
ON THE CHARACTER RING OF A FINITE GROUP

by

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Abstract. — Let G be a finite group and let k be a sufficiently large finite field. Let $\mathcal{R}(G)$ denote the character ring of G (i.e. the Grothendieck ring of the category of $\mathbb{C}G$ -modules). We study the structure and the representations of the commutative algebra $k \otimes_{\mathbb{Z}} \mathcal{R}(G)$.

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Introduction

Let G be a finite group. We denote by $\mathcal{R}(G)$ the *Grothendieck ring* of the category of $\mathbb{C}G$ -modules (it is usually called the *character ring* of G). It is a natural question to try to recover properties of G from the knowledge of $\mathcal{R}(G)$. It is clear that two finite groups having the same character table have the same Grothendieck rings and it is a Theorem of Saksonov [S] that the converse also holds. So the problem is reduced to an intensively studied question in character theory: recover properties of the group through properties of its character table.

In this paper, we study the k -algebra $k\mathcal{R}(G) = k \otimes_{\mathbb{Z}} \mathcal{R}(G)$, where k is a splitting field for G of positive characteristic p . It is clear that the knowledge of $k\mathcal{R}(G)$ is a much weaker information than the knowledge of $\mathcal{R}(G)$. The aim of this paper is to gather results on the representation theory of the algebra $k\mathcal{R}(G)$: although most of the results are certainly well-known, we have not found any general treatment of these questions. The blocks of $k\mathcal{R}(G)$ are local algebras which are parametrized by conjugacy classes of p -regular elements of G . So the simple $k\mathcal{R}(G)$ -modules are parametrized by conjugacy classes of p -regular elements of G . Moreover, the dimension of the projective cover of the simple module associated to the conjugacy class of the p -regular element $g \in G$ is equal to the number of conjugacy classes of p -elements in the centralizer $C_G(g)$. We also prove that the radical of $k\mathcal{R}(G)$ is the kernel of the decomposition map $k\mathcal{R}(G) \rightarrow k \otimes_{\mathbb{Z}} \mathcal{R}(kG)$, where $\mathcal{R}(kG)$ is the Grothendieck ring of the category of kG -modules (i.e. the ring of virtual Brauer characters of G).

We prove that the block of $k\mathcal{R}(G)$ associated to the p -element g is isomorphic to the block of $k\mathcal{R}(C_G(g))$ associated to 1 (such a block is called the *principal block*). This shows that the study of blocks of $k\mathcal{R}(G)$ is reduced to the study of principal blocks. We also show that the principal block of $k\mathcal{R}(G)$ is isomorphic to the principal block of $k\mathcal{R}(H)$ whenever H is a subgroup of p -index which controls the fusion of p -elements or whenever H is the quotient of G by a normal p -subgroup.

We also introduce several numerical invariants (Loewy length, dimension of Ext-groups) that are partly related to the structure of G . These numerical invariants are computed completely whenever G is the symmetric group \mathcal{S}_n (this relies on previous work of the author: the descending Loewy series of $k\mathcal{R}(\mathcal{S}_n)$ was entirely computed in [B]) or G is a dihedral group and $p = 2$. We also provide tables for these invariants for small groups (alternating groups A_n with $n \leq 12$, some small simple groups, groups $PSL(2, q)$ with q a prime power ≤ 27 , exceptional finite Coxeter groups).

NOTATION - Let \mathcal{O} be a Dedekind domain of characteristic zero, let \mathfrak{p} be a maximal ideal of \mathcal{O} , let K be the fraction field of \mathcal{O} and let $k = \mathcal{O}/\mathfrak{p}$. Let $\mathcal{O}_{\mathfrak{p}}$ be the localization of \mathcal{O} at \mathfrak{p} : then $k = \mathcal{O}_{\mathfrak{p}}/\mathfrak{p}\mathcal{O}_{\mathfrak{p}}$. If $x \in \mathcal{O}_{\mathfrak{p}}$, we denote by \bar{x} its image in $\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}\mathcal{O}_{\mathfrak{p}} = k$. Throughout this paper, we assume that k has characteristic $p > 0$ and that K and k are splitting fields for all the finite groups involved in this paper. If n is a non-zero natural number, n_p denotes the largest divisor of n prime to p and we set $n_p = n/n_p$.

If F is a field and if A is a finite dimensional F -algebra, we denote by $\mathcal{R}(A)$ its Grothendieck group. If M is an A -module, the radical of M is denoted by $\text{Rad } M$ and the class of M in $\mathcal{R}(A)$ is denoted by $[M]$. If S is a simple A -module, we denote by $[M : S]$ the multiplicity of S as a chief factor of a Jordan-Hölder series of M . The set of irreducible characters of A is denoted by $\text{Irr } A$.

We fix all along this paper a finite group G . For simplification, we set $\mathcal{R}(G) = \mathcal{R}(KG)$ and $\text{Irr } G = \text{Irr } KG$ (recall that K is a splitting field for G). The abelian group $\mathcal{R}(G)$ is endowed with a structure of ring induced by the tensor product. If $\chi \in \mathcal{R}(G)$, we denote by χ^* its dual (as a class function on G , we have $\chi^*(g) = \chi(g^{-1})$ for any $g \in G$). If R is any commutative ring, we denote by $\text{Class}_R(G)$ the space of class functions $G \rightarrow R$ and we set $R\mathcal{R}(G) = R \otimes_{\mathbb{Z}} \mathcal{R}(G)$. If X is a subset of G , we denote by $1_X^R : G \rightarrow R$ the characteristic function of X . If R is a subring of K , then we simply write $1_X = 1_X^R$. Note that 1_G is the trivial character of G . If $f, f' \in \text{Class}_K(G)$, we set

$$\langle f, f' \rangle_G = \frac{1}{|G|} \sum_{g \in G} f(g^{-1})f'(g).$$

Then $\text{Irr } G$ is an orthonormal basis of $\text{Class}_K(G)$. We shall identify $\mathcal{R}(G)$ with the sub- \mathbb{Z} -module (or sub- \mathbb{Z} -algebra) of $\text{Class}_K(G)$ generated by $\text{Irr } G$, and $K\mathcal{R}(G)$ with $\text{Class}_K(G)$. If $f \in \mathcal{O}_{\mathfrak{p}}\mathcal{R}(G)$, we denote by \bar{f} its image in $k\mathcal{R}(G)$.

If g and h are two elements of G , we write $g \sim h$ (or $g \sim_G h$ if we need to emphasize the group) if they are conjugate in G . We denote by g_p (resp. $g_{p'}$) the p -part (resp. the p' -part) of g . If X is a subset of G , we set $X_p = \{g_p \mid g \in X\}$ and $X_{p'} = \{g_{p'} \mid g \in X\}$. If moreover X is closed under conjugacy, the set of conjugacy classes contained in X is denoted by X/\sim . In this case, $1_X^R \in \text{Class}_R(G)$. The centre of G is denoted by $Z(G)$.

REMARK - We have recently discovered that some of the questions investigated in this paper were already studied by M. Deiml in his Ph.D. Thesis [D, Chapter 3]. More precisely, most of the results of our Section 2 were already proved by M. Deiml.

1. Preliminaries

1.A. Symmetrizing form. — Let

$$\begin{aligned} \tau_G : \mathcal{R}(G) &\longrightarrow \mathbb{Z} \\ \chi &\longmapsto \langle \chi, 1_G \rangle_G \end{aligned}$$

denote the canonical symmetrizing form on $\mathcal{R}(G)$. The dual basis of $\text{Irr } G$ is $(\chi)_{\chi \in \text{Irr } G}$. It is then readily seen that $(\mathcal{R}(G), \text{Irr } G)$ is a *based ring* (in the sense of Lusztig [L, Page 236]).

If R is any ring, we denote by $\tau_G^R : R\mathcal{R}(G) \rightarrow R$ the symmetrizing form $\text{Id}_R \otimes_{\mathbb{Z}} \tau_G$.

1.B. Translation by the centre. — If $\chi \in \text{Irr } G$, we denote by $\omega_\chi : Z(G) \rightarrow \mathcal{O}^\times$ the linear character such that $\chi(zg) = \omega_\chi(z)\chi(g)$ for all $z \in Z(G)$ and $g \in G$. If $z \in Z(G)$, we denote by $t_z : K\mathcal{R}(G) \rightarrow K\mathcal{R}(G)$ the linear map defined by $(t_z f)(g) = f(zg)$ for all $f \in K\mathcal{R}(G)$ and $g \in G$. It is clear that $t_{zz} = t_z \circ t_z$ for all $z, z \in Z(G)$ and that t_z is an automorphism of algebra. Moreover,

$$t_z \chi = \omega_\chi(z) \chi$$

for every $\chi \in \text{Irr } G$. Therefore, t_z is an isometry which stabilizes $\mathcal{OR}(G)$. If R is a subring of K such that $\mathcal{O} \subset R \subset K$, we still denote by $t_z : R\mathcal{R}(G) \rightarrow R\mathcal{R}(G)$ the restriction of t_z . Let $\bar{t}_z = \text{Id}_k \otimes_{\mathcal{O}} t_z : k\mathcal{R}(G) \rightarrow k\mathcal{R}(G)$. This is again an automorphism of k -algebra. If z is a p -element, then $\bar{t}_z = \text{Id}_{k\mathcal{R}(G)}$.

1.C. Restriction. — If $\pi : H \rightarrow G$ is a morphism of groups, then the *restriction through* π induces a morphism of rings $\text{Res}_\pi : \mathcal{R}(G) \rightarrow \mathcal{R}(H)$. If R is a subring of K , we still denote by $\text{Res}_\pi : R\mathcal{R}(G) \rightarrow R\mathcal{R}(H)$ the morphism $\text{Id}_R \otimes_{\mathbb{Z}} \text{Res}_\pi$. We denote by $\overline{\text{Res}}_\pi : k\mathcal{R}(G) \rightarrow k\mathcal{R}(H)$ the reduction modulo \mathfrak{p} of $\text{Res}_\pi : \mathcal{OR}(G) \rightarrow \mathcal{OR}(H)$. Recall that, if H is a subgroup of G and π is the canonical injection, then Res_π is just Res_H^G . In this case, $\overline{\text{Res}}_\pi$ will be denoted by $\overline{\text{Res}}_H^G$. Note the following fact:

(1.1) *If π is surjective, then $\overline{\text{Res}}_\pi$ is injective.*

Proof of 1.1. — Indeed, if π is surjective, then $\text{Res}_\pi : \mathcal{R}(G) \rightarrow \mathcal{R}(H)$ is injective and its image is a direct summand of $\mathcal{R}(H)$. \square

1.D. Radical. — First, note that, since $k\mathcal{R}(G)$ is commutative, we have

(1.2) *$\text{Rad } k\mathcal{R}(G)$ is the ideal of nilpotent elements of $k\mathcal{R}(G)$.*

So, if $\pi : H \rightarrow G$ is a morphism of finite groups, then

(1.3) $\overline{\text{Res}}_\pi(\text{Rad } k\mathcal{R}(G)) \subset \text{Rad } k\mathcal{R}(H)$.

The *Loewy length* of the algebra $k\mathcal{R}(G)$ is defined as the smallest natural number n such that $(\text{Rad } k\mathcal{R}(G))^n = 0$. We denote it by $\ell_p(G)$. By 1.1 and 1.3, we have:

(1.4) *If π is surjective, then $\ell_p(G) = \ell_p(H)$.*

2. Modules for $K\mathcal{R}(G)$ and $k\mathcal{R}(G)$

2.A. Semisimplicity. — Recall that $K\mathcal{R}(G)$ is identified with the algebra of class functions on G . If $C \in G/\sim$ and $f \in K\mathcal{R}(G)$, we denote by $f(C)$ the constant value of f on C . We now define $\text{ev}_C : K\mathcal{R}(G) \rightarrow K$, $f \mapsto f(C)$. It is a morphism of K -algebras. In other words, it is an irreducible representation (or character) of $K\mathcal{R}(G)$. We denote by \mathcal{D}_C the corresponding simple $K\mathcal{R}(G)$ -module ($\dim_K \mathcal{D}_C = 1$ and an element $f \in K\mathcal{R}(G)$ acts on \mathcal{D}_C by multiplication by $\text{ev}_C(f) = f(C)$). Now, 1_C is a primitive idempotent of $K\mathcal{R}(G)$ and it is easily checked that

(2.1) $K\mathcal{R}(G)1_C \simeq \mathcal{D}_C$.

Recall that

(2.2) $1_C = \frac{|C|}{|G|} \sum_{\chi \in \text{Irr } G} \chi(C^{-1}) \chi$

and

$$(2.3) \quad \sum_{C \in G/\sim} 1_C = 1_G.$$

Therefore:

Proposition 2.4. — *We have:*

- (a) $(\mathcal{D}_C)_{C \in G/\sim}$ is a family of representatives of isomorphism classes of simple $K\mathcal{R}(G)$ -modules.
- (b) $\text{Irr } K\mathcal{R}(G) = \{\text{ev}_C \mid C \in G/\sim\}$.
- (c) $K\mathcal{R}(G)$ is split semisimple.

We conclude this section by the computation of the Schur elements (see [GP, 7.2] for the definition) associated to each irreducible character of $K\mathcal{R}(G)$. Since

$$(2.5) \quad \tau_G^K = \sum_{C \in G/\sim} \frac{|C|}{|G|} \text{ev}_C,$$

we have by [GP, Theorem 7.2.6]:

Corollary 2.6. — *Let $C \in G/\sim$. Then the Schur element associated with the irreducible character ev_C is $\frac{|G|}{|C|}$.*

REMARK 2.7 - If $z \in Z(G)$, then t_z induces an isomorphism of algebras $K\mathcal{R}(G)1_C \simeq K\mathcal{R}(G)1_{z^{-1}C}$.

REMARK 2.8 - If $f \in K\mathcal{R}(G)$, then $f = \sum_{C \in G/\sim} f(C)1_C$.

EXAMPLE 2.9 - The map ev_1 will sometimes be denoted by deg , since it sends a character to its degree.

2.B. Decomposition map. — Let $d_p : \mathcal{R}(G) \rightarrow \mathcal{R}(kG)$ denote the decomposition map. If R is any commutative ring, we denote by $d_p^R : R\mathcal{R}(G) \rightarrow R\mathcal{R}(kG)$ the induced map. Note that $\mathcal{R}(kG)$ is also a ring (for the multiplication given by tensor product) and that d_p is a morphism of ring. Also, by [CR, Corollary 18.14],

$$(2.10) \quad d_p \text{ is surjective.}$$

Since $\text{Irr}(kG)$ is a linearly independent family of class functions $G \rightarrow k$ (see [CR, Theorem 17.4]), the map $\chi : k\mathcal{R}(kG) \rightarrow \text{Class}_k(G)$ that sends the class of a kG -module to its character is (well-defined and) injective. This is a morphism of k -algebras.

Now, if C is a conjugacy class of p -regular elements (i.e. $C \in G_p/\sim$), we define

$$\mathcal{S}_p(C) = \{g \in G \mid g_p \in C\}$$

(for instance, $\mathcal{S}_p(1) = G_p$). Then $\mathcal{S}_p(C)$ is called the p -section of C : this is a union of conjugacy classes of G . Let $\text{Class}_k^p(G)$ be the space of class functions $G \rightarrow k$ which are constant on p -sections. Then, by [CR, Lemma 17.8], $\text{Irr}(kG) \subset \text{Class}_k^p(G)$, so the image of χ is contained in $\text{Class}_k^p(G)$. But, χ is injective, $|\text{Irr}(kG)| = |G_p/\sim|$ (see [CR, Corollary 17.11]) and $\dim_k \text{Class}_k^p(G) = |G_p/\sim|$. Therefore, we can identify, through χ , the k -algebras $k\mathcal{R}(kG)$ and $\text{Class}_k^p(G)$. In particular,

$$(2.11) \quad k\mathcal{R}(kG) \text{ is split semisimple.}$$

2.C. Simple $k\mathcal{R}(G)$ -modules. — If $C \in G/\sim$, we still denote by $\text{ev}_C : \mathcal{OR}(G) \rightarrow \mathcal{O}$ the restriction of ev_C and we denote by $\overline{\text{ev}}_C : k\mathcal{R}(G) \rightarrow k$ the reduction modulo \mathfrak{p} of ev_C . It is easily checked that $\overline{\text{ev}}_C$ factorizes through the decomposition map $d_{\mathfrak{p}}$. Indeed, if $\text{ev}_C^k : k\mathcal{R}(kG) \rightarrow k$ denote the evaluation at C (recall that $k\mathcal{R}(kG)$ is identified, via the map χ of the previous subsection, to $\text{Class}_k^p(G)$), then

$$(2.12) \quad \overline{\text{ev}}_C = \text{ev}_C^k \circ d_{\mathfrak{p}}^k.$$

Let $\overline{\mathcal{D}}_C$ be the corresponding simple $k\mathcal{R}(G)$ -module. Let $\delta_{\mathfrak{p}} : \mathcal{R}(K\mathcal{R}(G)) \rightarrow \mathcal{R}(k\mathcal{R}(G))$ denote the decomposition map (see [GP, 7.4] for the definition). Then

$$(2.13) \quad \delta_{\mathfrak{p}}[\mathcal{D}_C] = [\overline{\mathcal{D}}_C].$$

The following facts are well-known:

Proposition 2.14. — *Let $C, C' \in G/\sim$. Then $\overline{\mathcal{D}}_C \simeq \overline{\mathcal{D}}_{C'}$ if and only if $C_p = C'_p$.*

Proof. — The “if” part follows from the following classical fact [CR, Proposition 17.5 (ii) and (iv) and Lemma 17.8]: if $\chi \in \mathcal{R}(G)$ and if $g \in G$, then

$$\chi(g) \equiv \chi(g_p) \pmod{\mathfrak{p}}.$$

The “only if” part follows from 2.12 and from the surjectivity of the decomposition map $d_{\mathfrak{p}}$. \square

Corollary 2.15. — *We have:*

- (a) $(\overline{\mathcal{D}}_C)_{C \in G_p/\sim}$ is a family of representatives of isomorphism classes of simple $k\mathcal{R}(G)$ -modules.
- (b) $\text{Irr } k\mathcal{R}(G) = \{\overline{\text{ev}}_C \mid C \in G_p/\sim\}$.
- (c) $\text{Rad } k\mathcal{R}(G) = \text{Ker } d_{\mathfrak{p}}^k$.
- (d) $k\mathcal{R}(G)$ is split.

Proof. — (a) follows from 2.13 and from the fact that the isomorphism class of any simple $k\mathcal{R}(G)$ -modules must occur in some $\delta_{\mathfrak{p}}[S]$, where S is a simple $K\mathcal{R}(G)$ -module. (b) follows from (a). (c) and (d) follow from (a), (b), 2.12 and 2.11. \square

Corollary 2.16. — $\dim_k \text{Rad}(k\mathcal{R}(G)) = |G/\sim| - |G_p/\sim|$.

Corollary 2.17. — $k\mathcal{R}(G)$ is semisimple if and only if p does not divide $|G|$.

EXAMPLE 2.18 - Since ev_1 is also denoted by deg , we shall sometimes denote by $\overline{\text{deg}}$ the morphism $\overline{\text{ev}}_1$. If G is a p -group, then Corollary 2.15 shows that $\text{Rad } k\mathcal{R}(G) = \text{Ker}(\overline{\text{deg}})$. In this case, if $1, \lambda_1, \dots, \lambda_r$ denote the linear characters of G and χ_1, \dots, χ_s denote the non-linear irreducible characters of G , then $(\overline{\lambda}_1 - 1, \dots, \overline{\lambda}_r - 1, \overline{\chi}_1, \overline{\chi}_s)$ is a k -basis of $\text{Rad } k\mathcal{R}(G)$.

2.D. Projective modules. — We now fix a conjugacy class C of p -regular elements (i.e. $C \in G_p/\sim$). Let

$$e_C = 1_{S_p(C)} = \sum_{D \in S_p(C)/H} 1_D.$$

If necessary, e_C will be denoted by e_C^G . If H is a subgroup of G , then

$$(2.19) \quad \text{Res}_H^G e_C^G = \sum_{D \in (C \cap H)/H} e_D^H.$$

Proposition 2.20. — *Let $C \in G_p/\sim$. Then $e_C \in \mathcal{O}_p\mathcal{R}(G)$.*

Proof. — Using Brauer's Theorem, we only need to prove that $\text{Res}_N^G e_C^G \in \mathcal{O}_p \mathcal{R}(N)$ for every nilpotent subgroup N of G . By 2.19, this amounts to prove the lemma whenever G is nilpotent. So we assume that G is nilpotent. Then $G = G_p \times G_p$, and G_p and G_p are subgroups of G . Moreover, $C \subset G_p$ and $\mathcal{S}_p(G) = C \times G_p$. If we identify $k\mathcal{R}(G)$ and $k\mathcal{R}(G_p) \otimes_K k\mathcal{R}(G_p)$, we have $e_C^G = 1_C^{G_p} \otimes_{\mathcal{O}_p} e_1^{G_p}$. But, by 2.2, we have that $e_C^{G_p} \in \mathcal{O}_p \mathcal{R}(G_p)$. On the other hand, $e_1^{G_p} = 1_{G_p} \in \mathcal{R}(G_p)$. The proof of the lemma is complete. \square

Corollary 2.21. — *Let $C \in G_p / \sim$. Then e_C is a primitive idempotent of $\mathcal{O}_p \mathcal{R}(G)$.*

Proof. — By Proposition 2.15 (a), the number of primitive idempotents of $k\mathcal{R}(G)$ is $|G_p / \sim|$. So the number of primitive idempotents of $\hat{\mathcal{O}}_p \mathcal{R}(G)$ is also $|G_p / \sim|$ (here, $\hat{\mathcal{O}}_p$ denotes the completion of \mathcal{O}_p at its maximal ideal). Now, $(e_C)_{C \in G_p / \sim}$ is a family of orthogonal idempotents of $\mathcal{O}_p \mathcal{R}(G)$ (see Proposition 2.20) and $1_G = \sum_{C \in G_p / \sim} e_C$. The proof of the lemma is complete. \square

Let $\bar{e}_C \in k\mathcal{R}(G)$ denote the reduction modulo $\mathfrak{p}_{\mathcal{O}_p}$ of e_C . Then it follows from 2.12 that

$$(2.22) \quad d_p^k \bar{e}_C = 1_{\mathcal{S}_p(C)}^k \in k\mathcal{R}(kG) \simeq \text{Class}_k^p(G).$$

Let $\mathcal{P}_C = \mathcal{O}_p \mathcal{R}(G) e_C$ and $\bar{\mathcal{P}}_C = k\mathcal{R}(G) \bar{e}_C$: they are indecomposable projective modules for $\mathcal{O}_p \mathcal{R}(G)$ and $k\mathcal{R}(G)$ respectively. Then

$$\mathcal{O}_p \mathcal{R}(G) = \bigoplus_{C \in G_p / \sim} \mathcal{P}_C$$

and

$$k\mathcal{R}(G) = \bigoplus_{C \in G_p / \sim} \bar{\mathcal{P}}_C.$$

Note also that

$$(2.23) \quad \dim_k k\mathcal{R}(G) \bar{e}_C = \text{rank}_{\mathcal{O}_p} \mathcal{O}_p \mathcal{R}(G) e_C = |\mathcal{S}_p(G) / \sim|.$$

Proposition 2.24. — *Let C and C' be two conjugacy classes of p -regular elements of G . Then:*

- (a) $[\bar{\mathcal{P}}_C : \bar{\mathcal{D}}_{C'}] = \begin{cases} |\mathcal{S}_p(C) / \sim| & \text{if } C = C', \\ 0 & \text{otherwise.} \end{cases}$
- (b) $\bar{\mathcal{P}}_C / \text{Rad } \bar{\mathcal{P}}_C \simeq \bar{\mathcal{D}}_C$.

Proof. — Let us first prove (a). By definition of e_C , we have

$$[K \otimes_{\mathcal{O}_p} \mathcal{P}_C] = \sum_{D \in \mathcal{S}_p(G) / \sim} [\mathcal{D}_D].$$

Also, by definition of the decomposition map $\delta_p : \mathcal{R}(K\mathcal{R}(G)) \rightarrow \mathcal{R}(k\mathcal{R}(G))$, we have

$$\delta_p[K \otimes_{\mathcal{O}_p} \mathcal{P}_C] = [\bar{\mathcal{P}}_C].$$

So the result follows from these observations and from 2.13. Now, (b) follows easily from (a). \square

2.E. More on the radical. — Let $\text{Rad}_p(G)$ denote the set of functions $f \in \mathcal{O}_p \mathcal{R}(G)$ whose restriction to G_p is zero. Note that $\text{Rad}_p(G)$ is a direct summand of the \mathcal{O}_p -module $\mathcal{O}_p \mathcal{R}(G)$. So, $k \text{Rad}_p(G) = k \otimes_{\mathcal{O}_p} \text{Rad}_p(G)$ is a sub- k -vector space of $k\mathcal{R}(G)$.

Proposition 2.25. — *We have:*

- (a) $\dim_k k \text{Rad}_p(G) = |G / \sim| - |G_p / \sim|$.
- (b) $k \text{Rad}_p(G)$ is the radical of $k\mathcal{R}(G)$.

Proof. — (a) is clear. (b) follows from 2.12 and from Corollary 2.15. \square

Corollary 2.26. — *Let e be the number such that p^e is the exponent of a Sylow p -subgroup of G . If $f \in \text{Rad } k\mathcal{R}(G)$, then $f^{p^e} = 0$.*

Proof. — Let $e = e_p(G)$. If $f \in K\mathcal{R}(G)$ and if $n \geq 1$, we denote by $f^{(n)} : G \rightarrow K$, $g \mapsto f(g^n)$. Then the map $K\mathcal{R}(G) \rightarrow K\mathcal{R}(G)$, $f \mapsto f^{(n)}$ is a morphism of K -algebras. Moreover (see for instance [CR, Corollary 12.10]), we have

$$(2.27) \quad \text{If } f \in \mathcal{R}(G), \text{ then } f^{(n)} \in \mathcal{R}(G).$$

Therefore, it induces a morphism of k -algebras $\theta_n : k\mathcal{R}(G) \rightarrow k\mathcal{R}(G)$. Now, let $F : k\mathcal{R}(G) \rightarrow k\mathcal{R}(G)$, $\lambda \otimes_{\mathbb{Z}} f \mapsto \lambda^p \otimes_{\mathbb{Z}} f$. Then F is an injective endomorphism of the ring $k\mathcal{R}(G)$. Moreover (see for instance [I, Problem 4.7]), we have

$$(2.28) \quad F \circ \theta_p(f) = f^p$$

for every $f \in k\mathcal{R}(G)$. Since F and θ_p commute, we have $F^e \circ \theta_{p^e}(f) = f^{p^e}$ for every $f \in k\mathcal{R}(G)$. Therefore, if $\chi \in \text{Rad}_p(G)$, we have

$$\bar{\chi}^{p^e} = F^e(\overline{\chi^{(p^e)}}).$$

But, by hypothesis, $g^{p^e} \in G_p$ for every $g \in G$. So, if $f \in \text{Rad}_p(G)$, then $f^{(p^e)} = 0$. Therefore, $\bar{f}^{p^e} = 0$. The corollary follows from this observation and from Proposition 2.25. \square

3. Principal block

If $C \in G_p / \sim$, we denote by $\mathcal{R}_p(G, C)$ the \mathcal{O}_p -algebra $\mathcal{O}_p\mathcal{R}(G)e_C$. As an $\mathcal{O}_p\mathcal{R}(G)$ -module, this is just \mathcal{P}_C , but we want to study here its structure as a ring, so that is why we use a different notation. If R is a commutative \mathcal{O}_p -algebra, we set $R\mathcal{R}_p(G, C) = R \otimes_{\mathcal{O}_p} \mathcal{R}_p(G, C)$. For instance, $k\mathcal{R}_p(G, C) = k\mathcal{R}(G)\bar{e}_C$, and $K\mathcal{R}_p(G, C)$ can be identified with the algebra of class functions on $\mathcal{S}_p(C)$.

The algebra $\mathcal{R}_p(G, 1)$ (resp. $k\mathcal{R}_p(G, 1)$) will be called the *principal block* of $\mathcal{O}_p\mathcal{R}(G)$ (resp. $k\mathcal{R}(G)$). The aim of this section is to construct an isomorphism $\mathcal{R}_p(G, C) \simeq \mathcal{R}_p(C_G(g), 1)$, where g is any element of C . We also emphasize the functorial properties of the principal block.

REMARK 3.1 - If $C \in G_p / \sim$ and if $z \in Z(G)$, then t_z induces an isomorphism of algebras $\mathcal{R}_p(G, C) \simeq \mathcal{R}_p(G, z_p^{-1}C)$ (see Remark 2.7). Consequently, \bar{t}_z induces an isomorphism of algebras $k\mathcal{R}_p(G, C) \simeq k\mathcal{R}_p(G, z^{-1}C)$.

3.A. Centralizers. — Let $C \in G_p / \sim_G$. Let $\text{proj}_C^G : K\mathcal{R}(G) \rightarrow K\mathcal{R}_p(G, C)$, $x \mapsto xe_C$ denote the canonical projection. We still denote by $\text{proj}_C^G : \mathcal{O}_p\mathcal{R}(G) \rightarrow \mathcal{R}_p(G, C)$, the restriction of proj_C^G and we denote by $\overline{\text{proj}}_C^G : k\mathcal{R}(G) \rightarrow k\mathcal{R}_p(G, C)$ its reduction modulo $\mathfrak{p}\mathcal{O}_p$.

Let us now fix $g \in C$. It is well-known (and easy) that the map $C_G(g)_p / \sim_{C_G(g)} \rightarrow \mathcal{S}_p(C) / \sim_G$ that sends the $C_G(g)$ -conjugacy class $D \in C_G(g)_p / \sim_{C_G(g)}$ to the G -conjugacy class containing gD is bijective. In particular,

$$(3.2) \quad |\mathcal{S}_p(C) / \sim_G| = |C_G(g)_p / \sim_{C_G(g)}|.$$

Now, let $d_g^G : K\mathcal{R}(G) \rightarrow K\mathcal{R}(C_G(g))$ be the map defined by:

$$(d_g^G f)(h) = \begin{cases} f(gh) & \text{if } h \in C_G(g)_p, \\ 0 & \text{otherwise,} \end{cases}$$

for all $f \in K\mathcal{R}(G)$ and $h \in C_G(g)$. Then $d_g^G f \in K\mathcal{R}_p(C_G(g), 1)$. It must be noticed that

$$(3.3) \quad d_g^G = \text{proj}_1^{C_G(g)} \circ t_g^{C_G(g)} \circ \text{Res}_{C_G(g)}^G = t_g^{C_G(g)} \circ \text{proj}_g^{C_G(g)} \circ \text{Res}_{C_G(g)}^G.$$

In particular, d_g^G sends $\mathcal{O}_p\mathcal{R}(G)$ to $\mathcal{R}_p(C_G(g), 1)$. We denote by $\text{res}_g : \mathcal{R}_p(G, C) \rightarrow \mathcal{R}_p(C_G(g), 1)$ the restriction of d_g^G to $\mathcal{R}_p(G, C)$. Let $\text{ind}_g : K\mathcal{R}_p(C_G(g), 1) \rightarrow K\mathcal{R}_p(G, C)$ be the map defined by

$$\text{ind}_g f = \text{Ind}_{C_G(g)}^G(t_{g^{-1}}^{C_G(g)} f)$$

for every $f \in K\mathcal{R}_p(C_G(g), 1)$. It is clear that $\text{ind}_g f \in \mathcal{R}_p(G, C)$ if $f \in \mathcal{R}_p(C_G(g), 1)$. Thus we have defined two maps

$$\text{res}_g : \mathcal{R}_p(G, C) \rightarrow \mathcal{R}_p(C_G(g), 1)$$

and

$$\text{ind}_g : \mathcal{R}_p(C_G(g), 1) \rightarrow \mathcal{R}_p(G, C).$$

We have:

Theorem 3.4. — *If $g \in G_p$, then res_g and ind_g are isomorphisms of \mathcal{O}_p -algebras inverse to each other.*

Proof. — We first want to prove that $\text{res}_g \circ \text{ind}_g$ is the identity morphism. Let $f \in K\mathcal{R}_p(C_G(g), 1)$. Let $f = t_{g^{-1}} f$ and let $x \in C_G(g)_p$. We just need to prove that

$$(?) \quad (\text{Ind}_{C_G(g)}^G f)(gx) = f(gx).$$

But, by definition,

$$(\text{Ind}_{C_G(g)}^G f)(gx) = \sum_{\substack{h \in [G/C_G(g)] \\ h(gx)h^{-1} \in C_G(g)}} f(h(gx)h^{-1}).$$

Here, $[G/C_G(g)]$ denotes a set of representatives of $G/C_G(g)$. Since f has support in $gC_G(g)_p$, we have $f(h(gx)h^{-1}) \neq 0$ only if the p -part of $h(gx)h^{-1}$ is equal to g , which happens if and only if $h \in C_G(g)$. This shows (?).

The fact that $\text{ind}_g \circ \text{res}_g$ is the identity can be proved similarly, or can be proved by using a trivial dimension argument. Since res_g is a morphism of algebras, we get that ind_g is also a morphism of algebras. \square

3.B. Subgroups of index prime to p . — If H is a subgroup of G , then the restriction map Res_H^G sends $\mathcal{R}_p(G, 1)$ to $\mathcal{R}_p(H, 1)$ (indeed, by 2.19, we have $\text{Res}_H^G e_1^G = e_1^H$).

Theorem 3.5. — *If H is a subgroup of G of index prime to p , then $\text{Res}_H^G : \mathcal{R}_p(G, 1) \rightarrow \mathcal{R}_p(H, 1)$ is a split injection of \mathcal{O}_p -modules.*

Proof. — Let us first prove that Res_H^G is injective. For this, we only need to prove that the map $\text{Res}_H^G : K\mathcal{R}_p(G, 1) \rightarrow K\mathcal{R}_p(H, 1)$. But $K\mathcal{R}_p(G, 1)$ is the space of functions whose support is contained in G_p . Since the index of H is prime to p , every conjugacy class of p -elements of G meets H . This shows that Res_H^G is injective.

In order to prove that it is a split injection, we only need to prove that the \mathcal{O}_p -module $\mathcal{R}_p(H, 1)/\text{Res}_H^G(\mathcal{R}_p(G, 1))$ is torsion-free. Let π be a generator of the ideal $\mathfrak{p}\mathcal{O}_p$. Let $\gamma \in \mathcal{R}_p(G, 1)$ and $\eta \in \mathcal{R}_p(H, 1)$ be such that $\pi\eta = \text{Res}_H^G \gamma$. We only need to prove that $\gamma/\pi \in \mathcal{R}_p(G, 1)$. By Brauer's Theorem, it is sufficient to show that, for any nilpotent subgroup N of G , we have $\text{Res}_N^G \gamma \in \pi\mathcal{O}_p\mathcal{R}(N)$.

So let N be a nilpotent subgroup. We have $N = N_p \times N_p'$ and, since the index of H in G is prime to p , we may assume that $N_p \subset H$. Since $\text{Res}_N^G \psi \in \mathcal{R}_p(N, 1) = \mathcal{O}_p\mathcal{R}(N_p) \otimes_{\mathcal{O}_p} e_1^{N_p'}$, we have

$$\begin{aligned} \text{Res}_N^G \gamma &= (\text{Res}_{N_p}^G \gamma) \otimes_{\mathcal{O}_p} e_1^{N_p'} \\ &= (\pi \text{Res}_{N_p}^H \eta) \otimes_{\mathcal{O}_p} e_1^{N_p'} \in \pi\mathcal{O}_p\mathcal{R}(N), \end{aligned}$$

as expected. \square

Corollary 3.6. — *If H is a subgroup of G of index prime to p , then the map $\overline{\text{Res}}_H^G : k\mathcal{R}_p(G, 1) \rightarrow k\mathcal{R}_p(H, 1)$ is an injective morphism of k -algebras.*

Corollary 3.7. — *If H is a subgroup of G of index prime to p which controls the fusion of p -elements, then $\text{Res}_H^G : \mathcal{R}_p(G, 1) \rightarrow \mathcal{R}_p(H, 1)$ is an isomorphism of \mathcal{O}_p -algebras.*

Proof. — In this case, $\dim_K K\mathcal{R}_p(G, 1) = \dim_K K\mathcal{R}_p(H, 1)$, so the result follows from Corollary 3.6. \square

EXAMPLE 3.8 - Let P be a Sylow p -subgroup of G and assume in this example that P is abelian. Then $N_G(P)$ controls the fusion of p -elements. It then follows from Corollary 3.7 that the restriction from G to $N_G(P)$ induces isomorphisms of algebras $\mathcal{R}_p(G, 1) \simeq \mathcal{R}_p(N_G(P), 1)$ and $k\mathcal{R}_p(G, 1) \simeq k\mathcal{R}_p(N_G(P), 1)$. In particular, $\ell_p(G, 1) = \ell_p(N_G(P), 1)$.

EXAMPLE 3.9 - Let N be a p -group, let H be a group acting on N and let $G = H \ltimes N$. Then H is of index prime to p and controls the fusion of p -elements of G . So Res_H^G induces isomorphisms of algebras $\mathcal{R}_p(G, 1) \simeq \mathcal{R}_p(H, 1)$ and $k\mathcal{R}_p(G, 1) \simeq k\mathcal{R}_p(H, 1)$. In particular, $\ell_p(G, 1) = \ell_p(H, 1)$.

3.C. Quotient by a normal p' -subgroup. — Let N be a normal subgroup of G . Let $\pi : G \rightarrow G/N$ denote the canonical morphism. Then the morphism of algebras $\text{Res}_\pi : \mathcal{R}_p(G/N) \rightarrow \mathcal{R}_p(G)$ induces a morphism of algebras $\text{Res}_\pi^{(1)} : \mathcal{R}_p(G/N, 1) \rightarrow \mathcal{R}_p(G, 1)$, $f \mapsto (\text{Res}_\pi f)e_1^G$. Note that $\text{Res}_\pi^{(1)} e_1^{G/N} = e_1^G$. We denote by $\overline{\text{Res}}_\pi^{(1)} : k\mathcal{R}_p(G/N, 1) \rightarrow k\mathcal{R}_p(G, 1)$ the morphism induced by $\text{Res}_\pi^{(1)}$. Then:

Theorem 3.10. — *With the above notation, we have:*

- (a) $\text{Res}_\pi^{(1)}$ is a split injection of \mathcal{O}_p -modules.
- (b) If N is prime to p , then $\text{Res}_\pi^{(1)}$ is an isomorphism.

Proof. — (a) The injectivity of $\text{Res}_\pi^{(1)}$ follows from the fact that $(G/N)_p = G_p N/N$. Now, let I denote the image of $\text{Res}_\pi^{(1)}$. Since $\text{Res}_\pi(\mathcal{O}_p \mathcal{R}(G/N))$ is a direct summand of $\mathcal{O}_p \mathcal{R}(G)$, we get that $\text{Res}_\pi(\mathcal{R}_p(G/N, 1)) = (\text{Res}_\pi^{(1)} e_1^{G/N}) \text{Res}_\pi(\mathcal{O}_p \mathcal{R}(G/N))$ is a direct summand of $\mathcal{O}_p \mathcal{R}(G)$. Since $I = e_1^G \text{Res}_\pi(\mathcal{R}_p(G/N, 1))$ and $e_1^G = e_1^G \text{Res}_\pi(e_1^{G/N})$, we get that $I = e_1^G \text{Res}_\pi(\mathcal{O}_p \mathcal{R}(G/N))$ is a direct summand of $\mathcal{O}_p \mathcal{R}(G)$, as desired.

(b) now follows from (a) and from the fact that the map π induces a bijection between G_p / \sim_G and $(G/N)_p / \sim_{G/N}$ whenever N is a normal p -subgroup. \square

4. Some invariants

We introduce in this section some numerical invariants of the k -algebra $k\mathcal{R}(G)$ (more precisely, of the algebras $k\mathcal{R}_p(G, C)$): Loewy length, dimension of the Ext-groups.

4.A. Loewy length. — If $C \in G_p / \sim$, we denote by $\ell_p(G, C)$ the Loewy length of the k -algebra $k\mathcal{R}_p(G, C)$. Then, by definition, we have

$$(4.1) \quad \ell_p(G) = \max_{C \in G_p / \sim} \ell_p(G, C).$$

On the other hand, by Theorem 3.4, we have

$$(4.2) \quad \text{If } C \in G_p / \sim \text{ and if } g \in C, \text{ then } \ell_p(G, C) = \ell_p(C_G(g), 1).$$

The following bound on the Loewy length of $k\mathcal{R}(G)$ is obtained immediately from 2.23 and 3.2:

$$(4.3) \quad \ell_p(G) \leq \max_{C \in G_p / \sim} |S_p(C) / \sim| = \max_{g \in G_p} |C_G(g)_p / \sim_{C_G(g)}|.$$

We set $S_p(G) = \max_{C \in G_p / \sim} |S_p(C) / \sim|$.

EXAMPLE 4.4 - The inequality 4.3 might be strict. Indeed, if $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, then $\ell_2(G) = 3 < 4 = S_2(G)$.

EXAMPLE 4.5 - If $S_p(G) = 2$, then $\ell_p(G) = 2$. Indeed, in this case, we have that p divides $|G|$, so $k\mathcal{R}(G)$ is not semisimple by Corollary 2.17, so $\ell_p(G) \geq 2$. The result then follows from 4.3.

4.B. Ext-groups. — If $i \geq 0$ and if $C \in G_p / \sim$, we set

$$\text{ext}_p^i(G, C) = \dim_{\mathbb{F}_p} \text{Ext}_{k\mathcal{R}(G)}^i(\bar{\mathcal{D}}_C, \bar{\mathcal{D}}_C).$$

Note that $\text{ext}_p^i(G, C) = \dim_{\mathbb{F}_p} \text{Ext}_{k\mathcal{R}_p(G, C)}^i(\bar{\mathcal{D}}_C, \bar{\mathcal{D}}_C)$. So, if $g \in C$, it follows from Theorem 3.4 that

$$(4.6) \quad \text{ext}_p^i(G, C) = \text{ext}_p^i(C_G(g), 1).$$

4.C. Subgroups, quotients. — The next results follows respectively from Corollaries 3.6, 3.7 and from Theorem 3.10:

Proposition 4.7. — *Let H be a subgroup of G of index prime to p and let N be a normal subgroup of G . Then:*

- (a) $\ell_p(G, 1) = \ell_p(H, 1)$.
- (b) *If H controls the fusion of p -elements, then $\ell_p(G, 1) = \ell_p(H, 1)$ and $\text{ext}_p^i(G, 1) = \text{ext}_p^i(H, 1)$ for every $i \geq 0$.*
- (c) $\ell_p(G/N, 1) = \ell_p(G, 1)$.
- (d) *If $|N|$ is prime to p , then $\ell_p(G, 1) = \ell_p(H, 1)$ and $\text{ext}_p^i(G, 1) = \text{ext}_p^i(H, 1)$ for every $i \geq 0$.*

4.D. Direct products. — We study here the behaviour of the invariants $\ell_p(G, C)$ and $\text{ext}_p^1(G, C)$ with respect to taking direct products. We first recall the following result on finite dimensional algebras:

Proposition 4.8. — *Let A and B be two finite dimensional k -algebras. Then:*

- (a) $\text{Rad}(A \otimes_k B) = A \otimes_k (\text{Rad } B) + (\text{Rad } A) \otimes_k B$.
- (b) *If $A/\text{Rad } A \simeq k$ and $B/\text{Rad } B \simeq k$, then*

$$\text{Rad}(A \otimes_k B)/\text{Rad}(A \otimes_k B)^2 \simeq (\text{Rad } A)/(\text{Rad } A)^2 \oplus (\text{Rad } B)/(\text{Rad } B)^2.$$

Proof. — (a) is proved for instance in [CR, Proof of 10.39]. Let us now prove (b). Let $\theta : (\text{Rad } A) \oplus (\text{Rad } B) \rightarrow \text{Rad}(A \otimes_k B)/\text{Rad}(A \otimes_k B)^2$, $a \oplus b \mapsto a \otimes_k \bar{1} + \bar{1} \otimes_k b$. By (a), θ is surjective and $(\text{Rad } A)^2 \oplus (\text{Rad } B)^2$ is contained in the kernel of θ . Now the result follows from dimension reasons (using (a)). \square

Proposition 4.9. — *Let G and H be two finite groups and let $C \in G_p / \sim$ and $D \in H_p / \sim$. Then*

$$\ell_p(G \times H, C \times D) = \ell_p(G, C) + \ell_p(H, D) - 1$$

and

$$\text{ext}_p^1(G \times H, C \times D) = \text{ext}_p^1(G, C) + \text{ext}_p^1(H, D).$$

Proof. — Write $A = k\mathcal{R}_p(G, C)$ and $B = k\mathcal{R}_p(H, D)$. It is easily checked that $k\mathcal{R}_p(G \times H, C \times D) = A \otimes_k B$. So the first equality follows from Proposition 4.8 (a) and from the commutativity of A and B . Moreover $A/(\text{Rad } A) \simeq k$ and $B/(\text{Rad } B) \simeq k$. In particular

$$\dim_k \text{Ext}_A^1(A/\text{Rad } A, A/\text{Rad } A) = \dim_k (\text{Rad } A)/(\text{Rad } A)^2.$$

So the second equality follows from Proposition 4.8 (b). \square

4.E. Abelian groups. — We compute here the invariants $\ell_p(G, 1)$ and $\text{ext}_p^1(G, 1)$ whenever G is abelian. If G is abelian, then there is a (non-canonical) isomorphism of algebras $k\mathcal{R}(G) \simeq kG$.

Let us first start with the cyclic case:

$$(4.10) \quad \text{if } G \text{ is cyclic, then } \ell_p(G) = |G|_p + 1 \text{ and } \text{ext}_p^1(G, 1) = \begin{cases} 1 & \text{if } p \text{ divides } |G|, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, by Proposition 4.9, we have: if G_1, \dots, G_n are cyclic, then

$$(4.11) \quad \ell_p(G_1 \times \dots \times G_n) = |G_1|_p + \dots + |G_n|_p - n + 1.$$

and

$$(4.12) \quad \text{ext}_p^1(G_1 \times \dots \times G_n) = |\{1 \leq i \leq n \mid p \text{ divides } G_i\}|.$$

5. The symmetric group

In this section, and only in this section, we fix a non-zero natural number n and a prime number p and we assume that $G = S_n$, that $\mathcal{O} = \mathbb{Z}$ and that $\mathfrak{p} = p\mathbb{Z}$. Let $\mathbb{F}_p = k$. It is well-known that \mathbb{Q} and \mathbb{F}_p are splitting fields for S_n . For simplification, we set $\mathcal{R}_n = \mathcal{R}(S_n)$ and $\overline{\mathcal{R}}_n = \mathbb{F}_p\mathcal{R}(S_n)$. We investigate further the structure of $\overline{\mathcal{R}}_n$. This is a continuation of the work started in [B] in which the description of the descending Loewy series of $\overline{\mathcal{R}}_n$ was obtained.

We first introduce some notation. Let $\text{Part}(n)$ denote the set of partitions of n . If $\lambda = (\lambda_1, \dots, \lambda_r) \in \text{Part}(n)$ and if $1 \leq i \leq n$, we denote by $r_i(\lambda)$ the number of occurrences of i as a part of λ . We set

$$\pi_p(\lambda) = \sum_{i=1}^n \left\lfloor \frac{r_i(\lambda)}{p} \right\rfloor$$

where, for $x \in \mathbb{R}$, $x \geq 0$, we denote by $[x]$ the unique natural number $m \geq 0$ such that $m \leq x < m + 1$. Note that $\pi_p(\lambda) \in \{0, 1, 2, \dots, [n/p]\}$ and recall that λ is *p-regular* (resp. *p-singular*) if and only if $\pi_p(\lambda) = 0$ (resp. $\pi_p(\lambda) \geq 1$). We denote by S_λ the Young subgroup canonically isomorphic to $S_{\lambda_1} \times \dots \times S_{\lambda_r}$, by 1_λ the trivial character of S_λ , and by c_λ an element of S_λ with only r orbit in $\{1, 2, \dots, n\}$. Let C_λ denote the conjugacy class of c_λ in S_n . Then the map $\text{Part}(n) \rightarrow S_n/\sim$, $\lambda \mapsto C_\lambda$ is a bijection. Let $W(\lambda) = N_{S_n}(S_\lambda)/S_\lambda$. Then

$$(5.1) \quad W(\lambda) \simeq \prod_{i=1}^n S_{r_i(\lambda)}.$$

In particular, $\pi_p(\lambda)$ is the *p-rank* of $W(\lambda)$, where the *p-rank* of a finite group is the maximal rank of an elementary abelian subgroup. Now, we set $\varphi_\lambda = \text{Ind}_{S_\lambda}^{S_n} 1_\lambda$. An old result of Frobenius says that

$$(5.2) \quad (\varphi_\lambda)_{\lambda \in \text{Part}(n)} \text{ is a } \mathbb{Z}\text{-basis of } \mathcal{R}_n$$

(see for instance [GP, Theorem 5.4.5 (b)]). Now, if $i \geq 1$, let

$$\text{Part}_p^i(n) = \{\lambda \in \text{Part}(n) \mid \pi_p(\lambda) = i\}$$

and

$$\text{Part}_p^i(n) = \{\lambda \in \text{Part}(n) \mid \pi_p(\lambda) = i\}.$$

Then, by [B, Theorem A], we have

$$(5.3) \quad (\text{Rad } \overline{\mathcal{R}}_n)^i = \bigoplus_{\lambda \in \text{Part}_p^i(n)} \mathbb{F}_p \overline{\varphi}_\lambda.$$

Let $\text{Part}_p(n)$ denote the set of partitions of n whose parts are prime to p . Then the map $\text{Part}_p(n) \rightarrow G_p / \sim, \lambda \mapsto C_\lambda$ is bijective. We denote by $\tau_p(\lambda)$ the unique partition of n such that $(c_\lambda)_p \in C_{\tau_p(\lambda)}$. If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$, the partition $\tau_p(\lambda)$ is obtained as follows. Let

$$\lambda = \underbrace{((\lambda_1)_p, \dots, (\lambda_1)_p)}_{(\lambda_1)_p \text{ times}}, \dots, \underbrace{((\lambda_r)_p, \dots, (\lambda_r)_p)}_{(\lambda_r)_p \text{ times}}.$$

Then $\tau_p(\lambda)$ is obtained from λ by reordering the parts. The map $\tau_p : \text{Part}(n) \rightarrow \text{Part}_p(n)$ is obviously surjective. If $\lambda \in \text{Part}_p(n)$, we set for simplification $\mathcal{R}_{n,p}(\lambda) = \mathcal{R}_{p\mathbb{Z}}(\mathbb{S}_n, C_\lambda)$ and $\overline{\mathcal{R}}_n(\lambda) = \mathbb{F}_p \mathcal{R}_{p\mathbb{Z}}(\mathbb{S}_n, C_\lambda)$. In other words,

$$\mathbb{Z}_{p\mathbb{Z}} \mathcal{R}_n = \bigoplus_{\lambda \in \text{Part}_p(n)} \mathcal{R}_{n,p}(\lambda)$$

and

$$\overline{\mathcal{R}}_n = \bigoplus_{\lambda \in \text{Part}_p(n)} \overline{\mathcal{R}}_n(\lambda)$$

are the decomposition of $\mathbb{Z}_{p\mathbb{Z}} \mathcal{R}_n$ and $\overline{\mathcal{R}}_n$ as a sum of blocks. We now make the result 5.3 more precise:

Proposition 5.4. — *If $\lambda \in \text{Part}_p(n)$ and if $i \geq 0$, then*

$$\dim_{\mathbb{F}_p}(\text{Rad } \overline{\mathcal{R}}_n(\lambda))^i = |\tau_p^{-1}(\lambda) \cap \text{Part}_p^i(n)|.$$

Proof. — If λ and μ are two partitions of n , we write $\lambda \subset \mu$ if \mathbb{S}_λ is conjugate to a subgroup of \mathbb{S}_μ . This defines an order on $\text{Part}(n)$. On the other hand, if $d \in \mathbb{S}_n$, we denote by $\lambda \cap^d \mu$ the unique partition ν of n such that $\mathbb{S}_\lambda \cap^d \mathbb{S}_\mu$ is conjugate to \mathbb{S}_ν . Then, by the Mackey formula for tensor product (see for instance [CR, Theorem 10.18]), we have

$$(1) \quad \varphi_\lambda \varphi_\mu = \sum_{d \in [\mathbb{S}_\lambda \backslash \mathbb{S}_n / \mathbb{S}_\mu]} \varphi_{\lambda \cap^d \mu}.$$

Here, $[\mathbb{S}_\lambda \backslash \mathbb{S}_n / \mathbb{S}_\mu]$ denotes a set of representatives of the $(\mathbb{S}_\lambda, \mathbb{S}_\mu)$ -double cosets in \mathbb{S}_n . This shows that, if we fixe $\lambda_0 \in \text{Part}(n)$, then $\bigoplus_{\lambda \supset \lambda_0} \mathbb{Z} \varphi_\lambda$ and $\bigoplus_{\lambda \subset \lambda_0} \mathbb{Z} \varphi_\lambda$ are sub- $\mathcal{R}(G)$ -module of $\mathcal{R}(G)$. We denote by $\mathcal{D}_\lambda^{\mathbb{Z}}$ the quotient of these two modules. Then

$$(2) \quad K \otimes_{\mathbb{Z}} \mathcal{D}_\lambda^{\mathbb{Z}} \simeq \mathcal{D}_{C_\lambda}.$$

This follows for instance from [GP, Proposition 2.4.4]. Consequently,

$$(3) \quad k \otimes_{\mathbb{Z}} \mathcal{D}_\lambda^{\mathbb{Z}} \simeq \overline{\mathcal{D}}_{C_\lambda}.$$

It then follows from Proposition 2.14 that

$$(4) \quad k \otimes_{\mathbb{Z}} \mathcal{D}_\lambda^{\mathbb{Z}} \simeq k \otimes_{\mathbb{Z}} \mathcal{D}_\mu^{\mathbb{Z}} \text{ if and only if } \tau_p(\lambda) = \tau_p(\mu).$$

Now the Theorem follows from easily from (3), (4) and 5.3. \square

Now, if $\lambda \in \text{Part}_p(n)$, then $C_{\mathbb{S}_n}(w_\lambda)$ contains a normal p -subgroup N_λ such that $C_{\mathbb{S}_n}(w_\lambda)/N_\lambda$ is isomorphic to $W(\lambda)$. We denote by 1^n the partition $(1, 1, \dots, 1)$ of n . It follows from Theorem 3.4 and Theorem 3.10 that

$$(5.5) \quad \mathcal{R}_{n,p}(\lambda) \simeq \mathcal{R}_{p\mathbb{Z}}(W(\lambda), 1) \simeq \bigotimes_{i=1}^n \mathcal{R}_{r_i(\lambda),p}(1^{r_i(\lambda)})$$

and

$$(5.6) \quad \overline{\mathcal{R}}_n(\lambda) \simeq \overline{\mathcal{R}}(W(\lambda), 1) \simeq \bigotimes_{i=1}^n \overline{\mathcal{R}}_{r_i(\lambda)}(1^{r_i(\lambda)}).$$

We denote by $\text{Log}_p n$ the real number x such that $p^x = n$. Then:

Corollary 5.7. — *If $\lambda \in \text{Part}_p(n)$, then*

$$\text{ext}_p^1(\mathbb{S}_n, C_\lambda) = \sum_{i=1}^n [\text{Log}_p r_i(\lambda)]$$

and

$$\ell_p(\mathbb{S}_n, C_\lambda) = \pi_p(\lambda) + 1.$$

Proof. — By 5.6 and by Proposition 4.9, both equalities need only to be proved whenever $\lambda = (1^n)$. So we assume that $\lambda = (1^n)$.

Let us show the first equality. By Proposition 5.4, we are reduced to show that $|\tau_p^{-1}(1^n) \cap \text{Part}_p^1(n)| = [\text{Log}_p n]$. Let $r = [\text{Log}_p n]$. In other words, we have $p^r \leq n < p^{r+1}$. If $1 \leq i \leq r$, write $n - p^i = \sum_{j=0}^r a_{ij} p^j$ with $0 \leq a_{ij} < p - 1$ (the a_{ij} 's are uniquely determined). Let

$$\lambda(i) = \underbrace{(p^r, \dots, p^r)}_{a_{ir} \text{ times}}, \dots, \underbrace{(p^i, \dots, p^i)}_{a_{ir} \text{ times}}, \underbrace{(p^{i-1}, \dots, p^{i-1})}_{(p+a_{i-1,r}) \text{ times}}, \underbrace{(p^{i-2}, \dots, p^{i-2})}_{a_{i-2,r} \text{ times}}, \dots, \underbrace{(1, \dots, 1)}_{a_{0r} \text{ times}}.$$

The result will follow from the following equality

$$(*) \quad \tau_p^{-1}(1^n) \cap \text{Part}_p^1(n) = \{\lambda(1), \lambda(2), \dots, \lambda(r)\}.$$

So let us now prove (*). Let $I = \{\lambda(1), \lambda(2), \dots, \lambda(r)\}$. It is clear that $I \subset \tau_p^{-1}(1^n) \cap \text{Part}_p^1(n)$. Now, let $\lambda \in \tau_p^{-1}(1^n) \cap \text{Part}_p^1(n)$. Then there exists a unique $i \in \{1, 2, \dots, r\}$ such that $r_{p^{i-1}}(\lambda) = p$. Moreover, $r_{p^{i-1}}(\lambda) < 2p$. So, if we set $r_{p^j} = r_{r_j}(\lambda)$ if $j \neq i - 1$ and $r_{p^{i-1}} = r_{p^{i-1}}(\lambda) - p$, we get that $0 \leq r_{p^j} \leq p - 1$ and $n - p^i = \sum_{j=0}^r r_{p^j} p^j$. This shows that $r_{p^j} = a_{ij}$, so $\lambda = \lambda(i)$.

Let us now show the second equality for the Corollary. By Proposition 5.4, we only need to show that $|\tau_p^{-1}(1^n) \cap \text{Part}_p^{[n/p]}(n)| = 1$. But in fact, it is clear that $\tau_p^{-1}(1^n) \cap \text{Part}_p^{[n/p]}(n) = \{1^n\}$. \square

Corollary 5.8. — *We have*

$$\dim_{\mathbb{F}_p}(\text{Rad } \overline{\mathcal{R}}_n(1^n))^{[n/p]} = 1$$

and

$$\dim_{\mathbb{F}_p} \text{Ext}_{\overline{\mathcal{R}}_n}^1(\overline{\mathcal{D}}_{1^n}, \overline{\mathcal{D}}_{1^n}) = [\text{Log}_p n].$$

In particular, $\ell_p(\mathcal{S}_n, 1) = \ell_p(\mathcal{S}_n) = [n/p]$.

Proof. — This is just a particular case of the previous corollary. The first equality has been obtained in the course of the proof of the previous corollary. \square

6. Dihedral groups

Let $n \geq 1$ and $m \geq 0$ be two natural numbers. We assume in this section, and only in this subsection, that $G = D_{2^n(2m+1)}$ is the dihedral group of order $2^n(2m+1)$ and that $p = 2$.

Proposition 6.1. — *If $n \geq 1$ and $m \geq 0$ are natural numbers, then*

$$\ell_2(D_{2^n(2m+1)}, 1) = \begin{cases} 2 & \text{if } n = 1, \\ 3 & \text{if } n = 2, \\ 2^{n-2} + 1 & \text{if } n \geq 3. \end{cases}$$

and

$$\text{ext}_2^1(D_{2^n(2m+1)}, 1) = \begin{cases} 1 & \text{if } n = 1, \\ 2 & \text{if } n = 2, \\ 3 & \text{if } n \geq 3. \end{cases}$$

Proof. — Let N be the normal subgroup of G of order $2m+1$. Then $G \simeq D_{2^n} \rtimes N$. So, by Proposition 4.7 (d), we may, and we will, assume that $m = 0$. If $n = 1$ or 2 the result is easily checked. Therefore, we may, and we will, assume that $n \geq 3$.

Write $h = 2^{n-1}$. We have

$$G = \langle s, t \mid s^2 = t^2 = (st)^h = 1 \rangle.$$

Let $H = \langle st \rangle$ and $S = \langle s \rangle$. Then $|H| = 2^{n-1} = h$ and $G = S \rtimes H$. We fix a primitive h -th root of unity $\zeta \in \mathcal{O}^\times$. If $i \in \mathbb{Z}$, we denote by ξ_i the unique linear character of H such that $\xi_i(st) = \zeta^i$. Then $\text{Irr } H = \{\xi_0, \xi_1, \dots, \xi_{h-1}\}$, and $\xi_0 = 1_H$.

Since $n \geq 3$, h is even and, if we write $h = 2h'$, then $h' = 2^{n-2}$ is also even. For $i \in \mathbb{Z}$, we set

$$\chi_i = \text{Ind}_H^G \xi_i.$$

It is readily seen that $\chi_i = \chi_{-i}$, that $\chi_{i+h} = \chi_i$ and that

$$(6.2) \quad \chi_i \chi_j = \chi_{i+j} + \chi_{i-j}.$$

Let ε (resp. ε_s , resp. ε_t) be the unique linear character of order 2 such that $\varepsilon(st) = 1$ (resp. $\varepsilon_s(s) = 1$, resp. $\varepsilon_t(t) = 1$). Then

$$\chi_0 = 1_G + \varepsilon,$$

$$\chi_h = \varepsilon_s + \varepsilon_t,$$

and, if h' does not divide i ,

$$\chi_i \in \text{Irr } G.$$

Moreover, $|\text{Irr } G| = h' + 3$ and

$$\text{Irr } G = \{1_G, \varepsilon, \varepsilon_s, \varepsilon_t, \chi_1, \chi_2, \dots, \chi_{h'-1}\}.$$

Finally, note that

$$(6.3) \quad \varepsilon_s \chi_i = \varepsilon_t \chi_i = \chi_{i+h'}.$$

Let us start by finding a lower bound for $\ell_2(G)$. First, notice that the following equality holds: for all $i, j \in \mathbb{Z}$ and every $r \geq 0$, we have

$$(6.4) \quad (\bar{\chi}_i + \bar{\chi}_j)^{2^r} = \bar{\chi}_{2^r i} + \bar{\chi}_{2^r j}.$$

Proof of 6.4. — Recall that $\bar{\chi}_i$ denotes the image of χ_i in $k\mathcal{R}(G)$. We proceed by induction on r . The case $r = 0$ is trivial. The induction step is an immediate consequence of 6.2. \square

Note also the following fact (which follows from Example 2.18):

$$(6.5) \quad \text{If } i \in \mathbb{Z}, \text{ then } \bar{\chi}_i \in \text{Rad } k\mathcal{R}(G).$$

Therefore,

$$(6.6) \quad \ell_2(G) \geq 2^{n-2} + 1.$$

Proof of 6.6. — By 6.4, we have immediately that $(\bar{\chi}_0 + \bar{\chi}_1)^{2^{n-2}} = \bar{\chi}_0 + \bar{\chi}_h \neq 0$ and, by 6.5, $\bar{\chi}_0 + \bar{\chi}_1 \in \text{Rad } k\mathcal{R}(G)$. \square

By Example 2.18, we have

$$(6.7) \quad (\bar{1}_G + \bar{\varepsilon}_s, \bar{\chi}_0, \bar{\chi}_1, \dots, \bar{\chi}_h) \text{ is a } k\text{-basis of } \text{Rad } k\mathcal{R}(G).$$

By 6.3 and 6.2, we get that

$$(6.8) \quad (\bar{\chi}_i + \bar{\chi}_{i+2})_{0 \leq i \leq h-2} \text{ is a } k\text{-basis of } (\text{Rad } k\mathcal{R}(G))^2.$$

This shows that $\text{ext}_p^1(G) = 3$, as expected. It follows that, if $n \geq 3$ and $2 \leq i \leq 2^{n-2} + 1$, then

$$(6.9) \quad \dim_k(\text{Rad } k\mathcal{R}(D_{2^n}))^i = 2^{n-2} + 1 - i$$

Proof of 6.9. — Let $d_i = \dim_k(\text{Rad } k\mathcal{R}(D_{2^n}))^i$. By 6.8, we have $d_2 = 2^{n-2} - 1$. By 6.6, we have $d_{2^{n-2}} = 1$. Moreover, $d_1 > d_2 > d_3 > \dots$. So the proof of 6.9 is complete. \square

In particular, we get:

$$(6.10) \quad \text{If } n \geq 3, \text{ then } (\text{Rad } k\mathcal{R}(D_{2^n}))^{2^{n-2}} = k(\bar{1}_{D_{2^n}} + \bar{\varepsilon} + \bar{\varepsilon}_s + \bar{\varepsilon}_t).$$

and $\ell_2(D_{2^n}) = 2^{n-2} + 1$, as expected. \square

7. Some tables

For $0 \leq i \leq \ell_p(G) - 1$, we set $d_i = \dim_k(\text{Rad } k\mathcal{R}(G))^i$. Note that $d_0 = |G/\sim|$ and $d_0 - d_1 = |G_p/\sim|$. In this section, we give tables containing the values $\ell_p(G)$, $\ell_p(G, 1)$, $S_p(G)$, $\text{ext}_p^1(G, 1)$ and the sequence (d_0, d_1, d_2, \dots) for various groups. These computations have been made using GAP3 [GAP3].

These computations show that, if G satisfies at least one of the following conditions:

- (1) $|G| \leq 200$;
- (2) G is a subgroup of S_8 ;
- (3) G is one of the groups contained in the next tables;

then $\ell_p(G, 1) = \ell_p(N_G(P), 1)$ (here, P denotes a Sylow p -subgroup of G). Note also that this equality holds if P is abelian (see Example 3.8).

Question. *Is it true that $\ell_p(G, 1) = \ell_p(N_G(P), 1)$?*

The first table contains the data for the exceptional Weyl groups, the second table is for the alternating groups A_n for $5 \leq n \leq 12$, the third table is for some small finite simple groups, and the last table is for the groups $PSL(2, q)$ for q a prime power ≤ 27 .

| G | $ G $ | p | $\ell_p(G)$ | $S_p(G)$ | d_0, d_1, d_2, \dots | $\ell_p(G, 1)$ | $\text{ext}_p^1(G, 1)$ |
|----------|--------------------------------------|-----|-------------|----------|-------------------------------|----------------|------------------------|
| $W(E_6)$ | 51840 | 2 | 5 | 10 | 25, 19, 9, 3, 1 | 5 | 3 |
| | $2^7 \cdot 3^4 \cdot 5$ | 3 | 4 | 5 | 25, 13, 4, 1 | 4 | 2 |
| | | 5 | 2 | 2 | 25, 2 | 2 | 1 |
| $W(E_7)$ | 2903040 | 2 | 7 | 24 | 60, 52, 35, 18, 7, 3, 1 | 7 | 4 |
| | $2^{10} \cdot 3^4 \cdot 5 \cdot 7$ | 3 | 4 | 5 | 60, 30, 8, 2 | 4 | 2 |
| | | 5 | 2 | 2 | 60, 6 | 2 | 1 |
| | | 7 | 2 | 2 | 60, 2 | 2 | 1 |
| $W(E_8)$ | 696729600 | 2 | 8 | 32 | 112, 100, 68, 36, 17, 7, 3, 1 | 8 | 5 |
| | $2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$ | 3 | 5 | 8 | 112, 65, 24, 7, 2 | 5 | 2 |
| | | 5 | 3 | 3 | 112, 17, 2 | 3 | 1 |
| | | 7 | 2 | 2 | 112, 4 | 2 | 1 |
| $W(F_4)$ | 1152 | 2 | 5 | 14 | 25, 21, 12, 4, 1 | 5 | 4 |
| | $2^7 \cdot 3^2$ | 3 | 3 | 4 | 25, 11, 2 | 3 | 2 |
| $W(H_3)$ | 120 | 2 | 3 | 4 | 10, 6, 1 | 3 | 2 |
| | $2^3 \cdot 3 \cdot 5$ | 3 | 2 | 2 | 10, 2 | 2 | 1 |
| | | 5 | 3 | 3 | 10, 4, 2 | 3 | 1 |
| $W(H_4)$ | 14400 | 2 | 4 | 7 | 34, 24, 9, 1 | 4 | 3 |
| | $2^6 \cdot 3^2 \cdot 5^2$ | 3 | 3 | 3 | 34, 11, 2 | 3 | 1 |
| | | 5 | 5 | 6 | 34, 20, 11, 4, 2 | 5 | 2 |

| G | $ G $ | p | $\ell_p(G)$ | $S_p(G)$ | d_0, d_1, d_2, \dots | $\ell_p(G, 1)$ | $\text{ext}_p^1(G, 1)$ |
|----------|--|-----|-------------|----------|------------------------|----------------|------------------------|
| A_5 | 60 | 2 | 2 | 2 | 5, 1 | 2 | 1 |
| | $2^2 \cdot 3 \cdot 5$ | 3 | 2 | 2 | 5, 1 | 2 | 1 |
| | | 5 | 3 | 3 | 5, 2, 1 | 3 | 1 |
| A_6 | 360 | 2 | 3 | 3 | 7, 2, 1 | 3 | 1 |
| | $2^3 \cdot 3^2 \cdot 5$ | 3 | 3 | 3 | 7, 2, 1 | 3 | 1 |
| | | 5 | 3 | 3 | 7, 2, 1 | 3 | 1 |
| A_7 | 2520 | 2 | 3 | 3 | 9, 3, 1 | 3 | 1 |
| | $2^3 \cdot 3^2 \cdot 5 \cdot 7$ | 3 | 3 | 3 | 9, 3, 1 | 3 | 1 |
| | | 5 | 2 | 2 | 9, 1 | 2 | 1 |
| | | 7 | 3 | 3 | 9, 2, 1 | 3 | 1 |
| A_8 | 20160 | 2 | 4 | 5 | 14, 6, 2, 1 | 4 | 2 |
| | $2^6 \cdot 3^2 \cdot 5 \cdot 7$ | 3 | 3 | 3 | 14, 6, 2 | 3 | 1 |
| | | 5 | 3 | 3 | 14, 3, 1 | 2 | 1 |
| | | 7 | 3 | 3 | 14, 2, 1 | 3 | 1 |
| A_9 | 181440 | 2 | 4 | 5 | 18, 8, 3, 1 | 4 | 2 |
| | $2^6 \cdot 3^4 \cdot 5 \cdot 7$ | 3 | 4 | 6 | 18, 10, 3, 1 | 4 | 3 |
| | | 5 | 3 | 3 | 18, 4, 1 | 2 | 1 |
| | | 7 | 2 | 2 | 18, 1 | 2 | 1 |
| A_{10} | 1814400 | 2 | 5 | 7 | 24, 12, 6, 2, 1 | 5 | 2 |
| | $2^7 \cdot 3^4 \cdot 5^2 \cdot 7$ | 3 | 4 | 6 | 24, 13, 4, 1 | 4 | 3 |
| | | 5 | 3 | 3 | 24, 4, 1 | 3 | 1 |
| | | 7 | 3 | 3 | 24, 3, 1 | 2 | 1 |
| A_{11} | 19958400 | 2 | 5 | 7 | 31, 17, 8, 3, 1 | 5 | 2 |
| | $2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$ | 3 | 4 | 5 | 31, 16, 6, 1 | 4 | 2 |
| | | 5 | 3 | 3 | 31, 6, 1 | 3 | 1 |
| | | 7 | 3 | 3 | 31, 4, 1 | 2 | 1 |
| | | 11 | 3 | 3 | 31, 2, 1 | 3 | 1 |
| A_{12} | 239500800 | 2 | 6 | 10 | 43, 25, 13, 6, 2, 1 | 6 | 2 |
| | $2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$ | 3 | 5 | 8 | 43, 22, 9, 2, 1 | 5 | 3 |
| | | 5 | 3 | 3 | 43, 10, 2 | 3 | 1 |
| | | 7 | 3 | 3 | 43, 5, 1 | 2 | 1 |
| | | 11 | 3 | 3 | 43, 2, 1 | 3 | 1 |

| G | $ G $ | p | $\ell_p(G)$ | $S_p(G)$ | d_0, d_1, d_2, \dots | $\ell_p(G, 1)$ | $\text{ext}_p^1(G, 1)$ |
|-------------|---|-----|-------------|----------|------------------------|----------------|------------------------|
| $GL(3, 2)$ | 168 | 2 | 3 | 3 | 6, 2, 1 | 3 | 1 |
| | $2^3 \cdot 3 \cdot 7$ | 3 | 2 | 2 | 6, 1 | 2 | 1 |
| | | 7 | 3 | 3 | 6, 2, 1 | 3 | 1 |
| $SL(2, 8)$ | 504 | 2 | 2 | 2 | 9, 1 | 2 | 1 |
| | $2^3 \cdot 3^2 \cdot 7$ | 3 | 5 | 5 | 9, 4, 3, 2, 1 | 5 | 1 |
| | | 7 | 4 | 4 | 9, 3, 2, 1 | 4 | 1 |
| $SL(3, 3)$ | 5616 | 2 | 5 | 5 | 12, 5, 3, 2, 1 | 5 | 1 |
| | $2^4 \cdot 3^3 \cdot 13$ | 3 | 3 | 3 | 12, 3, 1 | 3 | 1 |
| | | 13 | 5 | 5 | 12, 4, 3, 2, 1 | 5 | 1 |
| $SU(3, 3)$ | 6048 | 2 | 6 | 7 | 14, 9, 6, 4, 2, 1 | 6 | 2 |
| | $2^5 \cdot 3^3 \cdot 7$ | 3 | 3 | 3 | 14, 5, 1 | 3 | 1 |
| | | 7 | 3 | 3 | 14, 2, 1 | 3 | 1 |
| M_{11} | 7920 | 2 | 5 | 5 | 10, 5, 3, 2, 1 | 5 | 1 |
| | $2^4 \cdot 3^2 \cdot 5 \cdot 11$ | 3 | 2 | 2 | 10, 2 | 2 | 1 |
| | | 5 | 2 | 2 | 10, 1 | 2 | 1 |
| | | 11 | 3 | 3 | 10, 2, 1 | 3 | 1 |
| $PSp(4, 3)$ | 25920 | 2 | 4 | 5 | 20, 12, 5, 1 | 4 | 2 |
| | $2^6 \cdot 3^4 \cdot 5$ | 3 | 5 | 7 | 20, 14, 8, 3, 1 | 5 | 2 |
| | | 5 | 2 | 2 | 20, 1 | 2 | 1 |
| M_{12} | 95040 | 2 | 4 | 7 | 15, 9, 3, 1 | 4 | 3 |
| | $2^6 \cdot 3^3 \cdot 5 \cdot 11$ | 3 | 3 | 3 | 15, 4, 1 | 3 | 1 |
| | | 5 | 2 | 2 | 15, 2 | 2 | 1 |
| | | 11 | 3 | 3 | 15, 2, 1 | 3 | 1 |
| J_1 | 175560 | 2 | 2 | 2 | 15, 4 | 2 | 1 |
| | $2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$ | 3 | 2 | 2 | 15, 4 | 2 | 1 |
| | | 5 | 3 | 3 | 15, 6, 3 | 3 | 1 |
| | | 7 | 2 | 2 | 15, 1 | 2 | 1 |
| | | 11 | 2 | 2 | 15, 1 | 2 | 1 |
| | | 19 | 4 | 4 | 15, 3, 2, 1 | 4 | 1 |
| M_{22} | 443520 | 2 | 4 | 5 | 12, 5, 2, 1 | 4 | 2 |
| | $2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$ | 3 | 2 | 2 | 12, 2 | 2 | 1 |
| | | 5 | 2 | 2 | 12, 1 | 2 | 1 |
| | | 7 | 3 | 3 | 12, 2, 1 | 3 | 1 |
| | | 11 | 3 | 3 | 12, 2, 1 | 3 | 1 |
| J_2 | 604800 | 2 | 4 | 5 | 21, 11, 3, 1 | 4 | 2 |
| | $2^7 \cdot 3^3 \cdot 5^2 \cdot 7$ | 3 | 3 | 3 | 21, 7, 1, | 3 | 1 |
| | | 5 | 5 | 5 | 21, 10, 6, 2, 1 | 5 | 1 |
| | | 7 | 2 | 2 | 21, 1 | 2 | 1 |
| HS | 44352000 | 2 | 5 | 9 | 24, 15, 8, 3, 1 | 5 | 3 |
| | $2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$ | 3 | 2 | 2 | 24, 5 | 2 | 1 |
| | | 5 | 3 | 4 | 24, 8, 2 | 3 | 2 |
| | | 7 | 2 | 2 | 24, 1 | 2 | 1 |
| | | 11 | 3 | 3 | 24, 2, 1 | 3 | 1 |

| G | $ G $ | p | $\ell_p(G)$ | $S_p(G)$ | d_0, d_1, d_2, \dots | $\ell_p(G, 1)$ | $\text{ext}_p^1(G, 1)$ |
|--------------|----------------|-----|-------------|----------|----------------------------|----------------|------------------------|
| $PSL(2, 2)$ | 6 | 2 | 2 | 2 | 3, 1 | 2 | 1 |
| S_3 | 2.3 | 3 | 2 | 2 | 3, 1 | 2 | 1 |
| $PSL(2, 3)$ | 12 | 2 | 2 | 2 | 4, 1 | 2 | 1 |
| A_4 | $2^2.3$ | 3 | 3 | 3 | 4, 2, 1 | 3 | 1 |
| $PSL(2, 4)$ | 60 | 2 | 2 | 2 | 5, 1 | 2 | 1 |
| $PSL(2, 5)$ | $2^2.3.5$ | 3 | 2 | 2 | 5, 1 | 2 | 1 |
| A_5 | | 5 | 3 | 3 | 5, 2, 1 | 3 | 1 |
| $PSL(2, 7)$ | 168 | 2 | 3 | 3 | 6, 2, 1 | 3 | 1 |
| | $2^3.3.7$ | 3 | 2 | 2 | 6, 1 | 2 | 1 |
| | | 7 | 3 | 3 | 6, 2, 1 | 3 | 1 |
| $PSL(2, 8)$ | 504 | 2 | 2 | 2 | 9, 1 | 2 | 1 |
| | $2^3.3^2.7$ | 3 | 5 | 5 | 9, 4, 3, 2, 1 | 5 | 1 |
| | | 7 | 4 | 4 | 9, 3, 2, 1 | 4 | 1 |
| $PSL(2, 9)$ | 360 | 2 | 3 | 3 | 7, 2, 1 | 3 | 1 |
| A_6 | $2^3.3^2.5$ | 3 | 3 | 3 | 7, 2, 1 | 3 | 1 |
| | | 5 | 3 | 3 | 7, 2, 1 | 3 | 1 |
| $PSL(2, 11)$ | 660 | 2 | 2 | 2 | 8, 2 | 2 | 1 |
| | $2^2.3.5.11$ | 3 | 2 | 2 | 8, 2 | 2 | 1 |
| | | 5 | 3 | 3 | 8, 2, 1 | 3 | 1 |
| | | 11 | 3 | 3 | 8, 2, 1 | 3 | 1 |
| $PSL(2, 13)$ | 1092 | 2 | 2 | 2 | 9, 2 | 2 | 1 |
| | $2^2.3.7.13$ | 3 | 2 | 2 | 9, 2 | 2 | 1 |
| | | 7 | 4 | 4 | 9, 3, 2, 1 | 4 | 1 |
| | | 13 | 3 | 3 | 9, 2, 1 | 3 | 1 |
| $PSL(2, 16)$ | 4080 | 2 | 2 | 2 | 17, 1 | 2 | 1 |
| | $2^4.3.5.17$ | 3 | 3 | 3 | 17, 5, 2 | 2 | 1 |
| | | 5 | 5 | 5 | 17, 6, 4, 2, 1 | 3 | 1 |
| | | 17 | 9 | 9 | 17, 8, 7, 6, 5, 4, 3, 2, 1 | 9 | 1 |
| $PSL(2, 17)$ | 2448 | 2 | 5 | 5 | 11, 4, 3, 2, 1 | 5 | 1 |
| | $2^4.3^2.17$ | 3 | 5 | 5 | 11, 4, 3, 2, 1 | 5 | 1 |
| | | 17 | 3 | 3 | 11, 2, 1 | 3 | 1 |
| $PSL(2, 19)$ | 3420 | 2 | 2 | 2 | 12, 3 | 2 | 1 |
| | $2^2.3^2.5.19$ | 3 | 5 | 5 | 12, 4, 3, 2, 1 | 5 | 1 |
| | | 5 | 3 | 3 | 12, 4, 2 | 3 | 1 |
| | | 19 | 3 | 3 | 12, 2, 1 | 3 | 1 |
| $PSL(2, 23)$ | 6072 | 2 | 4 | 4 | 14, 5, 3, 1 | 3 | 1 |
| | $2^3.3.11.23$ | 3 | 3 | 3 | 14, 4, 1 | 2 | 1 |
| | | 11 | 6 | 6 | 14, 5, 4, 3, 2, 1 | 6 | 1 |
| | | 23 | 3 | 3 | 14, 2, 1 | 3 | 1 |
| $PSL(2, 25)$ | 7800 | 2 | 4 | 4 | 15, 5, 3, 1 | 3 | 1 |
| | $2^3.3.5^2.13$ | 3 | 3 | 3 | 15, 4, 1 | 2 | 1 |
| | | 5 | 3 | 3 | 15, 2, 1 | 3 | 1 |
| | | 13 | 7 | 7 | 15, 6, 5, 4, 3, 2, 1 | 7 | 1 |
| $PSL(2, 27)$ | 9828 | 2 | 2 | 2 | 16, 4 | 2 | 1 |
| | $2^2.3^3.7.13$ | 3 | 3 | 3 | 16, 2, 1 | 3 | 1 |
| | | 7 | 4 | 4 | 16, 6, 4, 2 | 4 | 1 |
| | | 13 | 7 | 7 | 16, 6, 5, 4, 3, 2, 1 | 7 | 1 |

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November 8, 2006

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