \in \mathcal{D}^{a-1}(\mathcal{I})_p$ and $\mathcal{D}^1(x) = \mathcal{O}_p$.

Lemma 5.15. — *In characteristic 0, a maximal contact element exists.*

Indeed, $\mathcal{O}_p = \mathcal{D}^a(\mathcal{I}) = \mathcal{D}^1(\mathcal{D}^{a-1}(\mathcal{I}))$ so there is $x \in \mathcal{D}^{a-1}(\mathcal{I})$ with $\mathcal{D}^1(x) = \mathcal{O}_p$.

Remark 5.16. — A maximal contact element does not necessarily exist in characteristic p : R. Narasimhan [27] gave the example of a hypersurface

$$V(x^2 + yz^3 + zw^3 + y^7w) \subset \mathbb{A}^4$$

in characteristic 2 with a curve of singular points having 4-dimensional tangent space at the origin. You can still resolve it, see [19]. See [28] for examples in any positive characteristic. Hironaka's replacement for maximal contact ($x = 0$ in this case) is a hypersurface that realizes the characteristic polyhedron, but I digress.

Exercise 5.17. — Find at least two maximal contact elements for each of $x^2 - y^2z$ and of $x^5 + x^3y^3 + y^{100}$ at the origin.

Note that $x \in \mathcal{D}^{a-1}(\mathcal{I})$ implies that

$$V(x) \supset V(\mathcal{D}^{a-1}(\mathcal{I})) = \{p : \text{ord}_{\mathcal{I}}(p) \geq a\}.$$

So:

Observation. — *A maximal contact hypersurface is a smooth hypersurface which schematically includes all nearby points of order a .*

This observation implies that a maximal contact hypersurface $\{x = 0\}$ is a candidate for induction on dimension, as it contains the worst locus in terms of order. It will provide a good choice for the first variable $x_1 = x$ appearing in a center, and the center itself will start with x_1^a with $a = \text{ord}_{\mathcal{I}}(p)$.

But what subscheme should we consider on $\{x = 0\}$?

The simplest idea would be to restrict $\mathcal{I}$ to $\{x = 0\}$, but the plane curve example $x^5 + x^3y^3 + y^{100}$ shows that this is too simple: $\mathcal{I}|_{V(x)} = (y^{100})$, which entirely misses the term x^3y^3 . Blowing up the center (x^5, y^{100}) makes the singularities worse — try it!

The vertex x^3y^3 of the Newton polyhedron does leave a mark when we take the ideal $\mathcal{D}^3(\mathcal{I})$ and restrict to $\{x = 0\}$, giving the ideal (y^3) . More generally, a vertex of the form x^cy^d , with $c < \text{ord}_{\mathcal{I}}(p)$, leaves a mark when we take the ideal $\mathcal{D}^c(\mathcal{I})$ and restrict to $\{x = 0\}$.

This suggests that, instead of taking the ideal $\mathcal{I}$ and restricting it to $V(x)$, we should consider all the ideals $\mathcal{D}^c(\mathcal{I})$, $c < a$ and restrict them, somehow keeping the derivative c in mind. This approach, using Rees algebras, is pursued in [36] and in [4]. The classical approach is to combine these ideals in one ideal — the coefficient ideal — which we now introduce.

5.6. Coefficient ideals. — Consider an ideal $\mathcal{I}$ and an integer a (which usually stands for $a = \text{ord}_{\mathcal{I}}(p)$). We consider the ideals $\mathcal{I}, \mathcal{D}^1(\mathcal{I}), \dots, \mathcal{D}^{a-1}(\mathcal{I})$ with corresponding weights (standing for expected orders at p) $a, a-1, \dots, 1$.

Definition 5.18 ([24]). — The coefficient ideal of $\mathcal{I}$ at p is

$$C(\mathcal{I}, a) := \sum_{\sum ib_i \geq a!} \prod_{i=0}^{a-1} (\mathcal{D}^{a-i}(\mathcal{I}))^{b_i}.$$

Remark 5.19. — Notice that if $\text{ord}_{\mathcal{I}}(p) = a$ then $\text{ord}_{C(\mathcal{I}, p)}(p) = a!$, so there is a rescaling factor of $(a-1)!$ to be taken into account. The canonical objects to be considered are $\mathbb{Q}$ -ideals, in which case one considers $\mathcal{I}$ which has order a along with $\mathcal{D}^{a-i}(\mathcal{I})^{a/i}$ also with order a , and then

$$C(\mathcal{I}, a)^{1/(a-1)!} = \sum_{i=1}^a \mathcal{D}^{a-i}(\mathcal{I})^{a/i}$$

also has order a . We will not use this canonical formalism here. It is used in a paper of Bierstone and Milman [10, §3.3]. Its associated Rees algebra is used in Włodarczyk's treatment [36] and in our foliations paper [4].

The coefficient ideal takes into account all possible terms that should contribute to the restriction of $\mathcal{I}$ to any maximal contact hypersurface. If x is a maximal contact element, we write $\mathcal{I}[x] := C(\mathcal{I}, a)|_{V(x)}$.

Exercise 5.20. — Let $\mathcal{I} = (f)$ be principal, and $f = x^a + c_1x^{a-1} + c_2x^{a-2} + \dots + c_a$. Show that when $c_1 = 0$ the element x is a maximal contact. In this case show that $c_i \in \mathcal{D}^{a-i}(\mathcal{I})|_{V(x)}$, and that a suitable power contributes to $\mathcal{I}[x]$. (See Section 7.5.)

Exercise 5.21. — Compute $C(\mathcal{I}, a)$ and $\mathcal{I}[x]$ in the example $x^2 - y^2z$ and $\mathcal{I}[x]$ for $x^5 + x^3y^3 + y^{100}$ at the origin. (See Section 7.6.)

5.7. Invariant and center. — We can now define both an invariant and a center inductively, assuming an invariant $\text{inv}_{\mathcal{I}}(p)$ and center $J_{\mathcal{I}}$ were defined in dimension $\dim(Y) - 1$.

Definition 5.22. — Suppose $\text{ord}_{\mathcal{I}}(p) = a_1$ with maximal contact element x_1 . Suppose

$$\text{inv}_{\mathcal{I}[x_1]} = (b_2, \dots, b_k)$$

with center

$$J_{\mathcal{I}[x_1]} = (\bar{x}_2^{b_2}, \dots, \bar{x}_k^{b_k}).$$

Keeping in mind the rescaling factor of Remark 5.19, set

$$a_i = b_i / (a_1 - 1)!$$

and let $x_2, \dots, x_k$ be arbitrary elements restricting to $\bar{x}_i$ on $V(x_1)$. Define

$$\text{inv}_{\mathcal{I}}(p) := (a_1, a_2, \dots, a_k)$$

and

$$J_{\mathcal{I}} := (x_1^{a_1}, x_2^{a_2}, \dots, x_k^{a_k}).$$

Exercise 5.23. — Compute the invariant and center in the examples $x^2 - y^2z$ and of $x^5 + x^3y^3 + y^{100}$ at the origin. (See Section 7.7.)

Exercise 5.24. — What should one do if $\mathcal{I}$ is a unit ideal? What should one do if $\mathcal{I}[x_1] = 0$?

We order invariants lexicographically, where a truncation of a sequence is larger: $(2) > (2, 3)$. The set of values it takes is easily shown to be well-ordered — while the entries are rational, their denominators are bounded in terms of the earlier entries.

Exercise 5.25. — Verify this last statement.

Remark 5.26. — There is one property of the invariant and center I do not address here — namely that they are well-defined. The only issue is the choice of a maximal contact element x_1 and the lifts x_i of $\bar{x}_i$. Proposition 5.29 below gives a reason why it is well defined.

5.8. Properties of the invariant. — For ideals, “resolution” is replaced by a sequence of weighted blowups so that the proper transform (or more precisely the controlled transform) is the unit ideal — this is called “principalization”. There is a reduction of embedded resolution to this task, which I do not discuss here. Also, Lemma 4.1 has an evident analogue for ideals, for which the following is relevant:

Proposition 5.27. — *The function $\text{inv}_{\mathcal{I}}$ is a smooth functorial singularity invariant.*

Proof. —

The invariant $\text{inv}_{\mathcal{I}}$ is a “singularity invariant” for ideals. — We have already noted that the value set of the invariant is well-ordered.

We show $\text{inv}_{\mathcal{I}}$ is upper semicontinuous. Note that $a_1 = \text{ord}_{\mathcal{I}}$ is upper semicontinuous, and its maximum locus is contained in $V(x_1)$. By induction $\text{inv}_{\mathcal{I}[x_1]}$ is upper semicontinuous on $V(x_1)$. It follows that $\text{inv}_{\mathcal{I}}$ is upper semicontinuous.

The invariant’s minimum value is (0) taken only for the unit ideal. For resolution of singularities, the invariant of the ideal of a subvariety of codimension d takes its minimum value $(1, \dots, 1)$, with d entries, at smooth points of X .

The invariant $\text{inv}_{\mathcal{I}}$ is functorial. — We have seen that $\text{ord}_{\mathcal{I}}$ is functorial for smooth morphisms. Let us now choose x_1 — it remains maximal contact when pulled back by a smooth morphism. By induction $\text{inv}_{\mathcal{I}[x_1]}$ is functorial for smooth morphisms, hence $\text{inv}_{\mathcal{I}}$ is functorial.

The maximal locus is smooth. — The maximal locus of $\text{ord}_{\mathcal{I}}$ lies inside $V(x_1)$. Inductively the maximal locus of $\text{inv}_{\mathcal{I}[x_1]}$ is $V(x_2, \dots, x_k)$. Hence the maximal locus of $\text{inv}_{\mathcal{I}}$ is $V(x_1, x_2, \dots, x_k)$, which is smooth. $\square$

5.9. Admissibility. — A center J is admissible for an ideal $\mathcal{I}$ if $v_J(f) \geq 1$ for all $f \in \mathcal{I}$.

Lemma 5.28. — *The center $J_{\mathcal{I}}$ is admissible for $\mathcal{I}$. More generally, if $f \in \mathcal{D}^1(\mathcal{I})$ then $v_J(f) \geq 1 - i/a_1$, and if $f \in C(\mathcal{I}, a_1)$ then $v_J(f) \geq (a_1 - 1)!$*

To see this, expand an element $f \in \mathcal{I}$ as

$$f = c_0 + c_1 x_1 + \dots + c_{a_1-1} x_1^{a_1-1} + c_{a_1} x_1^{a_1} + \dots,$$

where

$$c_i = \frac{1}{i!} \left(\frac{\partial^i f}{\partial x_1^i} \right)_{x_1=0} \in \mathcal{D}^i(\mathcal{I})|_{V(x_1)},$$

and use induction again.

The following is the real reason the center is well-defined:

Proposition 5.29. — *J is the unique center of maximal invariant which is admissible for $\mathcal{I}$.*

I will not go into the proof of this, though it is not difficult.

Remark 5.30. — One may define an invariant in positive characteristic in an analogous manner, by taking the unique center of maximal invariant which is admissible for $\mathcal{I}$. This leads to an invariant which does not satisfy 4.1(a) — this fails for Whitney’s umbrella in characteristic 2 — neither is it a smooth invariant, as one can see from the system $(x^2 - y^2z, z^{100})$. These issues are again raised in [36].

5.10. Reduced center and transform. — The center $J_{\mathcal{I}} := (x_1^{a_1}, x_2^{a_2}, \dots, x_k^{a_k})$ is not in the form we used for weighted blowups. There is, however, a positive integer ℓ such that

$$\bar{J}_{\mathcal{I}} := J_{\mathcal{I}}^{1/\ell} = (x_1^{1/w_1}, \dots, x_k^{1/w_k}),$$

with w_i positive integers. We use this rescaled center $\bar{J}_{\mathcal{I}}$ to prove Theorem 5.1.

We note that the associated monomial valuations satisfy $v_{\bar{J}} = \ell \cdot v_J$.

We have already shown condition 4.1(a) above. In order to consider the remaining condition 4.1(b), we need to look at the transform of $\mathcal{I}$.

Recall the change of variables $x_i = s^{w_i} x'_i$ for $i = 1, \dots, k$. Write $\sigma : B \rightarrow Y$.

We have that if $f(x_1, \dots, x_n) \in \mathcal{I}$ then $\sigma^* f = f(s^{w_1} x'_1, \dots, s^{w_k} x'_k, x_{k+1}, \dots, x_n)$. It follows from admissibility that $\sigma^* f = s^\ell f'$ for some regular function f' — use that $a_i w_i = \ell$.

If instead $f \in \mathcal{D}^i(\mathcal{I})$ then we can always write $\sigma^* f = s^{\ell - i w_1} f'$ for some regular function f' — we see why below. This implies a similar result for the coefficient ideal, where ℓ is replaced by $(a - 1)! \cdot \ell$.

Exercise 5.31. — Verify these statements on the transform of $x^2 - y^2z$ by the center $\bar{J}_{\mathcal{I}} = (x^{1/3}, y^{1/2}, z^{1/2})$, and of $x^5 + x^3y^3 + y^{100}$ with center $\bar{J}_{\mathcal{I}} = (x^{1/3}, y^{1/2})$. (See Section 7.8.)

5.11. Transforms of derivatives and coefficient ideals. — For an ideal $\mathcal{I}$ denote by $\mathcal{I}'$ its controlled transform $\{f' : f \in \mathcal{I}\}$.

Lemma 5.32 (See [14, Lem. 4.5], [35, Lem. 2.6.2], [10, Lem. 3.3]). — $(\mathcal{D}(\mathcal{I}))' \subset \mathcal{D}(\mathcal{I}')$.

Exercise 5.33. — Verify the lemma on the transform of $x^2 - y^2z$ by the center $\bar{J}_{\mathcal{I}} = (x^{1/3}, y^{1/2}, z^{1/2})$, and of $x^5 + x^3y^3 + y^{100}$ with center $\bar{J}_{\mathcal{I}} = (x^{1/3}, y^{1/2})$. (See Section 7.9.)

Proof of lemma. — It suffices to show that $(\mathcal{D}(\mathcal{I}))' \subset \mathcal{D}_{B/\mathbb{A}^1}(\mathcal{I}')$.

As indicated above, $\mathcal{I}' = (s^{-\ell} f(s^{w_j} x'_j) : f \in \mathcal{I})$.

It follows that $\mathcal{D}_{B/\mathbb{A}^1}(\mathcal{I}') = (\frac{\partial}{\partial x'_i} (s^{-\ell} f(s^{w_j} x'_j)) : f \in \mathcal{I}, i = 1, \dots, n)$, and applying the chain rule this equals $(s^{-\ell} \frac{\partial f}{\partial x_i} (s^{w_j} x'_j) \cdot s^{w_i})$, giving

$$\mathcal{D}_{B/\mathbb{A}^1}(\mathcal{I}') = \left(s^{-\ell + w_i} \frac{\partial f}{\partial x_i} (s^{w_j} x'_j) \right).$$

On the other hand $\mathcal{D}(\mathcal{I}) = \left(\frac{\partial f}{\partial x_i} (x_i) \right)$ giving

$$\mathcal{D}(\mathcal{I})' = \left(s^{-\ell + w_i} \frac{\partial f}{\partial x_i} (s^{w_j} x'_j) \right).$$

Note that $w_1 \geq w_i$ for all i , so

$$\mathcal{D}(\mathcal{I})' = \left(s^{-\ell+w_1} \frac{\partial f}{\partial x_i}(s^{w_j} x'_j) \right) \subset \left(s^{-\ell+w_i} \frac{\partial f}{\partial x_i}(s^{w_j} x'_j) \right) = \mathcal{D}_{B/\mathbb{A}^1}(\mathcal{I}'),$$

as needed. $\square$

Proposition 5.34. — $\mathcal{I}'^{(a_1-1)!} \subset C(\mathcal{I}, a_1)' \subset C(\mathcal{I}', a_1)$.

Proof. — The inclusion $\mathcal{I}^{(a_1-1)!} \subset C(\mathcal{I}, a_1)$ implies an inclusion of transforms $\mathcal{I}'^{(a_1-1)!} \subset C(\mathcal{I}, a_1)'$. Lemma 5.32 above implies the second inclusion. $\square$

Exercise 5.35. — Verify the proposition on the transform of $x^2 - y^2 z$ by the center $\bar{J}_{\mathcal{I}} = (x^{1/3}, y^{1/2}, z^{1/2})$, and of $x^5 + x^3 y^3 + y^{100}$ with center $\bar{J}_{\mathcal{I}} = (x^{1/3}, y^{1/2})$.

We note that the invariant and center of $\mathcal{I}'^{(a_1-1)!}$ equals that of $C(\mathcal{I}', a_1)$. Proposition 5.34 says that this coincides with the invariant of $C(\mathcal{I}, a_1)'$. In other words: to check the resolution criterion for $\mathcal{I}$ it suffices to check it for $C(\mathcal{I}, a_1)$.

This allows us to conflate $\mathcal{I}$ with its coefficient ideal. I'll ignore the rescaling factor that this requires. After all, I am telling you *how* it works, the complete proofs are detailed elsewhere.

5.12. Order and maximal contact element on B . —

Claim. — $\max_B \text{ord}_{\mathcal{I}'} = a_1$ and wherever this is attained, x'_1 is a maximal contact element.

For this discussion we assume $\mathcal{I}$ is already a coefficient ideal, so in particular $\partial_{x_1}^{a_1}(\mathcal{I}) = \mathcal{O}$ and $x_1 \in \partial_{x_1}^{a_1-1}(\mathcal{I})$. This is harmless by the previous proposition. (It can also be arranged by a change of variables.)

This means that there is an element $f \in \mathcal{I}$ such that $x_1 = \partial_{x_1}^{a_1-1}(f)$ and so $\partial_{x_1}^{a_1}(f) = 1$.

Plugging in we obtain that $\partial_{x'_1}^{a_1}(s^{-\ell} f(s^{w_i} x'_i)) = 1$, since $a_1 w_1 = \ell$. This means that the order everywhere is at most a_1 . Where it is, we indeed get $x_1 = s^{-\ell+w_1} \partial_{x'_1}^{a_1-1}(f(s^{w_i} x'_i))$ so $x'_1 = \partial_{x'_1}^{a_1-1}(s^{-\ell} f(s^{w_i} x'_i))$. So it is a maximal contact element.

Exercise 5.36. — Verify the claim on the transform of $x^2 - y^2 z$ by the reduced center $\bar{J}_{\mathcal{I}} = (x^{1/3}, y^{1/2}, z^{1/2})$. Verify the statement *on B* for $x^5 + x^3 y^3 + y^{100}$, and moreover show that the maximal order *on B_+* of the transform of $x^5 + x^3 y^3 + y^{100}$ with center $\bar{J}_{\mathcal{I}} = (x^{1/3}, y^{1/2})$ is < 5 . (See Section 7.10.)

We note that for plane curves the order always drops *on B_+* , equivalently *on the blowup*, as shown in [2, Thm. 1.1].

5.13. The invariant of $\mathcal{I}'$. —

Claim. — $\max_B \text{inv}_{\mathcal{I}'} = (a_1, \dots, a_k)$

We note that this implies 4.1(b), hence resolution of singularities.

As we have seen, the maximal order is a_1 and it is attained within the hypersurface $H' = \{x'_1 = 0\}$, since it is maximal contact. Note that H' is the degeneration to the normal cone of $H = \{x_1 = 0\}$, with center $\bar{J}_H = (x_2^{1/w_2}, \dots, x_k^{1/w_k})$.

We can now apply induction, so that, ignoring annoying rescaling factors, $(a_2, \dots, a_k)$ is the maximal invariant of $\mathcal{I}'[x'_1]$ with center $J_{H'} = (x_2^{a_2}, \dots, x_k^{a_k})$, so by definition the maximal invariant if $\mathcal{I}'$ is indeed $(a_1, \dots, a_k)$, with center $(x_1^{a_1}, \dots, x_k^{a_k})$.

We are done!

6. Enter foliations

6.1. Resolution of foliations. — A *foliation* is a coherent subsheaf $\mathcal{F} \subset \mathcal{D}_X$ which is closed under Lie brackets. We also consider *logarithmic foliations* $\mathcal{F} \subset \mathcal{D}_X^{\log}$, for instance when X carries a normal crossings divisor. A foliation is *smooth* if it is a subbundle of $\mathcal{D}_X$, and *logarithmically smooth* if it is a subbundle of $\mathcal{D}_X^{\log}$.

One major question is that of resolution of singularities of foliations, sometimes called “reduction of singularities” since smoothness is rarely achievable.

The question remains open in dimension ≥ 4 . There are contributions in the CIRM workshop which tell us about the history (I need to study it myself). A few highlights: Bendixson [7] and Seidenberg [32] showed that rank-1 foliations on a smooth surface can be reduced by point blowups to singularities now known as *simple* singularities. Cano [11] showed the analogous result for corank-1 foliations in dimension 3. Panazzolo [29] and McQuillan–Panazzolo [26] settled the case of rank-1 foliations in dimension 3 using weighted blowups.

Relative logarithmic resolution of singularities is shown in [1, Thm. 2.1], [5, Thm. 1.2.12]. This implies in particular log resolution of a foliation of the form $\mathcal{D}_{X/B}$.

Log smoothness after blowups is impossible already in 2-dimensional examples, as one sees with the foliation spanned by $(x + y)\partial_x + y^2\partial_y$.

6.2. Resolution inside foliated manifolds. — All results I am involved in from here on are in [4].

We focus on embedded resolution: say $(Y, \mathcal{F})$ is a foliated manifold and $X \subset Y$ is a subvariety. What can we do to resolve X in a way that respects $\mathcal{F}$?

There are two opposing cases of interest:

Theorem 6.1. — *Assume that X is $\mathcal{F}$ -invariant, namely $\mathcal{F}(\mathcal{I}_X) \subset \mathcal{I}_X$. Then the functorial resolution of X described in previous sections is $\mathcal{F}$ -invariant: the foliation $\mathcal{F}$ lifts in the resolution process to a foliation $\mathcal{F}'$ on Y' , and X' is $\mathcal{F}'$ -invariant.*

Indeed, in this case one can think of generators of $\mathcal{F}$ as infinitesimal automorphisms of Y preserving X , and by functoriality of resolution (in the formal or analytic setting) it is equivariant for such group actions. This was used by people using Hironaka’s resolution, for instance Kawamata [23, Lem. 3.6].

The following seems new:

Theorem 6.2. — *Assume that X is generically transverse to $\mathcal{F}$. Then there is a sequence of $\mathcal{F}$ -aligned blowups, with resulting $X' \subset Y', \mathcal{F}'$, such that X' is transverse to $\mathcal{F}'$.*

This brings up a question: what is an aligned blowup? Again there are two extreme cases: if the center J is $\mathcal{F}$ -invariant then it is $\mathcal{F}$ -aligned. But also important is the case when the x_i are transverse of $\mathcal{F}$. An $\mathcal{F}$ -aligned center is a mixture of the two: on formal completion it can be written as $(x_1^{a_1}, \dots, x_k^{a_k}, y_1^{b_1}, \dots, y_k^{b_k})$, where x_i are transverse and the partial center $(y_1^{b_1}, \dots, y_k^{b_k})$ is invariant.

The two theorems above are consequences of the following:

Theorem 6.3. — *For $X \subset Y, \mathcal{F}$ as above, there is a functorial sequence of $\mathcal{F}$ -aligned blowups, arriving at $X' \subset Y', \mathcal{F}'$, so that X' is smooth and $\mathcal{I}_X$ is an $\mathcal{F}'$ -aligned center itself.*

How do we prove this? We construct an invariant and a center, and show that the criterion 4.1 applies. One starts with defining the $\mathcal{F}$ -order, the analogue of order but only using derivatives from $\mathcal{F}$. If finite, one defines $\mathcal{F}$ -maximal contact and $\mathcal{F}$ -coefficient ideals as before. If the $\mathcal{F}$ -order is infinite, namely $\mathcal{I}$ is $\mathcal{F}$ -invariant, one reverts to usual derivatives, but with $\mathcal{F}$ -invariant being $\infty + \text{inv}_{\mathcal{I}}$, see examples below. At the end this all works nicely.

We compute transforms of ideals and of foliations on B_+ rather than on X' . Specifically for foliations, we work with $\mathcal{D}_{B_+/\mathbb{A}^1}^{\log}$, which provides descent to $\mathcal{D}_{Y'/\mathcal{A}^1}^{\log} = \mathcal{D}_{Y'}^{\log}$, where $\mathcal{A} = [\mathbb{A}^1/\mathbb{G}^m]$. One point to be careful about: even when $\mathcal{F}' \subset \mathcal{D}_{B_+/\mathbb{A}^1}$ is smooth, the corresponding foliation on Y' may be only logarithmically smooth.

6.3. Examples of foliated principalization. —

Example 6.4 (For Theorem 6.2). — Let $X = V(y^3 - x^2)$ and $\mathcal{F} = \text{Span}(\partial_y)$. We define the $\mathcal{F}$ -order in an analogous way to the order, using only $\mathcal{F}$. Then the $\mathcal{F}$ -order at the origin is 3 with $\mathcal{F}$ -maximal contact y . We have $\mathcal{I}[y] = (x^4)$ with rescaling factor $(3 - 1)! = 2$. Its $\mathcal{F}$ -order is infinite, and its $\mathcal{D}$ -order is 4, which rescales down to $4/2 = 2$. We therefore assign $\mathcal{F}$ -invariant $(3, \infty + 2)$ with $\mathcal{F}$ -center (y^3, x^2) . The blowup has equation $(y'^3 - x'^2)$ which is transverse to the transformed foliation $\mathcal{F}' = \text{Span}(\partial_{y'})$ as we have removed $x' = y' = 0$.

Example 6.5 (For Theorem 6.1). — Let $X = V(y^3 - x^2)$ and $\mathcal{F} = \text{Span}(2y\partial_y + 3x\partial_x)$. Since X is $\mathcal{F}$ -invariant the $\mathcal{F}$ -order is ∞ . As the usual invariant is $(2, 3)$ with center (x^2, y^3) we assign $\mathcal{F}$ -invariant $(\infty + 2, \infty + 3)$ still with center (x^2, y^3) . The blowup X' is regular, has equation $(y'^3 - x'^2)$ with $x' = y' = 0$ removed, and is invariant under the transformed foliation $\mathcal{F}' = \text{Span}(2y'\partial_{y'} + 3x'\partial_{x'})$.

Exercise 6.6. — The examples are computed on B_+ . Compute the transforms on étale charts for the blow-up.

6.4. Preservation and reduction in good foliation classes. —

Let us look back at the problem of reducing the singularities of a foliation.

Theorem 6.7. — *The transform under an $\mathcal{F}$ -aligned blowup of a log smooth foliation $\mathcal{F}$ is log smooth. The restriction of a log smooth foliation $\mathcal{F}$ to the support of an $\mathcal{F}$ -aligned center not contained in the logarithmic boundary is log smooth.*

There are further classes of foliations that are preserved under aligned blowups, for instance Belotto's class of *monomial foliations*.

Theorem 6.8. — *The transform under an $\mathcal{F}$ -aligned blowup of a monomial foliation $\mathcal{F}$ is monomial. The restriction to the support of an $\mathcal{F}$ -aligned center of a monomial foliation $\mathcal{F}$ is monomial.*

This has the following implication: a class $\mathcal{C}$ of saturated foliations that is stable under aligned blowups, log smooth products, and restrictions as above is said to be *thick*.

A $\mathcal{C}$ -reduction of a foliation $\mathcal{F}$ is a modification $X' \rightarrow X$ such that the saturated pullback $\mathcal{F}'$ is in $\mathcal{C}$.

Theorem 6.9. — *Say $\mathcal{F}$ is the saturated preimage under a dominant rational map of a foliation in a thick class $\mathcal{C}$. Then $\mathcal{F}$ admits a $\mathcal{C}$ -reduction. In particular,*

- *A totally integrable foliation has a log smooth and monomial reduction of singularities.*
- *A Darboux-integrable foliation has a log smooth and monomial reduction of singularities.*

Sketch of proof. — Consider the graph embedding $X \hookrightarrow X \rightarrow B = Y$ with the foliation-pullback $\mathcal{H} := \pi_B^* \mathcal{G}$. Note that $\mathcal{H}$ is in $\mathcal{C}$. Its restriction to X is $\mathcal{F}$. Consider the embedded resolution of X in the foliated manifold $(Y, \mathcal{H})$. The result is $X' \subset Y'$, $\mathcal{H}'$ such that $\mathcal{H}'$ is still in $\mathcal{C}$ and X itself is a, $\mathcal{H}'$ -aligned center. So the restriction $\mathcal{F}'$ is in $\mathcal{C}$.

Now a totally integrable foliation is the foliation-pullback of the zero foliation, which is both log smooth and monomial. A Darboux-integrable foliation is by definition the foliation pullback of a monomial foliation. One shows that a monomial foliation has a modification that is both log smooth and monomial. $\square$

This makes it interesting to investigate the following:

Problem 6.10. — *Show that the transform under an $\mathcal{F}$ -aligned blowup of a log canonical foliation $\mathcal{F}$ is log canonical.*

Example 6.11. — Let $\mathcal{F} = \text{Span}(3y^2\partial_x - 2x\partial_y)$. This is the relative foliation $\mathcal{D}_{\mathbb{A}^2/\mathbb{A}^1}$ of the map $\mathbb{A}^2 \xrightarrow{y^3+x^2} \mathbb{A}^1$, hence it is the foliation pullback of the zero foliation. As the zero foliation is log smooth, we claim that we can transform it to a log smooth foliation.

The foliation $\mathcal{F}$ is thus the restriction via the graph $(x, y) \mapsto (x, y, y^3 + x^2)$ of the foliation $\mathcal{H} = \text{Span}(\partial_x, \partial_y)$ on the xyz -space $\mathbb{A}^3$.

The ideal of the graph is $(z - y^3 - x^2)$, with $\mathcal{H}$ -ord $_{\mathcal{I}} = 2$ at the origin. The restricted coefficient ideal is $\mathcal{I}[x] = (z - y^3)$ with $\mathcal{H}[x]$ -ord $_{\mathcal{I}[x]} = 3$ at the origin and $\mathcal{H}[x]$ -maximal contact y . Finally the remaining coefficient ideal is (z^2) with rescaling factor 2, having infinite $\mathcal{H}[x, y]$ -order and order 2. Rescaling back, we obtain the invariant $(2, 3, \infty + 1)$ with center (x^2, y^3, z) .

The weighted blowup has equation $(z' - y'^3 - x'^2)$ on the $\mathbb{G}_m$ -chart B_+ , where the origin is excluded. It is transversal to the transformed foliation

$$\mathcal{H}' = \text{Span}(\partial_{x'}, \partial_{y'}) \subset \mathcal{D}_{B_+/\mathbb{A}_s^1} = \mathcal{D}_{B_+/\mathbb{A}_s^1}^{\log}.$$

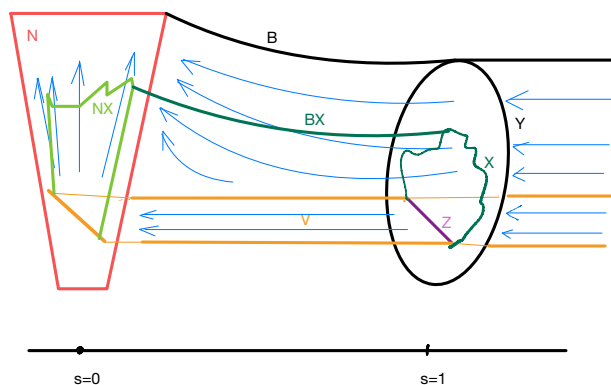
The resulting closed subscheme is the corresponding weighted blowup of the plain. Again on the chart B_+ the restricted foliation $\mathcal{F}' = \text{Span}(3y'^2\partial_{x'} - 2x'\partial_{y'})$ is nonsingular. This implies that it is logarithmically smooth on X' .

Exercise 6.12. — Explicitly compute the resulting foliation on étale charts for the weighted blowup.

The examples and exercises show in no uncertain terms that the correct place to compute transforms is B_+ .

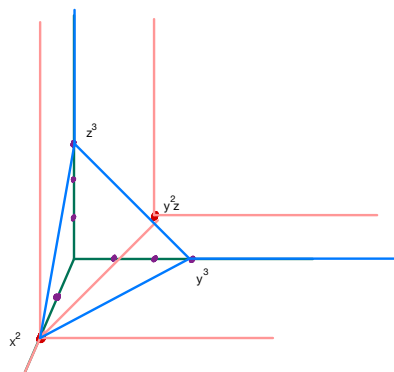
7. Suggestions for selected exercises

7.1. Exercise 2.2. —

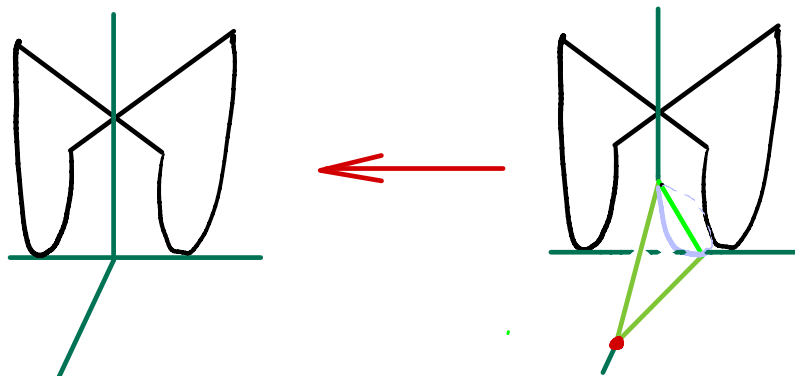


(See [3].)

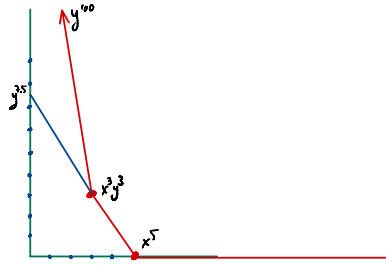
7.2. Exercise 5.3. — The Newton polyhedron of $x^2 - y^2z$ in pink, the center (x^2, y^3, z^3) behind in blue.



7.3. Exercise 5.4. — The red dot is the μ_3 point $s = z' = y' = 0$. The green triangle is the exceptional weighted projective plane $s = 0$. The light green line is the μ_2 -locus $s = x' = 0$. The purple curve is the intersection of the umbrella with the exceptional divisor — the μ_2 action identifies it with its mirror image, drawn with a lighter touch.



7.4. Exercise 5.6. —



7.5. Exercise 5.20. — Consider the derivative $\partial^{a-1}(f)/(\partial x)^{a-1} = a! \cdot x$. It shows that x is indeed a maximal contact element. Now $\partial^{a-2}(f)/(\partial x)^{a-2} = (a!/2)x^2 + (a-2)!c_2$, so c_2 appears in $\mathcal{D}^{a-2}(\mathcal{I})|_{V(x)}$.

7.6. Exercise 5.21. — Since $\mathcal{D}(x^2 - y^2z) = (x, y^2, yz)$ we have $C(\mathcal{I}, 2) = (x^2, y^2z, y^4)$ and $\mathcal{I}[x] = (y^2z, y^4)$. The coefficient ideal of the second equation has several terms, but to compute $\mathcal{I}[x]$ we may ignore x terms, so we get $((y^3)^{120/2}) = (y^{180})$.

7.7. Exercise 5.23. — For $x^2 - y^2z$ we have $x_1 = x$ and $\mathcal{I}[x] = (y^2z, y^4)$. This ideal has order 3 with two independent maximal contacts y, z , so, to make a long story short, we have $\text{inv}_{\mathcal{I}[x]} = (3, 3)$ with center $\mathcal{J}_{\mathcal{I}[x]} = (y^3, z^3)$. The rescaling factor is $(2-1)! = 1$ so we have $\text{inv}_{\mathcal{I}}(p) = (2, 3, 3)$ with center (x^2, y^3, z^3) .

For $x^5 + x^3y^3 + y^{100}$ we have $\mathcal{I}[x] = (y^{180})$, with invariant (180) and center (y^{180}) . The rescaling factor is $4! = 24$, with $180/24 = 15/2$. Therefore $\text{inv}_{\mathcal{I}}(p) = (5, 15/2)$ with center $J_{\mathcal{I}} = (x^5, y^{15/2})$.

7.8. Exercise 5.31. — For $x^2 - y^2z$, we computed $J = (x^2, y^3, z^3)$ and $\bar{J}_{\mathcal{I}} = (x^{1/3}, y^{1/2}, z^{1/2})$, so that $\ell = 6$. We have $\sigma^*(x^2 - y^2z) = (s^3x')^2 = (s^2y')^2(s^2z') = s^6(x'^2 + y'^2z') = s^\ell f'$ as needed. Considering $\mathcal{D}(\mathcal{I}) = (x, y^2, yz)$ we have similarly that $\sigma^*\mathcal{D}(\mathcal{I}) = (s^3x', s^4y'^2, s^4y'z') = s^{6-3}(x', sy'^2, sy'z')$ as needed.

For $x^5 + x^3y^3 + y^{100}$ the factor is $\ell = 15$, and $\sigma^*f = s^{15}(x'^5 + x'^3y'^3 + s^{185}y'^{100})$.

7.9. Exercise 5.33. — For $x^2 - y^2z$, we have computed $\mathcal{I}' = (x'^2 + y'^2z')$ and $\mathcal{D}(\mathcal{I})' = (x', sy'^2, sy'z')$. Since $\mathcal{D}(\mathcal{I}') = (x', y'^2y'z')$, the inclusion holds.

7.10. Exercise 5.36. — For $x^2 - y^2z$ with transform $x'^2 - y'^2z'$, the equation is the same, so indeed the order is 2 with maximal contact x' .

For $x^5 + x^3y^3 + y^{100}$, we computed the transform $x'^5 + x'^3y'^3 + s^{185}y'^{100}$. The rightmost term does not intervene: indeed the order of this equation is 5 with maximal contact x' .

When removing the vertex $x' = y' = 0$, the locus where x' is a unit is nonsingular, and the maximal order where y' is a unit is achieved where $x' = s = 0$. Writing it as $x'^3(y'^3 + x'^2) + s^{185}y'^{100} = x'^3U_1 + s^{185}U_2$, with U_1, U_2 units, we see that the maximal order on B_+ is $3 < 5$.

Acknowledgments. — I thank my collaborators Belotto da Silva, Temkin and Włodarczyk for this exciting project and their part in securing the above support. I thank IHES, the Hebrew University, CIRM and BNF for their hospitality during 2024-2025. I thank the participants of the CIRM workshop *Foliations, birational geometry and applications*, 3-7 February, 2025, where many stimulating discussions occurred, and Kirti Joshi for his comments. Finally thanks are due to the referee for help in improving the text.

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